Aide Mémoire

Subject: Useful formulas for sediment transport and morphology in rivers and channels

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1. General basic equations

Three-dimensional (3D):

The general 3D mass-balance equations for a unit volume of 2-phase (fluid and sediment) flow (after time-averaging of turbulence and applying the eddy-viscosity concept) are :

For fluid:

$$\frac{\partial(1-c_s)}{\partial t}+\frac{\partial u(1-c_s)}{\partial x}+\frac{\partial v(1-c_s)}{\partial y}+\frac{\partial w(1-c_s)}{\partial z}+ \\ +\frac{\partial}{\partial x}\left[\varepsilon_{fx}\frac{\partial c_s}{\partial x}\right]+\frac{\partial}{\partial y}\left[\varepsilon_{fy}\frac{\partial c_s}{\partial y}\right]+\frac{\partial}{\partial z}\left[\varepsilon_{fz}\frac{\partial c_s}{\partial z}\right]=0 \tag{1}$$

For sediment (convection-diffusion equation):

$$\frac{\partial c_s}{\partial t}+\frac{\partial u c_s}{\partial x}+\frac{\partial v c_s}{\partial y}+\frac{\partial (w-w_s)c_s}{\partial z}+ \\ -\frac{\partial}{\partial x}\left[\varepsilon_{sx}\frac{\partial c_s}{\partial x}\right]-\frac{\partial}{\partial y}\left[\varepsilon_{sy}\frac{\partial c_s}{\partial y}\right]-\frac{\partial}{\partial z}\left[\varepsilon_{sz}\frac{\partial c_s}{\partial z}\right]=0 \tag{2}$$

where:

c_s	=	local mean sediment volume concentration (suspension)
u, v, w	=	fluid-velocity in x,y,z -direction
w_s	=	fall velocity of sediment particles
ε_{fx}, ε_{fy}, ε_{fz}	=	fluid mixing coefficient in x,y,z-direction
ε_{sx}, ε_{sy}, ε_{sz}	=	sediment mixing coefficient in x,y,z -direction (assume $\varepsilon_s \approx \varepsilon_f$)

Or after rewriting:

$$\frac{\partial c_s}{\partial t}+u\frac{\partial c_s}{\partial x}+v\frac{\partial vc_s}{\partial y}+w\frac{\partial c_s}{\partial z}-\frac{\partial}{\partial x}\left[\varepsilon_{sx}\frac{\partial c_s}{\partial x}\right]-\frac{\partial}{\partial y}\left[\varepsilon_{sy}\frac{\partial c_s}{\partial y}\right]= \\ = w_s\left(1-c_s\right)^{\alpha}\frac{\partial c_s}{\partial z}+\frac{\partial}{\partial z}\left[\varepsilon_{sz}\frac{\partial c_s}{\partial z}\right] \quad (3)$$

where

α	=	coefficient used for large concentrations (small concentrations: $\alpha = 1$, else $\alpha \approx 4$ to 5)

Two-dimensional depth-averaged (2DH)

The general 2D continuity equations for sediment, after depth-averaging, become:

Sediment balance:

$$\left(1-\varepsilon_p\right)\frac{\partial z_b}{\partial t}+\frac{\partial s_{bx}}{\partial x}+\frac{\partial s_{by}}{\partial y}+\frac{\partial hC_s}{\partial t}+\frac{\partial UhC_s}{\partial x}+\frac{\partial VhC_s}{\partial y}=0 \quad (4)$$

where:

C_s	=	depth-averaged suspended sediment concentration
h	=	water depth
s_{bx}, s_{by}	=	bed-load transport per unit of width in x,y-direction
U, V	=	depth-averaged flow velocity in x,y -direction
ε_p	=	porosity of bed (void volume / total volume) (for sand 0.4 ± 0.05)

Modified Galappatti's equation for suspended sediment:

$$T_a'\frac{\partial C_s}{\partial t}+L_a'U\frac{\partial c_s}{\partial x}+L_a'V\frac{\partial vc_s}{\partial y}=\frac{w_s}{h}\left(C_{se}-C_s\right) \quad (5)$$

where:

T_a'	=	dimensionless adaptation time scale (roughly between 0.5 and 1)
L_a'	=	dimensionless adaptation time scale (approximately equal to T_a')
C_s	=	depth-averaged equilibrium bed-load concentration
C_{se}	=	depth-averaged equilibrium suspended sediment concentration (following from transport equation)

The adaptation time scale can be obtained from the figure below, as function of the ratios w_s/u_* and $u_*/u=(\sqrt{g})/C$. Dependent on the type of boundary condition used at the bed for depth-integration a value for T_a' can be selected from the graphs below. The concentration type is more suitable during erosion, while the other works better during sedimentation.

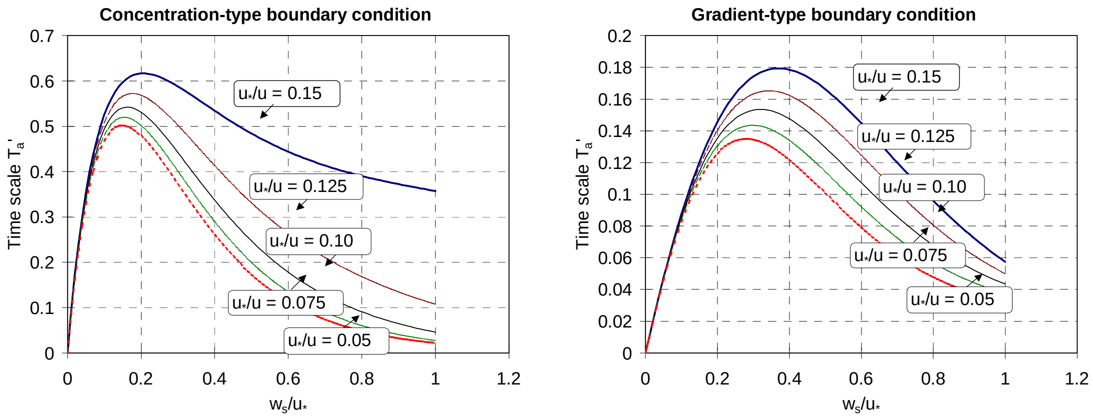


Figure 1 Galappatti's adaptation time-scale for concentration-type boundary condition (left) and gradient-type boundary conditions (right)

<u>One-dimensional cross-sectional averaged (1D)</u>

The one-dimensional basic equation for sediment can be obtained directly from the 2DH equations, resulting in:

$$\left(1-\varepsilon_p\right)\frac{\partial z_b}{\partial t}+\frac{\partial s_{bx}}{\partial x}+\frac{\partial hC_s}{\partial t}+\frac{\partial UhC_s}{\partial x}=0 \qquad (6)$$

And with the original Galappatti's equation for suspended sediment:

$$\frac{\partial C_s}{\partial t}+U\frac{\partial c_s}{\partial x}=\frac{w_s}{T_a'h}\left(C_{se}-C_s\right) \qquad (7)$$

Time scale T_a' can be obtained from figure 1.

2. Sediment characteristics

Definitions and standards:

- Apply the Phi-size for plotting the sediment size distribution (based on assumption of log-normal distributions):

$$\phi_i=\log_2\left(D_i\right) \quad \text{with} \quad D_i \text{ in millimeter} \qquad (8)$$

- Mode = most common grain size in the population (with two humps: bimodal mixture)
- Median grain size = D_{50} = grain size for which 50% of the sample is finer.
- Mean grain size = D_m = average grain size of the deposit with $D_m=\sum p_i D_i$
- Sieve diameter: if a particle falls through sieve D_A and remains on next sieve D_B then the sieve diameter $D = (D_A D_B)^{1/2}$. Sieving is used for gravel and sand > 70 µm.
- Geometric mean size (first moment of size frequency distribution):

1. $D_g = \sqrt{D_{84} D_{16}}$

(9)

2. $D_g = 2^{-\phi_m} \quad with \quad \phi_m = \sum_{i=1}^{N} \phi_{mi} p_i$

where ϕ_m is the arithmic mean phi-size, ϕ_{mi} is the class middle (in between sieve mesh sizes) of a size fraction, and p_i is probability in size fraction

- Geometric standard deviation (second moment of size frequency distribution; how large is variation around the mean):

1. $\sigma_g = 0.5\left(\frac{D_{84}}{D_{50}} + \frac{D_{50}}{D_{16}}\right)$

(10)

2. $\sigma_g = 2^{\sigma_m} \quad with \quad \sigma_m = \sum_{i=1}^{N} (\phi_{mi} - \phi_m)^2 \; p_i$

where ϕ_m is the arithmic mean phi-size (see above), ϕ_{mi} is the class middle (in between sieve mesh sizes) of a size fraction, and p_i is probability in size fraction

- *Specific weight* of a deposit is the density or weight per unit volume, expressed as dry weight or dry density including the pores:

$$\rho_d = \left(1 - \varepsilon_p\right)\rho_s \tag{11}$$

with

ε_p = porosity (void volume / total volume) (for sand 0.4 ± 0.05):

ρ_s = specific weight or density of sediment particles (usually order of 2650 kg/m^3, quartz)

- *Sediment concentrations* are often expressed in milligrams per liter (C_{mgl} in mg/l) or in parts per million (C_{ppm} in ppm). Conversion formulas:

- If $C_{mgl} < 16{,}000$ mg/l then: $C_{mgl} = C_{ppm}$ (12)

- If $C_{mgl} > 16{,}000$ mg/l then: $C_{ppm} = \frac{10^6}{\frac{10^6}{C_{mgl}} + 1 - \frac{1}{\sigma_s}} \quad and \quad C_{mgl} = \frac{10^6}{\frac{10^6}{C_{ppm}} - 1 + \frac{1}{\sigma_s}}$ (13)

In which σ_s is the specific gravity of sediment (in the order of 2.65)

- Grain size classification

Classification according to American Geophysical Union				
Sediment	**Millimetres**		**Sediment**	**Millimetres**
Very large boulders	4096 - 2048		Very coarse sand	2.0 - 1.0
Large boulders	2048 - 1024		Coarse sand	1.0 - 0.5
Medium boulders	1024 - 512		Medium sand	0.5 - 0.25
Small boulders	512 - 256		Fine sand	0.25 - 0.125
Large cobbles	256 - 128		Very fine sand	0.125 - 0.062
Small cobbles	128 - 64		Coarse silt	0.062 - 0.031
Very coarse gravel	64 - 32		Medium silt	0.031 - 0.016
Coarse gravel	32 - 16		Fine silt	0.016 - 0.008
Medium gravel	16 - 8		Very fine silt	0.008 - 0.004
Fine gravel	8 - 4		Coarse clay	0.004 - 0.002
Very fine gravel	4 - 2		Medium clay	0.002 - 0.001
			Fine clay	0.0010 - 0.0005
			Very fine clay	0.0005 - 0.00024

- Fall velocity w_s [m/s] for sediment (van Rijn, 1984)

1. For Stokes range ($Re=w_sD/\nu < 1$ or $D \leq 100$ µm): $w_s = \frac{1}{18}\frac{\Delta g D^2}{\nu}$ (14)

2. For $100 < D \leq 1000$ µm: $w_s = \frac{10\nu}{D}\left(\sqrt{1+\frac{0.01\Delta g D^3}{\nu^2}}-1\right)$ (15)

3. For $D > 1000$ µm: $w_s = 1.1\sqrt{\Delta g D}$ (16)

With

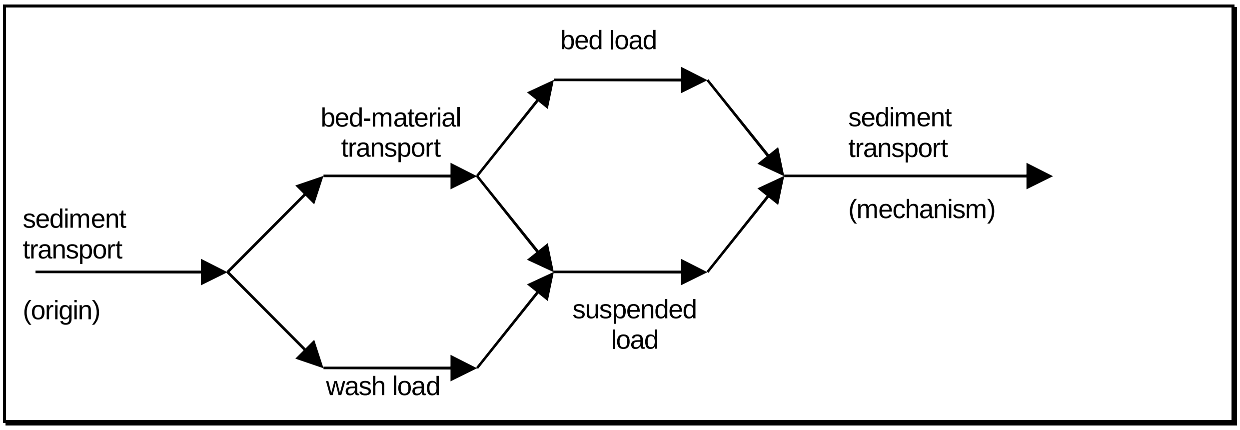
Δ = relative density of sediment, defined as $(\rho_s-\rho_w)/\rho_w$ (usually about 1.65)

ν = kinematic viscosity for clear fluid (usually $1 \cdot 10^{-6}$ m^2/s)

3. Sediment transport

Classification

- To study sediment transport processes it is possible to classify the transport processes according to origin and mechanisms as in the following scheme (Jansen et al., 1979):



Shields and initiation of motion (Shields curve)

- The Shields parameter is defined as the ration between flow forces and gravity forces on a sediment particle, and can be expressed as:

$$\theta = \frac{\tau_b}{(\rho_s - \rho)\, gD} = \frac{u_*^2}{\Delta gD} = \frac{u^2}{C^2 \Delta D} = \frac{h \cdot i}{\Delta D} \qquad \text{(17a)}$$

where

D	=	characteristic grain size
i	=	slope
u_*	=	shear velocity $= (\tau_b / \rho)^{½} = u \cdot g^{½}/C$
Δ	=	relative density = $(\rho_s - \rho)/\rho \approx$ 1.65 for quartz
ρ	=	density of water ≈ 1000 kg/m^3
ρ_s	=	density of sediment ≈ 1650 kg/m^3 for quartz

- Initiation of motion according to Shields: the threshold for initiation of motion is defined by the critical Shields parameter in the Shields curve. The Shields curve expresses the relation between the Shields parameter θ_c and the grain Reynolds number Re_*, see figure:

$$Re_* = \frac{u_* D}{\nu} \quad ; \quad \theta_c = \frac{\tau_{cr}}{(\rho_s - \rho)\, gD} = \frac{u_{*,cr}^2}{\Delta gD} \qquad \text{(17b)}$$

where

Re_*	=	particle Reynolds number
$u_{*,cr}$	=	critical shear velocity $= (\tau_{cr} / \rho)^{½}$
Δ	=	relative density = $(\rho_s - \rho)/\rho \approx$ 1.65 for quartz
ν	=	kinematic viscosity, usually $1 \cdot 10^{-6}$ m^2/s
ρ	=	density of water ≈ 1000 kg/m^3
ρ_s	=	density of sediment ≈ 1650 kg/m^3 for quartz

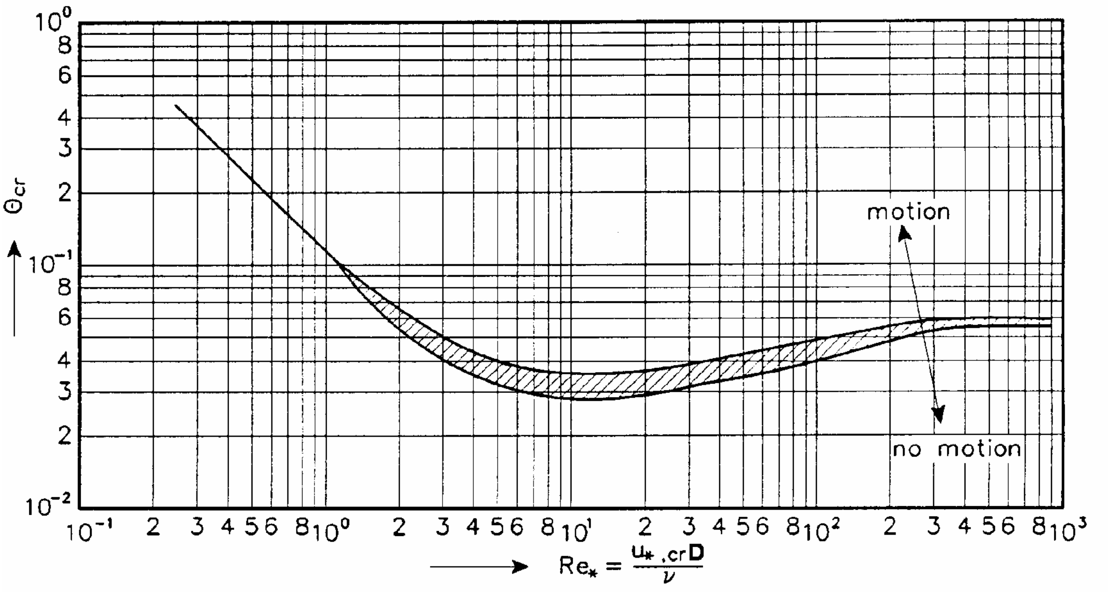


Figure Shields' diagram for initiation of motion (Shields parameter versus grain Reynolds number)

Van Rijn (1984) gives an approximation of the Shields curve as follows:

$$\left.\begin{aligned}
\theta_{cr} &= \frac{0.24}{D_*} && \text{if} \quad D_* \le 4 \\
\theta_{cr} &= \frac{0.14}{D_*^{0.64}} && \text{if} \quad 4 < D_* \le 10 \\
\theta_{cr} &= \frac{0.04}{D_*^{0.10}} && \text{if} \quad 10 < D_* \le 20 \\
\theta_{cr} &= 0.013 D_*^{0.29} && \text{if} \quad 20 < D_* \le 150 \\
\theta_{cr} &= 0.055 && \text{if} \quad D_* > 150
\end{aligned}\right\} \text{ with } D_* = D_{50}\left(\frac{\Delta g}{\nu^2}\right)^{1/3} \qquad (18)$$

Transport formula of Meyer-Peter and Müller (1984)

The empirical formula of Meyer-Peter and Müller (MP-M) for bed-load transport is written as

$$\Phi_s = 8\left(\psi - 0.047\right)^{3/2} \qquad (19)$$

with $\mu = \left(C/C_{90}\right)^{3/2}$ (ripple factor) (20)

$$\Phi_s = \frac{s}{D^{3/2}\sqrt{g\Delta}} \quad \text{(transport parameter)} \qquad (21)$$

$$\psi = \mu\theta = \frac{\mu h i}{\Delta D_{50}} = \mu \frac{u^2}{C^2 \Delta D_{50}} \quad \text{(flow parameter)} \qquad (22)$$

$$D = \overline{D} = \sum^{i}\left(p_i D_i\right) / \sum^{i} p_i \qquad (23)$$

$$C_{90} = 18 \cdot \log(12\,h/D_{90}) \qquad (24)$$

i = energy gradient (bed gradient in case of uniform flow conditions)

s = bed-load transport per unit of width (without pores, otherwise multiply with $1/(1-\varepsilon_p) \approx 1.66$)

It is valid for situations in which $w_s/u_* > 1$, $D_m > 0.4$ mm, and $\mu\theta < 0.2$.

For the MP-M formula an approximate power function $s = m\,u^n$ can be defined, with n equal to

$$n = \frac{3}{1-0.047\Psi^{-1}} \qquad (25)$$

Note: can only be used if C-value is known.

Transport formula of Engelund and Hansen (1967)

The formula of Engelund and Hansen (1967) for total load (bed- and suspended bed-material load) is written as:

$$\phi_s = 0.05\,\Psi^{5/2} \qquad (26)$$

with

$$\mu = \left(C^2/g\right)^{2/5} \text{ (ripple factor)} \qquad (27)$$

$$\Phi_s = \frac{s}{D_{50}^{3/2}\sqrt{g\Delta}} \text{ (transport parameter)} \qquad \text{(cf. eq. 21)}$$

$$\psi = \mu\theta = \frac{\mu h i}{\Delta D_{50}} \text{ (flow parameter)} \qquad \text{(cf. eq. 22)}$$

i = energy gradient (bed gradient in case of uniform flow conditions)

s = bed-load transport per unit of width (without pores, otherwise multiply with $1/(1-\varepsilon_p) \approx 1.66$)

The formula can also be written as (useful for unsteady flow):

$$s = 0.05\sqrt{g\Delta D_{50}^3}\left(\frac{u_*}{\sqrt{g\Delta D_{50}}}\right)^3\left(\frac{u}{\sqrt{g\Delta D_{50}}}\right)^2 = 0.05\frac{u^5}{\sqrt{g}C^3\Delta^2 D_{50}} \qquad (28)$$

Engelund and Hansen is valid for situations in which $w_s/u_* < 1$, $0.19 < D_{50} < 0.93$ mm, and $0.07 < \theta < 6$.

Transport formula of van Rijn (1984)

The transport formula of van Rijn (1984) distinguishes between a bed-load part (s_b) and a suspended-load part (s_s):

$$s = s_b + s_s \qquad (29)$$

A transport stage parameter T is defined as:

$$T = \frac{\tau_b' - \tau_{b\,cr}}{\tau_{b\,cr}} = \frac{(C/C')^2\,\theta - \theta_{cr}}{\theta_{cr}} \qquad (30)$$

with

τ_b' = bed-shear related to grains
$\tau_{b\,cr}$ = critical bed-shear according ot the Shields curve

Bed-shear τ_b' is written, with C'=C_{90} (see Meyer-Peter and Müller) as

$$\tau_b' = \left(\frac{C}{C'}\right)^2 \tau_b \qquad (31)$$

A dimensionless particle parameter D_* is defined as

$$D_* = D_{50}\left(\frac{\Delta g}{\nu^2}\right)^{\frac{1}{3}} \qquad (32)$$

<u>Bed-load transport</u> then follows from

$$\Phi_b = 0.053\frac{T^{2.1}}{D_*^{0.3}} \; voor\; T < 3$$

$$\Phi_b = 0.1\frac{T^{1.5}}{D_*^{0.3}} \; voor\; T \geq 3 \qquad (33)$$

with a transport parameter

$$\Phi_b = \frac{s_b}{\sqrt{g\Delta D_{50}{}^3}} \qquad (34)$$

and s_b is bed-load transport without pores, particles in the range 200-2000 µm.

<u>Suspended-load transport</u> then follows from

$$s_s = Fuhc_a \qquad (35)$$

with

u = depth-averaged flow velocity

h = water depth

F = integration factor

C_a = reference concentration at level a measured from the bed

a = reference level, assumed to be 1/2*bed-form-height or equal to equivalent roughness height k_s, with minimum value 0.01*h.

The reference concentration c_a (excluding pores) is written as

$$c_a = 0.015 \frac{D_{50}}{a} \frac{T^{1.5}}{D_*^{0.3}} \tag{36}$$

And the integrattion factor F is written as:

$$F = \frac{\left(\frac{a}{h}\right)^{Z'} - \left(\frac{a}{h}\right)^{1.2}}{\left(1 - \frac{a}{h}\right)^{Z'} (1.2 - Z')} \tag{37}$$

in which

$$Z' = \frac{w_s}{\left(1 + 2\left(\frac{w_s}{u_*}\right)^2\right) \kappa u_*} + 2.5 \left(\frac{w_s}{u_*}\right)^{0.8} \left(\frac{c_a}{0.65}\right)^{0.4} \quad \text{for} \quad 0.01 \le \frac{w_s}{u_*} \le 1 \tag{38}$$

Where κ = constant of Von Karman (=0.4). The fall velocity w_s is computed (using euations 14 to 16) from the representative grain size of suspended sediment D_s, following from

$$\frac{D_s}{D_{50}} = 1 + 0.011 \left(\frac{1}{2} \left(\frac{D_{84}}{D_{50}} + \frac{D_{50}}{D_{16}} \right) - 1 \right) (T - 25) \tag{39}$$

<u>Transport formula of Ackers and White (1973)</u>

The formula of Ackers and White (1973) starts from the definition of D_{gr} :

$$D_{gr} = D_{35} \left[\frac{g\Delta}{\nu^2} \right]^{1/3} \tag{40}$$

for coarse sediment, which is only considered here if $D_{gr} > 60$.

Then can be deduced that

$$G_{gr} = 0.025 \left[\frac{F_{gr}}{0.17} - 1 \right]^{1.5} \tag{41}$$

with

$$F_{gr} = \frac{1}{\sqrt{g\Delta D_{35}}} \; \frac{u}{\sqrt{32}\,\log\dfrac{10h}{D_{35}}} \tag{42}$$

and

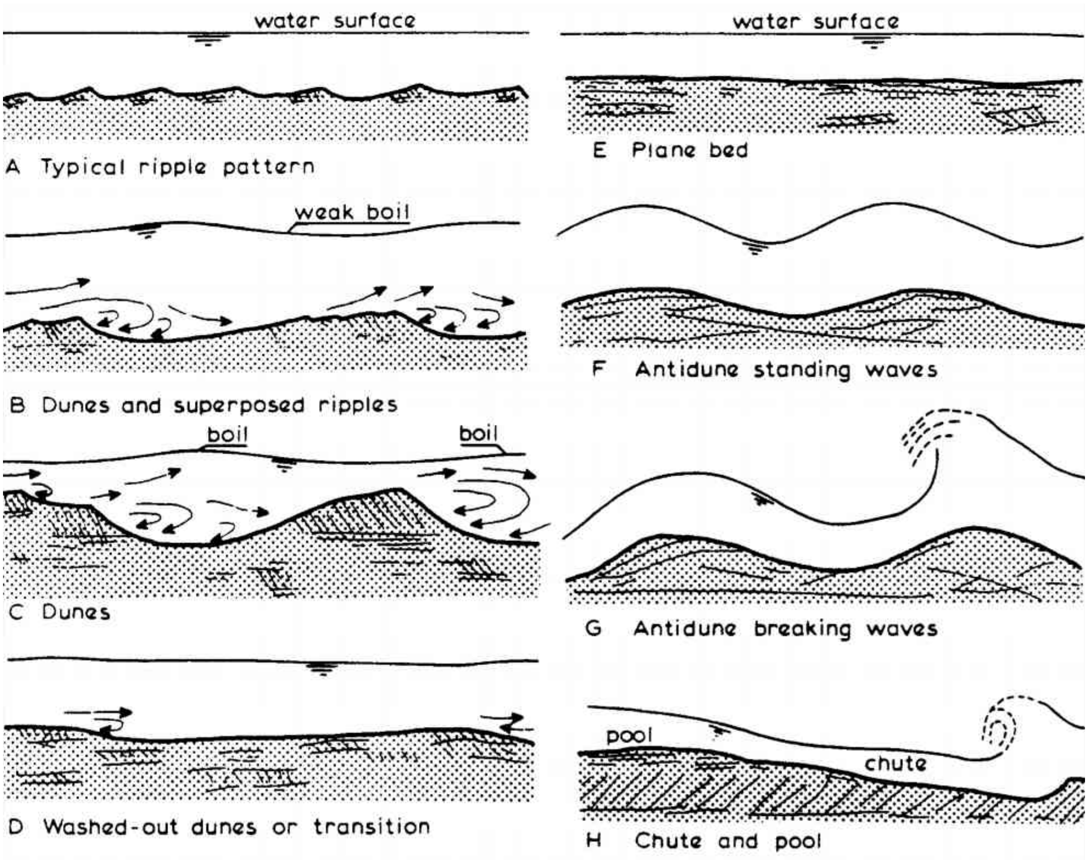
$$X_* = \frac{(\Delta+1)D_{35}\,G_{35}}{h}\left\{\frac{C}{\sqrt{g}}\right\} \tag{43}$$

and finally

$$s = \frac{X_* q \rho}{\rho_s(1-\varepsilon)} \qquad \text{met } \; \varepsilon \approx 0.4 \tag{44}$$

4. Bed forms and alluvial roughness

Types and classification are shown below:



Flow regime	Bedform	Bed-material concentration (ppm)	Mode of sediment transport	Type of roughness	Roughness $\frac{C}{\sqrt{g}}$
Lower regime	Ripples	10 – 200	Discrete steps	Form roughness pre-dominate	7.8 – 12.4
	Ripples on dunes	100 – 1,200			7.0 – 13.2
	Dunes	200 – 2,000			
Transition	Washed-out dunes	1,000 – 3,000		Variable	7.0 – 20.2
Upper regime	Plane bed	2,000 – 6,000	continu	Grain-roughness pre-dominates	16.3 – 20
	Antidunes	> 2,000			10.3 - 20
	Chutes and pools	> 2,000			9.4 – 10.7

Alluvial roughness predictor of White c.s (1980)

The roughness predictor of White C.S. (1980), here only presented for coarse material ($D_{gr} = D_{35}[g\Delta/\nu^2]^{1/3} > 60$), follows an iteration procedure. Given is bed slope i , a discharge per unit of width q, grain characteristics (ρ_s and D_{35}), and fluid characteristics (ρ and ν). The water depth (h) is estimated, from which can be computed that $u_* = \sqrt{ghi}$.

This results in

$$F_{fg} = \frac{u_*}{\sqrt{g\Delta D}} \tag{45}$$

Then determine F_{gr} from:

$$\frac{F_{gr} - 0.17}{F_{fg} - 0.17} = 1.0 - 0.76\left[1.0 - \frac{1}{\exp\left[\{\log D_{gr}\}^{1.7}\right]}\right] \tag{46}$$

Compute the flow velocity u from

$$F_{gr} = \frac{1}{\sqrt{g\Delta D}} \frac{u}{\sqrt{32} \cdot \log\{10h/D\}} \tag{47}$$

Make an improved estimate for the depth $h=q/u$ until h is sufficiently accurate.

Compute C from

$$C = \frac{u}{u_*}\sqrt{g} = \frac{u}{\sqrt{hi}} \tag{48}$$

τ) in a component due to bed forms (τ'') and a component due to roughness of sediment particles (τ'). Hence,

$$hi = (hi)'' + (hi)' \tag{49}$$

with $i = i'' = i'$, the depth (h) is subdivided into h'' and h'.

In 1967 Engelund en Hansen propose (with $\psi' = h'i / \Delta D$ en $\psi = hi / \Delta D$):

$$\Psi' = 0.06 + 0.4\Psi^2 \tag{50}$$

However, based on measurements, Engelund and Fredsøe (1982) propose:

$$\Psi' = 0.06 + 0.3\Psi^{3/2} \tag{51}$$

With given q, i, Δ and D the iteration procedures is as follows:

(i) Estimate u, after which h' can be computed from

$$\frac{u}{u'_*} = \frac{u}{\sqrt{g'hi}} = 9.45\left\{\frac{h'}{k}\right\}^{'1/8} \tag{52}$$

i.e., for roughness without bed forms.

Recommended value for $k = 2.5 \cdot D_s$ (sedimentation diameter).

(ii) Determine $\psi' = h'i / \Delta D$

(iii) Determine ψ from eq. (50) or (51) respectively. From ψ follows h.

(iv) Make an improvement estimation of u from $q = u \cdot h$ and repeat the procedure.

(v) With $u = C\sqrt{hi}$ the resulting C-value can then be determined.

Roughness predictor of Van Rijn (1987)

The bed form height H can be estimated from the following equation:

$$H = 0.11\,h\left(\frac{D_{50}}{h}\right)^{0.3}\left(1-\mathrm{e}^{-0.5\mathrm{T}}\right)\left(25-T\right) \quad \text{if} \quad 0 < T < 25$$
$$H = 0 \quad \text{if} \quad T \leq 0 \quad \text{or} \quad T \geq 25 \tag{53}$$

In which:

T	=	transport stage parameter, defined by equation (30)
h	=	water depth

Furthermore, the bed form length L can be defined as:

$$\mathrm{L} = 7.3\,\mathrm{h} \tag{54}$$

Based on these formules the total equivalent Nikuradse roughness height k_s is defined by van Rijn as:

$$k_s = k_s' + 1.1H\left(1 - e^{-25\frac{H}{L}}\right) \tag{55}$$

In which

$k_s' = 3\,D_{90}$ is the equivalent Nikuradse roughness height of grains.

Using the equivalent Nikuradse roughness height the Chézy coefficient for total roughness becomes

$$C = 18\log\left(\frac{12\,h}{k_s}\right) \tag{56}$$

5. Morphological time scales and celerities

The celerities express the propagation speed of small disturbances (infinitely small waves) on the water surface or on the bed. For the one-dimensional equations three celerities can be deduced:

- a negative and a positive celerity ($c_{1,2}$) associated to the propagation of long waves on the water surface ($\approx u \pm \sqrt{[gh]}$)
- a positive celerity associated to the propagation of small bed-waves, with an approximate value:

$$c_3 = \left(u_m \psi\right) \Big/ \left(1 - Fr^2\right) \text{ where } \psi = \frac{ds/du}{h} \approx n \cdot \frac{s}{q} \qquad (57)$$

with Fr = Froude number of the main-channel flow, h = average water depth in main channel, q = main-channel discharge per unit of width, s = sediment-transport rate per unit of width, u_m = flow velocity in main channel, n = power of the transport formula (if $s = mu^n$).

Theoretically the morphological time-scale T is defined as

$$T = \varepsilon \frac{L \cdot h}{s} \qquad (58)$$

With ε = definitionfactor dependent on the actual time-scale definition, with an order of magnitude of 1.0; and L = characteristic length scale.

Different types of phenomena are usually considered for definition of the time scale:

- Wave propagation: for small length scales hydraulic roughness is negligible, and morphological changes have a wave-character. The propagation speed of the wave (with small amplitude) is defined as:

$$c = \frac{u}{h}\left(\frac{ds}{du}\right) = \frac{n \cdot s}{h\left(1 - Fr^2\right)} \qquad (59)$$

With n = non-linearity of the transport formula (e.g., see formula 25), and Fr is the Froude number. The time scale T_g is defined as the time in which the wave propagates of a distance L_g :

$$T_g = \frac{L_g}{c} = \frac{L_g h\left(1 - Fr^2\right)}{n \cdot s} \qquad (60)$$

- Diffusion: for changes over long lengths (L_d > {2 to 3}·h/i) the bed-profile development has a diffusion character with a diffusion coefficient

$$D = \frac{u}{3i}\left(\frac{ds}{du}\right) = \frac{n \cdot s}{3i} \qquad (61)$$

With i = slope. The time scale T_d is defined as the time in which, at a distance L_d upstream of a measure, 50 % of the final erosion or sedimentation due to this measure has occurred

$$T_d = \frac{L_d^2}{D} = \frac{3L_d^2 \cdot i}{n \cdot s} \qquad (62)$$

- Relaxation of 2D (depth-averaged) processes: for processes that adapt gradually to an equilibrium condition, resulting in a diffusion-type of adaptation of the 2D morphology. The time-scale T_{2D} is defined as the time in which along one bank 50% of the total erosion or sedimentation occurs, caused by a measure taken on the other bend. It is written as:

$$T_{2D} = \frac{B^2 \, \mathrm{f}(\theta)}{\pi^2 s} = \frac{\lambda_s h}{s} \tag{63}$$

Where B = river width, f(θ) = function of the Shields parameter which expresses the effect of transverse bed-slope on the direction of sediment transport (see section 6), and λ_s = adaptation length for 2D morphology according to Struiksma et al (1985), defined as:

$$\lambda_s = \frac{\mathrm{f}(\theta)}{\pi^2}\left(\frac{B}{h}\right)B \tag{64}$$

6. Two-dimensional (2D) morphology

Due to curvature of flow (in river bends) the morphology is determined by near-bed flow defelction by helical flow and by gravity effects on particles by transverse slope. These effects are reproduced by correcting the sediment-transport direction as follows:

$$\tan(\beta) = \frac{\sin(\alpha) - \mathrm{f}(\theta,\eta)^{-1}\dfrac{\partial z}{\partial \eta}}{\sin(\alpha) - \mathrm{f}(\theta,\xi)^{-1}\dfrac{\partial z}{\partial \xi}} \tag{65}$$

In which $\partial z/\partial \eta$ and $\partial z/\partial \xi$ are bed-level slope in transverse (η) and in flow (ξ) direction respectively, and f(θ) = function of the Shields parameter which expresses the effect of transverse bed-slope on the direction of sediment transport.

In river bend redistribution of flow and sediment occur, which leads to an axi-symmetric solution in a long bend (or a bend with strongly damped morphology) with constant radius. The transverse bed slope β of the axi-symmetric situation is defined as:

$$\tan(\beta) = A\,\mathrm{f}(\theta)\frac{h}{R} \tag{66}$$

In which R = radius of curvature of the bend flow, and A = spiral flow coefficient defined as:

$$A = \frac{2\varepsilon}{\kappa^2}\left(1 - \frac{\sqrt{g}}{\kappa C}\right) \tag{67}$$

Where ε = calibration factor (~1), κ = Von Karman coefficient (≈0.4), g = gravity acceleration, and C = Chézy value. The function f(θ) was defined by Talmon et al (1995) as:

$$\mathrm{f}(\theta) = 9\left(\frac{D}{h}\right)^{0.3}\sqrt{\theta} \tag{68}$$

In practice often the function is written as f(θ)=0.85√θ for natural channels (or about 1.7√θ for laboratory flumes).

The shape of a point bar in river bend is determined by the wave length L_p and damping length L_D expressed by Struiksma et al (1985) as:

$$2\pi \frac{\lambda_w}{L_p} = \frac{1}{2}\sqrt{(n+1)\,IP^{-1} - IP^{-2} - \left(\frac{n-3}{2}\right)^2} \quad (68)$$

$$\frac{\lambda_w}{L_D} = \frac{1}{2}\left(IP^{-1} - \frac{n-3}{2}\right) \quad (69)$$

Where n = non-linearity of the transport formula (e.g., see formula 25), λ_w = adaptation length for 2D flow, defined as $\lambda_w = C^2h/(2g)$, λ_s = adaptation length for 2D morphology according to Struiksma et al (1985), defined as equation 64. IP is the interaction parameter defined as:

$$IP = \frac{\lambda_s}{\lambda_w} \approx\sim 2\frac{g}{C^2}\left(\frac{B}{h}\right)^2\left(\frac{D}{h}\right)^{0.3}\sqrt{\theta} \quad (70)$$