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# A model for a viscous preflush prior to gelation in a porous medium

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**Abstract**

In this paper we propose and analyse a simplified mathematical model on the effect of a fully miscible preflush on gel-placement in an oil reservoir. The approach is based on a segregated flow model even if the two phases are fully miscible. For the rate of a constant injection rate condition, fully implicit solutions can be constructed. Saturation profiles consist of shocks and fingering zones. For constant pressure conditions we construct a semi-explicit relation for the shock position and gel penetration depth.

keywords: self-similar solution, analytic solution, transport in porous media

# 1 Introduction

Polymer gels are used in a wide variety of fields, e.g. improved oil and gas recovery, confinement of ground water contaminants, water treatment plant control and filtration. One motivation is that mature oil fields suffer from excessive water production. Large water production causes serious environmental problems concerning water waste control. Moreover, operation cost increase and large oil reserves remain unproduced. A major cause of large water production is water-chanelling through high permeability layers in reservoirs.

To minimize water production, polymer-gels in aqueous solution are injected in the near well-bore region, aiming at a decrease of the permeability of high permeability regions. The gel adsorbs at the pore surface of the porous medium and consequently the permeability decreases. To improve the efficiency of the gel-placement a pre-treatment of the reservoir prior to gel-placement is often necessary. This treatment is in the form of injection of a fluid with a very high viscosity. This treatment is referred to as a viscous preflush and

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aims at a diversion of the polymer-gels from the low permeability regions. It is our aim to analyse a simplified model for this viscous preflush.

We assume that the porous medium is initially saturated with water. Since the viscosity of the viscous preflush fluid is much higher than the viscosity of water, the transition region between the fluid and water becomes negligible and effectively results in a shock. When the thin gel, whose viscosity is in the same order of magnitude as the viscosity of water, displaces the far more viscous preflush fluid, a fingering zone occurs between the gel and viscous fluid. This has been observed by [4]. They mainly observed fingering in the high permeability layer. In this viscous fingering area the effective water viscosity is between the pure water viscosity and the viscosity of the viscous preflush fluid. For this effective viscosity, we use a quarter power mixing rule as has been done by Koval [6]. We note for completeness that there are many other choices for this effective viscosity. Ourt model is based on a displacement model of miscible fluids derived by Koval. Stavland analysed this viscous preflush mainly experimentally and numerically [7].

In the present we aim at an analytic treatment of the mathematical problem. Stavland [7] also obtained an analytic solution for the injection of a viscous fluid. However, the subsequent inlet of a much less viscous fluid was not incorporated in their analytic solution. The immediate change of the condition at the inlet is referred to as a 'variable inlet condition'. Modelling transport in porous media with variable inlet conditions has, among others, been done earlier by Karakas [8] and Chen [9]. Karakas et al present a mathematical model describing combined injection of chemical components and hot water, aiming at a displacement of viscous oil. Chen et al consider Barenblatt self-similar solutions for the porous medium equation with a variable inlet condition according to a power-law in time.

This paper deals with an analysis of transport of miscible fluids with high viscosity contrast combined with an abruptly changing inlet condition. First, we obtain a self-similar solution for the saturation profile at consecutive times for the case of a constant injection rate. Subsequently, we apply this solution to a porous medium over which a constant pressure drop is applied. The analysis can serve as a tool for a check of numerical solutions obtained when modelling a viscous preflush treatment.

The paper is organised as follows. First we formulate the model equations. Subsequently, we solve the equations. This is followed by some examples and an application. We end up with some conclusions.

# 2 The Model

In this section we formulate a model for flow of two fully miscible fluids in porous media. We consider an open rectangular two dimensional domain $\Omega := \{(y,z) \in \mathbb{R}^2 : 0 < r_w < y < R, 0 < z < H\}$. The horizontal and vertical co-ordinates are respectively denoted by $y$ and $z$. At $t = 0$, $t$ denotes time, injection of a viscous fluid with viscosity $\mu_1$, which is constant, is started at the well at $y = r_w$ into the domain. The domain is treated as initially fully saturated with water. Injection of the viscous fluid takes place during the time-interval $t \in [0, T)$. For $t \geq T$ water with dissolved gel is injected. Since the viscosity

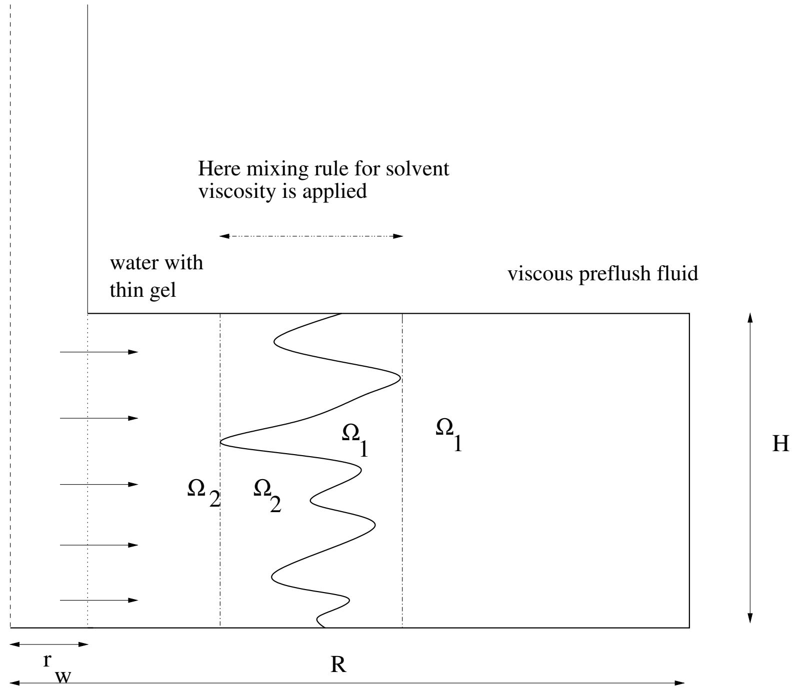


Figure 1: A sketch of the region near the fingering zone in the reservoir.

of the viscous fluid is several orders of magnitude larger than the viscosity of water and viscosity of water with dissolved gels, we use the simplification that the presence of gel does not effect the water viscosity. This can be used as a first order approximation. The viscosity of water with and without gel is denoted by $\mu_2$. This viscosity, $\mu_2$, depends on the presence of the viscous preflush fluid. Polymer/gel-wall interactions are neglected and the flow is incompressible. We disregard molecular diffusion and dispersion effects. The flow is modelled as a segregated flow (see Figure 1).

## 2.1 Transport

Let the porous medium in the domain $\Omega$ be homogeneous and isotropic with porosity $\phi$. Assuming the flow to be uniformly distributed accross the thickness of the reservoir, the *average* specific discharge of each phase ($q_i$) is given for $(y, z) \in \Omega$ and $t > 0$:

$$q_i = \frac{Q_i}{H}, \qquad i \in \{1, 2\}, \tag{2.1}$$

where $Q_i$ ($\mathrm{m}^3/\mathrm{s}$) represents the total volumetric flow of fluid $i$ in the domain $\Omega$.

## 2.2 Equations

For completeness we formulate the volume balance equations for the viscous phase in a layer of the reservoir. The geometry of a layer has been sketched in Figure 1. The fingering zone, as shown there, is a mixing on a macroscopic scale. We reduce the problem to a well-posed one-dimensional problem by averaging over the height $H$ of the reservoir. We use a procedure similar to the method of van Duijn and Strack [5]. They apply this to model flow of salt and fresh water in a porous medium. Furthermore, we assume that there is no flow through horizontal boundaries.

Let $S_i$ denote the saturation of phase $i$, then the mass balance equations are for $(y, z) \in \Omega$, $t > 0$:

$$\phi \frac{\partial S_i}{\partial t} + \underline{\nabla} \cdot \underline{q}_i = 0, \qquad i \in \{1, 2\}. \tag{2.2}$$

Here $\underline{q}_i = (q_{i,y}, q_{i,z})^T$. In the present work we are interested in averaged quantities over the height $H$ to reduce above equation to a one-dimensional problem. First we define an area element $dydz$ and we integrate above equation over the rectangular area, $V$, with vertices $(y, 0), (y + \Delta y, 0), (y + \Delta y, H), (y, H)$ to get

$$\int_0^H \int_y^{y+\Delta y} \phi \frac{\partial S_i}{\partial t} dydz + \int_0^H \int_y^{y+\Delta y} \underline{\nabla} \cdot \underline{q_i} dydz = 0.$$

Subsequently, we apply Gauss' Theorem for a planar domain to obtain

$$\phi \int_0^H \int_y^{y+\Delta y} \frac{\partial S_i}{\partial t} dydz + \oint_{\partial V} \underline{q_i} \cdot \underline{n} ds = 0,$$

where we treat $\phi$ as a constant and $\partial V := \overline{V} \setminus V$ represents the boundary of rectangle $R$. For the integral over $y$ in the first term, we apply the Mean Value Theorem for integrals. For the second term the condition $q_z = 0$ on $z \in \{0, H\}$ and $\underline{n} = (1, 0)^T$ on $y = y$ and $\underline{n} = (-1, 0)^T$ on $y = y + \Delta y$ are applied, this gives

$$\Delta y \phi \frac{\partial}{\partial t} \int_0^H S_i(\xi, z) dz + \int_0^H (q_{i,y}(y + \Delta y, z) - q_{i,y}(y, z)) dz = 0,$$

for a certain $\xi \in (y, y + \Delta y)$.

Since we integrate over the co-ordinate $z$, we interchange the differentiation and integration. Subsequently, the Mean Value Theorem is applied again for the second term to get

$$\Delta y \phi \frac{\partial}{\partial t} \int_0^H S_i(\xi, z) dz + \Delta y \int_0^H \frac{\partial q_{i,y}}{\partial y} (\eta, z) dz = 0,$$

for a certain $\eta \in (y, y + \Delta y)$.

We let $\Delta y \to 0$ and divide above equation by $\Delta y H$ and (re)define

$$u_i = u_i(y) := \frac{1}{H} \int_0^H S_i(y, z) dz,$$

$$q_i = q_i(y) := \frac{1}{H} \int_0^H q_{i,y}(y, z) dz,$$

With $\xi, \eta \to y$ as $\Delta y \to 0$, this gives

$$\phi \frac{\partial u_i}{\partial t} + \frac{\partial q_i}{\partial y} = 0. \tag{2.3}$$

Summation of above equation over $i \in \{1, 2\}$, use of $u_1 + u_2 = 1$ and subsequent integration over $y$ gives

$$q_1 + q_2 = Q(t) \tag{2.4}$$

Let $\Omega_1$ and $\Omega_2$, $\Omega_1 \cup \Omega_2 \cup (\overline{\Omega}_1 \cap \overline{\Omega}_2) = \Omega$, be the non-overlapping open subdomains of $\Omega$ that respectively contain the viscous fluid and water (overlines refer to the closures of the subdomains), then Darcy Law gives

$$q_{i,z} = \begin{cases} q_{i,y} = -\dfrac{k_0 S_i}{\mu_i} \dfrac{\partial p}{\partial y}, & (y, z) \in \Omega_i, \\ 0, & (y, z) \in \Omega \setminus \Omega_i. \end{cases}$$

We use the concept of vertical equilibrium, meaning that we disregard viscous forces in the vertical direction ($z$). This means that we have hydrostatic equilibrium, i.e.

$$p(y, z) = p_0(y) + \rho g z,$$

and consequently

$$\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial y}\right) = 0 = \frac{\partial}{\partial y}\left(\frac{\partial p}{\partial z}\right).$$

Then the averaged specific discharge reads as

$$q_i = -\frac{k_0 u_i}{\mu_i} \frac{\partial p}{\partial y}.$$

Combination of above equation with equation (2.4), subsequent substitution into equation (2.3), and defining
$x := \phi(y - r_w)$ with $\tilde{\Omega} := \{x \in \mathbb{R} : 0 < x < L\}$, where $L := \phi(R - r_w)$ gives

$$\begin{aligned} q_{i,x} &= \frac{Q(t)}{1 + \frac{u_2 \mu_1}{u_1 \mu_2}} =: Q(t) f(u_1) \\ \frac{\partial u_i}{\partial t} &+ Q(t) \frac{\partial f}{\partial x}(u_1) = 0 \\ q_i &= -\frac{k_0 u_i}{\mu_i \phi} \frac{\partial p}{\partial x} \end{aligned} \tag{2.5}$$

Here $f(u_i)$ is referred to as the flux-function. We reduced the two dimensional problem to a one dimensional problem. Here $Q(t)$ is interpreted as a velocity.

Initially the porous medium is saturated with water and no preflush fluid is present in the reservoir. Between $t = 0$ and $t = T$ the viscous fluid is injected. After $t = T$, water is injected again. Hence we are faced with the following initial and boundary conditions

$$(IB)\begin{cases} u_1(x,0) = 0, \quad 0 < x < L \\ \\ u_1(0,t) = \begin{cases} 1, & 0 < t < T \\ 0, & t \geq T. \end{cases} \end{cases}$$

The boundary condition at $x = 0$ is referred to as a variable-inlet condition. Karakas et al [8] analyse the injection of a chemical slug under a variable inlet condition.

We deal with two fully miscible fluids. We use a mixing rule, see Koval [6], for the water viscosity in the area where $0 < u_1 < 1$. Let $\mu_w$ be the viscosity of *pure* water, then we use the following function for $E = E(u_1) := \dfrac{\mu_1}{\mu_2(u_1)}$

$$E = E(u_1) = \begin{cases} E_1, & u_1 \in \{0,1\}, \\ \\ \left(0.78 + 0.22E_1^{1/4}\right)^4, & u_1 \in (0,1), \end{cases}$$

where $E_1 := \dfrac{\mu_1}{\mu_w}$. The fluxfunction $f = f(u_1)$ becomes

$$f = f(u_1) = \frac{u_1}{u_1 + (1 - u_1)E}. \tag{2.6}$$

# 3 Analysis

Since $E(u_1)$ is constant on $u_1 \in (0,1)$, see equation 2.6, the flux-function is continuous and differentiable on $u_1 \in (0,1)$. For $\mu_1 > \mu_w$ we have $1 < E \leq E_1$. Since $E > 1$ it can be shown using equation (2.6) that $f''(u_1) > 0$ for all $0 < u_1 < 1$. On the contrary if $\mu_1 < \mu_2$ then $f''(u_1) < 0$ for all $0 \leq u_1 \leq 1$. Note that the case $\mu_1 = \mu_2$, i.e. $f(u_1) = u_1$, is similar to single phase flow. Since for $0 < t < T$ we have $u_1 = 1$ at $x = 0$, it is clear from the entropy condition [12] that the interface between the viscous fluid and water is a stable shock. For completeness, we illustrate this by the method of characteristics. For more details on this standard method we suggest to consult for instance the books of Smoller [12] and Rhee et al [11]. Let $u_1(x,t) = u_1(x(t),t)$ be the solution over characteristics, then we choose

$$\frac{du_1(x(t),t)}{dt} = \frac{\partial u_1(x(t),t)}{\partial t} + \frac{\partial u_1(x(t),t)}{\partial x}\frac{dx(t)}{dt} \equiv 0.$$

Combination of above relation with equation (2.5) implies if $u_1$ is continuous and differentiable near $x(t)$

$$\frac{dx(t)}{dt} = f'(u_1) \cdot Q(t).$$

This gives the reciprocal slope of characteristics in the x,t-plane. From differentiation of equation (2.6) follows that $\lim_{u_1 \to 0^+} f'(u_1) = \frac{1}{E_2} < 1 < E_2 = \lim_{u_1 \to 1^-} f'(u_1)$, $f(u_1)$ is convex, and hence characteristics originating from the $x$ and $t$-axis (for $0 < t < T$) intersect (see Figure 2). This intersection results in a stable shock at position $s(t)$ (see Figure 2, between the viscous preflush fluid and water, which is the reverse case of the rarefaction as obtained from the Koval model), travelling with velocity

$$\dot{s}(t) = \frac{f(u_1(s_+(t),t)) - f(u_1(s_-(t),t))}{u_1(s_+(t),t) - u_1(s_-(t),t)} \cdot Q(t) = \frac{Q(t)}{(u_1(s_-(t),t) + (1 - u_1(s_-(t),t)) \cdot E)}. \tag{3.1}$$

Above condition is referred to as the Rankine-Hugoniot condition [12]. Note that $E = E(u_1)$ which has been defined previously. If $u_1(s_-(t),t) = 1$ then $\dot{s}(t) = Q(t)$ and if $u_1(s_-(t),t) \in (0,1)$ then $E = E_2$. It follows that $\dot{s}(t) < Q(t)$ for $0 < u_1(s_-(t),t) < 1$ and that $\dot{s}(t) \to \dfrac{Q(t)}{E_2}$ as $u_1(s_-(t),t) \to 0$. We will show later in this section that the rarefaction travels faster than the shock and that under certain condition the rarefaction overtakes the shock. So there exists a $\tau > T$ such that $\dot{s}(t) = Q(t)$ for all $0 < t < \tau$ and $\dfrac{Q(t)}{E_2} < \dot{s}(t) < Q(t)$ for $t > \tau$. This point has been indicated in Figure 2.

## 3.1 The saturation profile for $t > T$

For $t > T$ (injection of water and gel) the injection saturation is changed and from the entropy condition [12], [11] a rarefaction between water with dissolved gel and the viscous fluids results (see Figure 2), whereas the shock between the viscous fluid and water, $s(t)$, continues to move. In the rarefaction part the saturation is continuous and we give a self-similar solution for this region, where $0 < u_1 < 1$. Both fluids are fully miscible. However, due to absence of molecular diffusion, the fluids do not actually mix on a microscale. Since equation (2.2) holds on an upscaled macro-scale, the continuous part of the solution is interpreted as a fingering zone [6], which is modelled as mixing on an integrated scale. We seek solutions of the form

$$u_1(x,t) = \overline{u}_1(\eta), \quad \eta := \frac{x}{\int_T^t Q(\chi) d\chi}.$$

Note that this $\eta$ is not the same as used in Section 2. A generalisation for all curvilinear co-ordinates can be done easily. We have chosen to omit this now since it does not change the nature of the problem. Substitution of this transformation into equation (2.5) gives

$$\overline{u}_1' = 0 \quad \text{or} \quad \eta = f'(u_1).$$

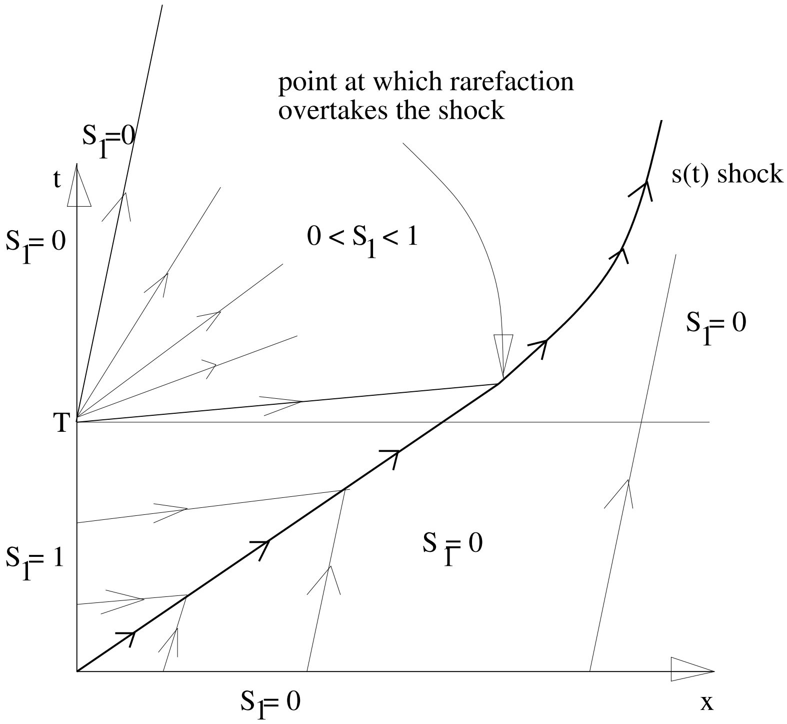


Figure 2: Sketch of characteristics in the $x, t$-plane.

The first solution corresponds to a constant state solution. A shock (with velocity from equation (3.1)) occurs at the interface between the viscous fluid and water, where a high viscous fluid displaces a low viscous fluid. The second solution represents a rarefaction (fingering) part between water and the viscous fluid. Here we have for the viscosity ratio $E = E_2$. Assuming that both fluids are fully miscible, we combine equation (2.6) with above relation to obtain

$$\overline{u}_1(\eta) = \frac{\sqrt{\frac{E_2}{\eta}} - E_2}{1 - E_2}. \tag{3.2}$$

Above relation holds for the rarefaction part of the saturation profile, i.e. $\overline{u}_1(\eta)$ is continuous for a region $\eta_L < \eta < \eta_R$. The bounds $\eta_L$ and $\eta_R$ are to be determined. We are interested in bounded solutions $\overline{u}_1(\eta) \in [0, 1]$, see (IB). From above relation follows that $\overline{u}(\eta)$ increases strictly monotonically since $\mu_1 > \mu_2$. Therefore there exists a pair $(\eta_L, \eta_R)$, $\eta_L < \eta_R$, such that $\overline{u}_1(\eta_L) = 0$ and $\overline{u}_1(\eta_R) = 1$. Using above relation gives for $\eta_L$ and $\eta_R$

$$\eta_L = \frac{1}{E_2} < 1, \quad \eta_R = E_2 > 1.$$

note that $E_2 > 1$. We introduce the positions $x_L(t)$ and $x_R(t)$ such that respectively $u_1(x_L(t), t) = 0$ and $u_1(x_R(t), t) = 1$ for all $T < t < \tau$, where bound $\tau$ is to be determined.

Since $E > 1$, it is clear that for $T < t < \tau$

$$\dot{x}_L(t) = \frac{Q(t)}{E_2} < Q(t) = \dot{s}(t) < E_2\, Q(t) = \dot{x}_R(t),$$

where $x_L(T) = 0 = x_R(T)$.

With $s(t) = \int_0^t Q(\chi)d\chi$ and $x_L(T) = 0 = x_R(T)$, we obtain $x_L(t) = \frac{1}{E_2} \cdot \int_T^t Q(\chi)d\chi$, $x_R(t) = E_2 \cdot \int_T^t Q(\chi)d\chi$, since $E_2 > 1$ we have $E_2 Q(t) > \dot{s}(t)$ and hence there exists a $t = \tau$ such that $x_R(t)$ reaches the shock position, i.e.

$$\int_T^\tau E_2 Q(\chi)d\chi = \int_0^\tau Q(\chi)d\chi = s(\tau).$$

This implies

$$E_2(s(\tau) - s(T)) = s(\tau) \Leftrightarrow (1 - E_2)s(\tau) = E_2 s(T),$$

and

$$\int_T^\tau Q(\chi)d\chi = \frac{\int_0^T Q(\chi)d\chi}{E_2 - 1}.$$

Here $\tau$ is the time at which the rare-faction overtakes the shock. The shock is overtaken within the reservoir, e.g. domain $\tilde{\Omega}$, iff $s(\tau) < L$, i.e. $E_2 s(T) < L(E_2 - 1)$. Summarised, we showed:

**Proposition 1:** *Let $Q(t) > 0$ be integrable over $t > 0$, then if $E_2 s(T) < L(E_2 - 1)$ then*

1. *there exists a $\tau > T$ such that*

$$s(\tau) = \frac{E_2 s(T)}{E_2 - 1},$$

2. *here $\tau$ also satisfies*

$$\int_T^\tau Q(\chi)d\chi = \frac{\int_0^T Q(\chi)d\chi}{E_2 - 1}.$$

□

Note that $s(T) = \int_0^T Q(\chi)d\chi$ and for $t < \tau$ we have $s(t) = \int_0^t Q(\chi)d\chi$.

For $t > \tau$ the saturation at $x \rightarrow s_-(t)$ is given by (e.g. equation (3.2))

$$u_1(s_-(t), t) = \frac{\sqrt{\dfrac{E_2 \int_T^\tau Q(\chi)d\chi}{s(t)}} - E_2}{1 - E_2}.$$

Since $s(t) < E_2 \int_T^\tau Q(\chi)d\chi$ for $t > \tau$ we have $u_1(s_-(t), t) < 1$. Substitution of above expression into equation (3.1) and using $E = E_2$ for $u_1(s_-(t), t) < 1$ gives for $t > \tau$

$$\dot{s}(t) = \sqrt{\frac{s(t)}{E_2 \int_T^t Q(\chi)d\chi}}\, Q(t). \tag{3.3}$$

We require for physical reasons that the shock position $s(t)$ is a continuous function of time $t$. Hence continuity of $s(t)$ at $t = \tau$ and solving equation (3.3) imply for the solution of (3.3):

$$\begin{aligned} s(t) = (\sqrt{s(\tau)} + \sqrt{\tfrac{1}{E_2}}\{\left[\textstyle\int_T^t Q(\chi)d\chi\right]^{\frac{1}{2}} + \\ - \left[\textstyle\int_T^\tau Q(\chi)d\chi\right]^{\frac{1}{2}}\})^2, \quad t > \tau. \end{aligned} \tag{3.4}$$

Above relation causes the convex shape of the shock-curve, $s(t)$, for $t > T$ in the $x,t$-plane (see Figure 2). For compactness of notation we define the subintervals $J_1$, $J_2$ and $J_3$ as follows:

- $J_1 := \left[0, \left(\frac{1}{E_2}\int_T^t Q(\chi)d\chi\right)_+\right]$,
- $J_2 := \left((\frac{1}{E_2}\int_T^t Q(\chi)d\chi)_+, \min\left\{s(t), E_2\int_T^t Q(\chi)d\chi\right\}\right)$,
- $J_3 := \left[\left(E_2\int_T^t Q(\chi)d\chi\right)_+, s(t)\right]$

Here we use the convention $(.)_+ := \max\{0, .\}$. Above subintervals physically correspond to respectively the gel, gel-preflush fluid and preflush fluid region.

Using equation (3.2) and integrated expressions for $x_L(t)$, $x_R(t)$ we arrive at the following expression which describes the saturation profile for all $x > 0$ and $t > 0$:

$$u_1(x,t) = \begin{cases} 0, & x \in J_1, t > T \\ \dfrac{\sqrt{\frac{E_2\int_T^t Q(\chi)d\chi}{x}} - E_2}{1 - E_2}, & x \in J_2, t > T \\ 1, & x \in J_3, \quad t < \tau, \\ 0, & x \geq s(t), \end{cases}$$

where the shock position $s(t)$ is given by

$$s(t) = \begin{cases} \int_0^t Q(\chi)d\chi, & t < \tau, \\ \left(\sqrt{s(\tau)} + \sqrt{\frac{1}{E_2}}\{\left[\int_T^t Q(\chi)d\chi\right]^{\frac{1}{2}} - \left[\int_T^\tau Q(\chi)d\chi\right]^{\frac{1}{2}}\}\right)^2, & t \geq \tau. \end{cases}$$

The time $\tau$ is defined by Proposition 1. Differentiation of $s = s(t)$ with respect to time near $t = \tau$ gives $s'(t) \to Q(\tau)$ as $t \to \tau_-$ and $s'(t) \to \dfrac{\sqrt{s(\tau)}Q(\tau)}{\sqrt{\int_T^\tau Q(\chi)d\chi E_2}} = Q(\tau)$ as $t \to \tau^+$, hence $s'(t)$ is continuous at $t = \tau$ if $Q(t)$ is continuous in $t = \tau$. We see that $\dot{s}(t) = Q(t)$ for all $0 < t < T$ and as $t \to \infty$ we have $s(t) \to \frac{1}{E_2} \int_T^t Q(\chi) d\chi$ for $Q(t) > 0$. Note that hence $s'(t) \to \frac{Q(t)}{E_2}$ as $t \to \infty$. Summarised, this gives:

**Proposition 2:** *Let $Q(t) > 0$ be integrable over $t > 0$, then*

1. *the shock-speed, $s'(t)$, is continuous in $t = \tau$ if $Q(t)$ is continuous in $t = \tau$,*
2. *let $\frac{s(t)}{L} < 1$, $L \to \infty$ then $s'(t) \to \frac{Q(t)}{E_2}$ as $t \to \infty$* $\square$.

In the second part of above proposition we assume that $L$ is sufficiently large. In subsequent subsections the procedure to determine $\tau$ is pointed out.

## 3.2 The pressure drop

From integration of the lower equation in system (2.5) we obtain for the pressure difference for $t > 0$

$$\Delta p = -\frac{\mu_1 Q}{k_0 \phi} \cdot \int_{\tilde{\Omega}} \frac{dx}{(u_1(1 - E(u_1)) + E(u_1))}, \tag{3.5}$$

where $\Delta p$ (Pa) represents the pressure difference between the locations $y = r_w$ and $y = R$ (over the reservoir). Furthermore, note that $E = E(u_1)$ in above equation.

## 3.3 Constant Injection Rate

For this case $Q(t) = Q$, hence from Proposition 1 we find that $\tau = \frac{E_2 T}{E_2 - 1} = \frac{T}{1 - 1/E_2} > T$ and $s(\tau) = \frac{E_2 Q T}{E_2 - 1}$. Further, for $Q$ constant the shock position $s(t)$ is given by

$$s(t) = \begin{cases} Qt, \quad t < \tau, \\ \left( \sqrt{s(\tau)} + \sqrt{\frac{Q}{E_2}} \left\{ (t - T)^{\frac{1}{2}} - (\tau - T)^{\frac{1}{2}} \right\} \right)^2, t \geq \tau. \end{cases}$$

We note that the shock speed is continuous at $t = \tau$ (see also Proposition 2). We give an example of the solution in Figure 3. Input parameters are $E_2 = 2$, $T = 4$. The curves correspond to times $t = 2$, 6, 10 and 20. In this example we set $Q(t) = 1$ at all times. Note that the figure corresponds to linear (planar) geometry. The shock speed stays constant for $0 < t \leq \tau$ and subsequently decreases for $t > \tau$. Note further that the shock speed is continuous. Furthermore, it is clear that $s(t) \to \frac{Q}{E_2}$ as $t \to \infty$.

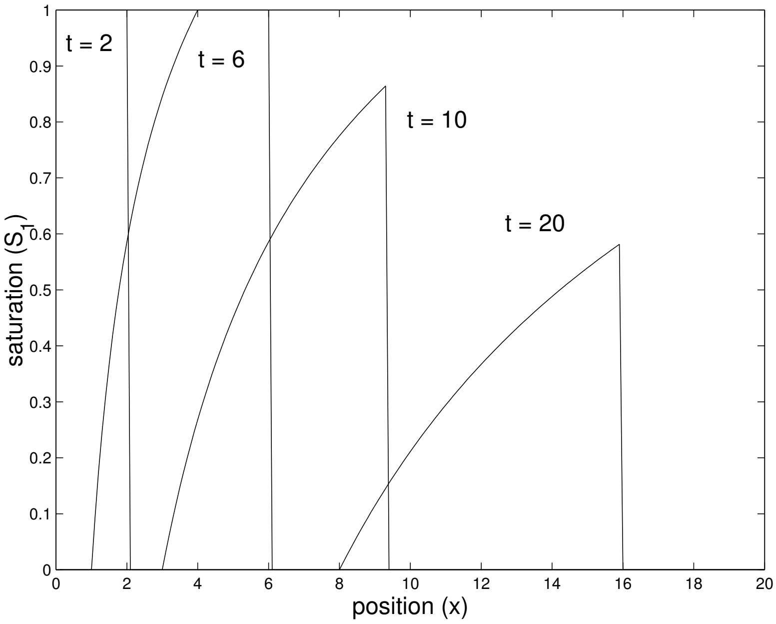


Figure 3: Saturation profiles at consecutive times for the case of constant injection rate.

### 3.4 Constant Pressure Condition

For the case that $\Delta p(t) = \Delta p$, one obtains from equation (3.5) for $t > 0$

$$\begin{aligned} Q(t) &= -\frac{k_0 \Delta p \phi}{\mu_1 I}, \\ I &:= \int_{\tilde{\Omega}} \frac{dx}{(u_1(x,t)(1 - E(u_1)) + E(u_1))} \end{aligned} \tag{3.6}$$

This can be applied to the case of several parallel layers with different permeability over which a pressure drop, which is equal over all layers, is applied.

To determine $I$ when $E > 1$, we split the time into three respective intervals: the 'preflush injection interval' ($0 < t \leq T$), the 'early gel injection interval' ($T < t \leq \tau$) and the 'late gel injection interval' ($t > \tau$). The saturation profile and shock speed during these time intervals are obtained from the preceding analysis. These time intervals are treated subsequently.

#### 3.4.1 The Preflush Injection Interval

During this interval ($0 < t \leq T$) viscous fluid is injected and we notice from the preceding subsection that there is a stable shock travelling with speed $Q(t)$ in the $x,t$-plane. Since there is no mixing we have $E = E_1$. Hence combining equation (3.6) with

$$u_1(x,t) = \begin{cases} 1, & x < s(t) \\ 0, & x > s(t), \end{cases}$$

yields

$$I = s(t) + (L - s(t))\frac{1}{E_1} \tag{3.7}$$

Substitution of above expession into equation (3.6) gives, with $s'(t) = Q(t)$

$$-\frac{k_0 \Delta p \phi}{\mu_1 s'(t)} = s(t)(1 - \frac{1}{E_1}) + \frac{L}{E_1}. \tag{3.8}$$

Separation of variables and subsequent solving of the algebraic equation yields

$$s = \frac{-L + \sqrt{L^2 - 2E_1(E_1 - 1)\alpha t}}{E_1 - 1},$$

where $\alpha := \frac{k_0 \Delta p \phi}{\mu_1}$. Above expression relates the shock position directly to time.

### 3.4.2 The Early Gel Injection Interval

During this interval ($T < t \leq \tau$) water with gel is injected and fingering takes place. The subintervals $J_1, J_2$ and $J_3$ are now given by

- $J_1 = \left[0, \frac{1}{E_2}\int_T^t Q(\chi)d\chi\right]$
- $J_2 = \left(\frac{1}{E_2}\int_T^t Q(\chi)d\chi, E_2\int_T^t Q(\chi)d\chi\right)$
- $J_3 = \left[E_2\int_T^t Q(\chi)d\chi, s(t)\right]$

Now the saturation $S_1(x,t)$ is given by for $E(S_1) \neq 1$

$$u_1(x,t) = \begin{cases} 0, & x \in J_1, \\ \dfrac{\sqrt{\frac{E_2\int_T^t Q(\chi)d\chi}{x}} - E_2}{1 - E_2}, & x \in J_2, \\ 1, & x \in J_3, \\ 0, & x \geq s(t), \end{cases}$$

with $s(t) = \int_0^t Q(\chi)d\chi$.

Using above saturation relation the integral $I$ can be computed using straightforward integration, yielding

$$I = \int_T^t Q(\chi)d\chi \cdot \left\{ \frac{1}{E_1 E_2} + \frac{2(E_2^3 - 1)}{3E_2^2} - E_2 \right\} + \frac{L-s}{E_1} + s(t).$$

When we define, for convenience, $f(t) := \int_T^t Q(\chi)d\chi$, $\beta := \frac{E_1 - 1}{E_1} + \frac{1}{E_1 E_2} + \frac{2(E_2^3 - 1)}{3E_2^2} - E_2$ and $\gamma := \frac{L - s(T)}{E_1} + s(T)$, then

$$-\frac{\alpha}{f'} = \beta f + \gamma.$$

Separation of variables gives a quadratic equation in $f$. Using $f(T) = 0$ gives as solution

$$f = f(t) = \frac{-\gamma + \sqrt{\gamma^2 - 2\alpha\beta(t-T)}}{\beta} = \int_T^t Q(\chi)d\chi$$

The shock position is determined by $s(t) = s(T) + f(t)$. The velocity $Q(t)$ is determined by differentiation of above equation with respect to $t$. Furthermore, from the equations follows the value of $\tau$, which is the time at which the shock is overtaken by the rarefaction:

$$\tau = T + \frac{\gamma^2 - (\frac{s(T)}{E_2 - 1}\beta + \gamma)^2}{2\alpha\beta}.$$

### 3.4.3 The Late Gel Injection Interval

During this interval ($t \geq \tau$) still water with gel is injected. However, this interval differs from preceding interval since the fingering zone overtook the shock, i.e. $E_2(s(t) - s(T)) > s(t)$ and $\dot{s}(t) < Q(t)$ for $t > \tau$. The subintervals $J_1, J_2$ and $J_3$ become

- $J_1 = \left[0, \frac{1}{E_2} \int_T^t Q(\chi)d\chi\right)$
- $J_2 = \left[\frac{1}{E_2} \int_T^t Q(\chi)d\chi, s(t)\right]$

For this case the saturation profile is given by:

$$u_1(x,t) = \begin{cases} 0, & x \in J_1, \\ \dfrac{\sqrt{\frac{E_2 \int_T^t Q(\chi)d\chi}{x}} - E_2}{1 - E_2}, & x \in J_2, \\ 0, & x \geq s(t), \end{cases}$$

The shock position $s(t)$ is related to $Q(t)$ by equation (3.4). Defining again $f(t) := \int_T^t Q(\chi)d\chi$, gives for the integral $I$:

$$I = \frac{f}{E_1E_2} + \frac{L}{E_1} - \frac{\left(\sqrt{s(\tau)} + \sqrt{\frac{1}{E_2}}(\sqrt{f} - \sqrt{f(\tau)})\right)^2}{E_1} +$$

$$+\frac{2}{3\sqrt{E_2 f}}\left\{(\sqrt{s(\tau)} + \sqrt{\frac{1}{E_2}}(\sqrt{f} - \sqrt{f(\tau)}))^3 - (\frac{f}{E_2})^{3/2}\right\}$$

Substitution of this expression into equation (3.6) gives following ordinary differential equation in $f$.

$$-\frac{\alpha}{I(f)} = f'(t), \qquad t > \tau.$$

Here the integral $I$ is a function of $f$. Subsequently above equation can be solved by separation of variables. An implicit solution in terms of an algebraic equation is then obtained. We do not give this solution due to its complexity. A zero-point iteration method is necessary to solve the algebraic equation.

We see that when $x_L(t) > L$ then $u_1 = 0$ for $x \in \tilde{\Omega}$ and hence $Q(t) = -\dfrac{\alpha E_1}{L}$.

# 4 Experiments

In Figure 4 we show the shock-position $s(t)$, rare-faction bounds $x_L(t)$, $x_R(t)$ (see Section 2) as a function of time. The time $\tau$ is here the position at which the curves for $s(t)$ and $x_R(t)$ intersect. Note that the curve for $x_R(t)$ loses its use after intersection with the curve for $s(t)$.

In the figures that follow we use $L = 40$, $\mu_1 = 250$, $\mu_2 = 1$, $k_0 = 1$, $\Delta p = -1$, $\phi = 0.5$, $T = 2000$, unless stated otherwise.

In Figure 5 we show the shock position as a function of time for different values of the viscosity of the preflush fluid. It is clear that a high viscosity gives little penetration. However, the relation between the shock position at a certain time and the preflush viscosity is non-linear.

In Figure 6 we plot the shock position as a function of time for different values of the permeability of the porous medium. This situation reflects the case where several layers with different permeabilities are treated simultaneously with the same pressure drop over the width of the layer. Again, the relation between the shock position at a certain time and the permeability of the layer is non-linear.

# 5 Conclusions and remarks

We presented a simple model for the injection of a highly viscous fluid into a porous medium where a less viscous fluid was initially present. Furthermore, we describe the case where

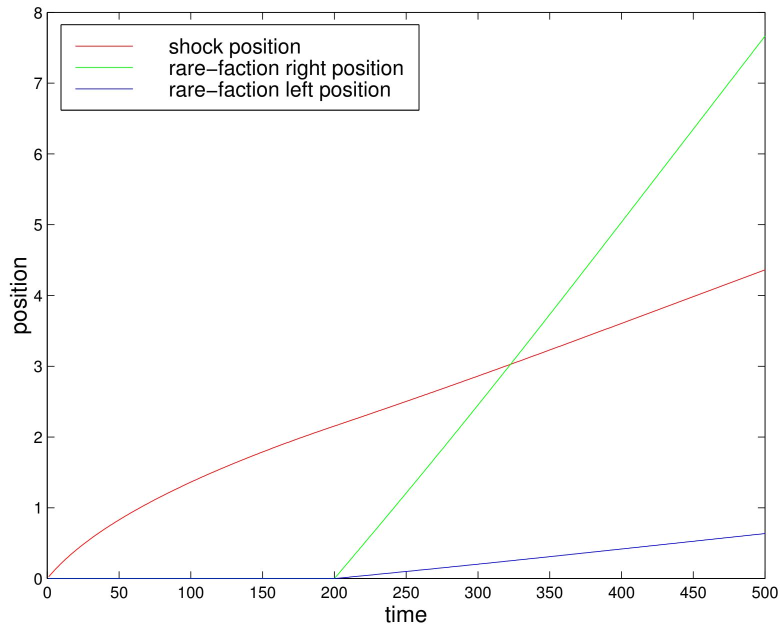


Figure 4: The shock position, $s(t)$, the left and right bounds ($x_L(t)$ and $x_R(t)$) as a function of time. We used $L = 40$, $\mu_1 = 1$, $\mu_2 = 50$, $k_0 = 1$, $\Delta p = -1$, $\phi = 1$ and $T = 200$.

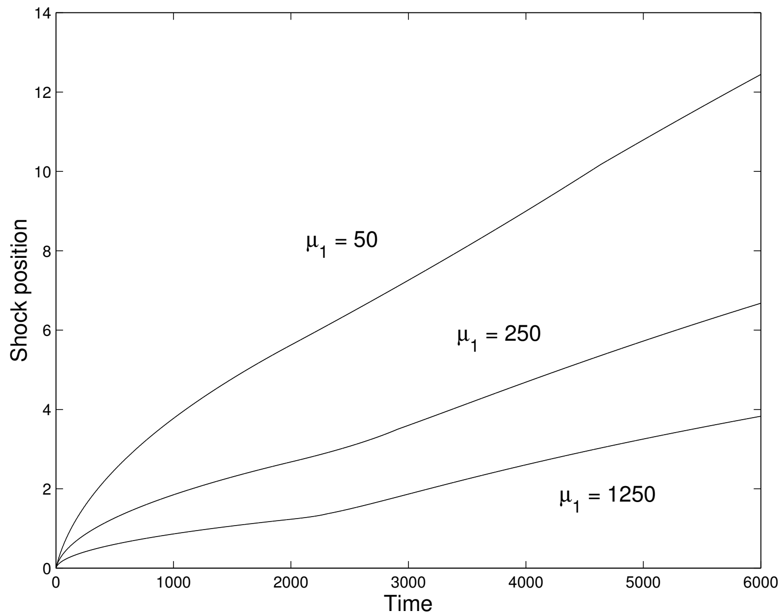


Figure 5: The shock position $s(t)$ as a function of time for different values of the viscosity of the preflush fluid $\mu_1$. Further input data have been given in the text.

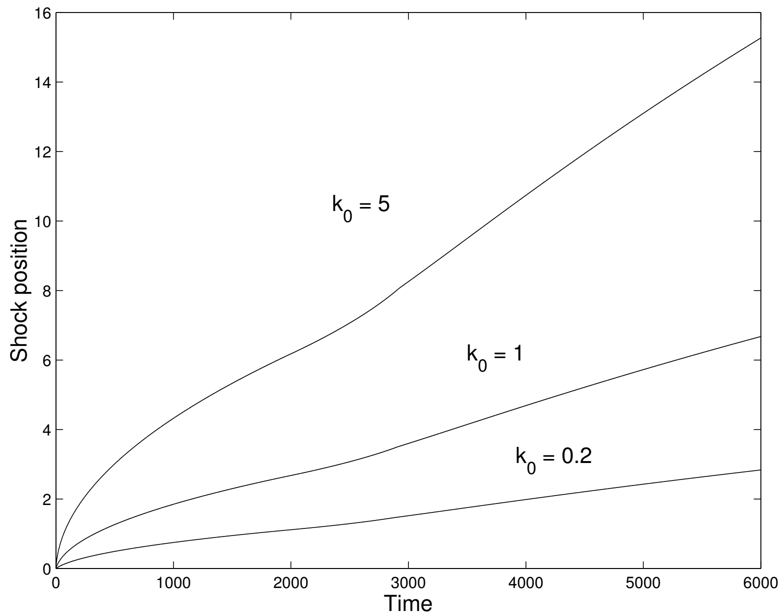


Figure 6: The shock position $s(t)$ as a function of time for different values of the permeability of the reservoir. Further input data have been given in the text.

subsequently the less viscous fluid is injected again. A semi-explicit analytic solution is given. This solution can be used as a test-case for numerical computations to solve first order hyperbolic equations.

The solution consists of a shock, a sharp interface, between the viscous fluid and the less viscous fluid. A rarefaction, a smooth transition, occurs between the less viscous and viscous fluid. Since the rarefaction holds for the averaged saturation over the height of the porous medium, it is interpreted as a fingering zone.

The solution as given in this paper is only applicable to rectangular geometries. Curvilinear geometries are considered in future. Furthermore, effects due to gravity are neglected here. These effects are to be taken into account in future numerical modelling.

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# References

[1] Y.C. Yortsos. A theoretical analysis of vertical flow equilibrium, Transport in Porous Media **18**, (1995) 107-129.

[2] M. Blunt, M. Christie. How to predict viscous fingering in three component flow, Transport in porous media **12**, (1993) 207-236.

[3] Z-X. Chen, G.S. Bodvarsson, P.A. Witherspoon, Y.C. Yortsos. An integral equation formulation for the unconfined flow of ground water with variable inlet conditions, Transport in porous media **18**, (1995) 15-36.

[4] K.S. Sorbie, R.S. Seright. Gel placement in heterogeneous systems with cross-flow, Society of Petroleum Engineers **SPE 24192**, (1992) 1-13.

[5] C.J. van Duijn, O. Strack. Private Communication, (2000).

[6] E.J. Koval. A method for predicting the performance of unstable miscible displacement in heterogeous media, SPE reprint series **8**, (1963) 145-154.

[7] A. Stavland, A. Lohne. Diversion by viscous preflush, Welgel report **3**, (1999).

[8] M. Karakas, S. Saneie, Y. Yortsos. Displacement of a viscous oil by the combined injection of hot water and and chemical additive, Society of Petroleum Engineering Reservoir Engineering, **July**, (1986) 391-402.

[9] Z-X. Chen, G.S. Bodvarsson, P.A. Witherspoon, Y. Yortsos. An integral formulation for the unconfined flow of groundwater with variable inlet conditions, Transport in porous media, **18**, (1995) 15-36.

[10] L.W. Lake. *Enhanced oil recovery* (Prentance-Hall, Englewood Cliffs 1989)

[11] H.K. Rhee, R. Aris, N.R. Amundson. *First-order partial differential equations, Theory and application of hyperbolic systems of quasi-linear equations* (Prentance-Hall, Englewood Cliffs 1969)

[12] J. Smoller. *Shock waves and reaction-diffusion equations* (Springer, New York 1983)