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S.B. Waluya and W.T. van Horssen

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ON APPROXIMATIONS OF FIRST INTEGRALS FOR A SYSTEM OF WEAKLY NONLINEAR, COUPLED HARMONIC OSCILLATORS

S. B. Waluya † and W.T. van Horssen

Department of Applied Mathematical Analysis,
Faculty of Information Technology and Systems, Delft University of Technology,
Mekelweg 4, 2628 CD Delft,
The Netherlands

(S.B.Waluya@its.tudelft.nl, W.T.vanHorssen@its.tudelft.nl)

Abstract. In this paper a system of weakly nonlinear, coupled harmonic oscillators will be studied. It will be shown that the recently developed perturbation method based on integrating vectors can be used to approximate first integrals and periodic solutions. To show how this perturbation method works the method is applied to a system of weakly nonlinear, coupled harmonic oscillators with 1:3 and 3:1 internal resonances. Not only approximations of first integrals will be given, but it will also be shown how, in rather efficient way, the existence and stability of time-periodic solutions can be obtained from these approximations. In addition some bifurcation diagrams for a set of values of the parameters will be presented.

Key words. Integrating factor, integrating vector, first integral, perturbation method, asymptotic approximation of first integral, periodic solution, coupled harmonic oscillators, bifurcation point.

1. Introduction. In [14, 15, 16] a perturbation method based on integrating factors and vectors has been presented for regularly perturbed systems of ordinary differential equations (ODEs). When approximations of integrating vectors have been obtained an approximation of a first integral can be given. Also an error-estimate for this approximation of a first integral can be given on a time-scale. It has also been shown in [11, 15, 16] how in a rather efficient way the existence and stability of time-periodic solutions can be obtained from these approximations for the first integrals. In this paper it will be shown how the perturbation method can be applied to systems of weakly nonlinear, coupled harmonic oscillators. In the literature many mathematical models have been considered describing the dynamics of systems with two degrees of freedom. Verros and Natsiavas [6] considered the dynamics of symmetric self-excited oscillators with an one-to-two internal resonance. Natsiavas [12] studied also the free vibrations of a weakly nonlinear oscillator using a multiple time scales perturbation method. Haaker and van der Burgh [13] used an averaging method to study nonlinear rational galloping for two mechanically coupled seesaw oscillators in steady crossflow. Mitsi, Natsiavas and Tsiafis [9] considered a class weakly nonlinear oscillators with symmetric restoring forces. The weakly nonlinear resonant response of systems with multiple degrees of freedom to simple harmonic excitations has been extensively studied by Nayfeh and Mook [1]. Bajaj, Chang and Johnson [2], and others have studied forced weakly nonlinear oscillations with two degrees of freedom as model for autoparametric vibration absorbers with resonant excitations. In this paper the recently developed perturbation method based on integrating factors and vectors will be used to approximate first integrals and periodic solutions for the following weakly

† On leave from: Mathematics Department Semarang State University, Indonesia

nonlinear system

$$\begin{cases} \ddot{X} + \left(\omega_1^2 + \epsilon\delta_1\right) X = \epsilon\left[-a_{1,0}\dot{X} + a_{0,1}\dot{Y} + a_{2,0}\dot{X}^2 - a_{1,1}\dot{X}\dot{Y} + a_{0,2}\dot{Y}^2 - a_{0,3}\dot{Y}^3\right], \\ \ddot{Y} + \left(\omega_2^2 + \epsilon\delta_2\right) Y = \epsilon\left[-b_{1,0}\dot{X} + b_{0,1}\dot{Y} + b_{2,0}\dot{X}^2 - b_{1,1}\dot{X}\dot{Y} - b_{0,2}\dot{Y}^2 - b_{0,3}\dot{Y}^3\right], \end{cases}$$
(1.1)

where $X = X(t)$, $Y = Y(t)$, and where ϵ is a small parameter satisfying $0 < \epsilon \ll 1$. The constants $a_{i,j}, b_{i,j}$ have the following properties: $a_{1,0}, a_{2,0}, a_{0,2}, b_{0,1}, b_{1,1}, b_{0,3}$ are positive, and $a_{0,1}, a_{1,1}, a_{0,3}, b_{1,0}, b_{2,0}, b_{0,2}$ have the same sign. As follows from measurements of aerodynamic coefficients in a wind-tunnel these signs are relevant for the description of a galloping phenomenon. The dot represents differentiation with respect to t. The constants δ_1, δ_2 are detuning parameters. We consider in this paper the 1:3 and 3:1 internal resonances, that is, $\omega_1 : \omega_2$=1:3 or 3:1. The frequencies ω_1 and ω_2 are assumed to be constants. In this paper not only asymptotic approximations of first integrals are constructed but also asymptotic approximations of periodic solutions. The presented results include existence, uniqueness, and stability properties of the periodic solutions. In [4] van der Beek uses (1.1) without detuning parameters δ_1, δ_2 as mathematical model to describe flow-induced vibrations of two weakly non-linear, coupled harmonic oscillators in a uniform windfield. The model problem originates from the phenomenon of galloping of overhead transmission lines on which ice has accreted. These conductors can become aerodynamically unstable, resulting in large amplitude oscillations with low frequencies. The oscillator consists of a rigid cylinder with small ridge and a number of springs mounted in a frame. The oscillator is constructed in such a way that the cylinder-spring system has two degrees of freedom, i.e. oscillation in the direction and oscillations perpendicular to the windfield. A more detailed description is given in [4], whereas a short summary is included in the Appendix A. The internal resonances that will be studied in this paper have not been studied in [4]. This paper is organized as follows. In section 2 of this paper the perturbation method based on integrating vectors and an asymptotic theory will be given briefly. It will be shown in section 3 of this paper how approximations of first integrals can be constructed for systems of weakly nonlinear, coupled harmonic oscillators. In section 4 it will be shown how the existence and stability of time-periodic solutions can be obtained. We will also present some bifurcation diagrams for a set of values of the parameters. Finally in section 5 of this paper some conclusions will be drawn and some remarks will be made.

2. Integrating vectors and an asymptotic theory. In this section we briefly outline the perturbation method based on integrating vectors as given in [11, 15, 16]. We consider the following system of n first order ODEs

$$\frac{d\underline{y}}{dt} = \underline{f}(\underline{y}, t; \epsilon), \tag{2.1}$$

where ϵ is a small parameter, and where the function $\underline{f}$ has the form $\underline{f}(\underline{y}, t; \epsilon) = \underline{f}_0(\underline{y}, t) + \epsilon \underline{f}_1(\underline{y}, t)$. An integrating vector $\underline{\mu} = \underline{\mu}(\underline{y}, t; \epsilon)$ for system (2.1) has to satisfy

$$\begin{cases} \frac{\partial \mu_i}{\partial y_j} = \frac{\partial \mu_j}{\partial y_i}, \qquad 1 \le i < j \le n, \\ \frac{\partial \underline{\mu}}{\partial t} = -\nabla(\underline{\mu} \cdot \underline{f}). \end{cases} \tag{2.2}$$

Assume that $\underline{\mu}$ can be expanded in a power series in ϵ, that is, $\underline{\mu}(\underline{y}, t; \epsilon) = \underline{\mu}_0(\underline{y}, t) + \epsilon\underline{\mu}_1(\underline{y}, t) + \ldots + \epsilon^m \underline{\mu}_m(\underline{y}, t) + \ldots$. We determine an integrating vector up to $\mathcal{O}(\epsilon^m)$. An approximation F_{app} of F in the first integral $F = constant$ can be obtained from:

$$\begin{cases} \nabla F_{app} &= \underline{\mu}_0 + \epsilon\underline{\mu}_1 + \ldots + \epsilon^m \underline{\mu}_m, \\ \frac{\partial F_{app}}{\partial t} &= -\left[\left(\underline{\mu}_0 + \epsilon\underline{\mu}_1 + \ldots + \epsilon^m \underline{\mu}_m\right) \cdot \underline{f}\right]_*, \end{cases} \tag{2.3}$$

where the * indicates that terms of order ϵ^{m+1} and higher have been neglected. Then we obtain $F_{app}(\underline{y}, t; \epsilon) = F_0(\underline{y}, t) + \epsilon F_1(\underline{y}, t) + \ldots + \epsilon^m F_m(\underline{y}, t)$. It should be observed that an approximation up to $\mathcal{O}(\epsilon^m)$ of an integrating vector $\underline{\mu}$ has been used to obtain an exact ODE up to $\mathcal{O}(\epsilon^{m+1})$, that is,

$$\begin{aligned} \frac{dF_{app}}{dt} &= \left[\left(\underline{\mu}_0 + \epsilon \cdot \underline{\mu}_1 + \ldots + \epsilon^m \underline{\mu}_m\right) \cdot \underline{f}\right]_{**} \\ &= \epsilon^{m+1} R_{m+1}(\underline{y}, t, \underline{\mu}_0, \ldots, \underline{\mu}_m; \epsilon), \end{aligned} \tag{2.4}$$

where the ** indicates that only terms of order ϵ^{m+1} and higher are included. How well F_{app} approximates $F(\underline{y}, t; \epsilon) = constant$ can be determined from (2.4), that is, error estimates can be given on time-scales depending on the boundedness properties of R_{m+1}.

3. Approximations of First Integrals. In this section we will show how the perturbation method based on integrating vectors can be applied to approximate first integrals for a system of weakly nonlinear, coupled harmonic oscillators. In the first part of this section we will consider system (1.1) with a 1:3 internal resonance and the second part with a 3:1 internal resonance.

3.1. The 1:3 internal resonance case. Consider the mathematical model which describes the flow-induced vibrations of an oscillator with two degrees of freedom in a uniform windfield with a 1:3 internal resonance

$$\begin{cases} \ddot{X} + (1 + \epsilon\delta_1) X = \epsilon\left[-a_{1,0}\dot{X} + a_{0,1}\dot{Y} + a_{2,0}\dot{X}^2 - a_{1,1}\dot{X}\dot{Y} + a_{0,2}\dot{Y}^2 - a_{0,3}\dot{Y}^3\right], \\ \ddot{Y} + (9 + \epsilon\delta_2) Y = \epsilon\left[-b_{1,0}\dot{X} + b_{0,1}\dot{Y} + b_{2,0}\dot{X}^2 - b_{1,1}\dot{X}\dot{Y} - b_{0,2}\dot{Y}^2 - b_{0,3}\dot{Y}^3\right]. \end{cases} \tag{3.1}$$

To analyze system (3.1) the equations are first written as a system of first order ODEs. Let $X = X_1, \dot{X} = X_2, Y = X_3, \dot{Y} = X_4$, from (3.1) we then obtain

$$\begin{cases} \dot{X}_1 &= X_2, \\ \dot{X}_2 &= -X_1 + \epsilon\left[-\delta_1 X_1 - a_{1,0}X_2 + a_{0,1}X_4\right. \\ &\qquad \left.+ a_{2,0}X_2^2 - a_{1,1}X_2X_4 + a_{0,2}X_4^2 - a_{0,3}X_4^3\right], \\ \dot{X}_3 &= X_4, \\ \dot{X}_4 &= -9X_3 + \epsilon\left[-\delta_2 X_3 - b_{1,0}X_2 + b_{0,1}X_4\right. \\ &\qquad \left.+ b_{2,0}X_2^2 - b_{1,1}X_2X_4 - b_{0,2}X_4^2 - b_{0,3}X_4^3\right]. \end{cases} \tag{3.2}$$

By using the transformation $X_1 = r_1\cos(\theta_1), X_2 = r_1\sin(\theta_1), X_3 = r_2\cos(3\theta_2)$ and $X_4 = 3r_2\sin(3\theta_2)$, system (3.2) then becomes

$$\begin{cases} \frac{dr_1}{dt} = \epsilon g_1(r_1,r_2,\theta_1,\theta_2) & = & f_1(r_1,r_2,\theta_1,\theta_2), \\ \frac{d\theta_1}{dt} = -1+\epsilon g_2(r_1,r_2,\theta_1,\theta_2) & = & f_2(r_1,r_2,\theta_1,\theta_2), \\ \frac{dr_2}{dt} = \epsilon g_3(r_1,r_2,\theta_1,\theta_2) & = & f_3(r_1,r_2,\theta_1,\theta_2), \\ \frac{d\theta_2}{dt} = -1+\epsilon g_4(r_1,r_2,\theta_1,\theta_2) & = & f_4(r_1,r_2,\theta_1,\theta_2), \end{cases} \tag{3.3}$$

where g_1, g_2, g_3, and g_4, are given in Appendix B formula (B.2). Multiplying the first, the second, the third, and the fourth equation in (3.3) by μ_1, μ_2, μ_3, and μ_4 respectively, it follows from (2.2) that the integrating factors μ_1, μ_2, μ_3, and μ_4 have to satisfy ($\mu_i = \mu_i(r_1,r_2,\theta_1,\theta_2,t)$ for $i = 1,2,3$, and 4)

$$\begin{cases} \frac{\partial\mu_1}{\partial\theta_1} = \frac{\partial\mu_2}{\partial r_1}, \\ \frac{\partial\mu_1}{\partial r_2} = \frac{\partial\mu_3}{\partial r_1},\ \frac{\partial\mu_2}{\partial r_2} = \frac{\partial\mu_3}{\partial\theta_1}, \\ \frac{\partial\mu_1}{\partial\theta_2} = \frac{\partial\mu_4}{\partial r_1},\ \frac{\partial\mu_2}{\partial\theta_2} = \frac{\partial\mu_4}{\partial\theta_1},\ \frac{\partial\mu_3}{\partial\theta_2} = \frac{\partial\mu_4}{\partial r_2}, \\ \frac{\partial\mu_1}{\partial t} = -\frac{\partial}{\partial r_1}\left(\mu_1 f_1 + \mu_2 f_2 + \mu_3 f_3 + \mu_4 f_4\right), \\ \frac{\partial\mu_2}{\partial t} = -\frac{\partial}{\partial\theta_1}\left(\mu_1 f_1 + \mu_2 f_2 + \mu_3 f_3 + \mu_4 f_4\right), \\ \frac{\partial\mu_3}{\partial t} = -\frac{\partial}{\partial r_2}\left(\mu_1 f_1 + \mu_2 f_2 + \mu_3 f_3 + \mu_4 f_4\right), \\ \frac{\partial\mu_4}{\partial t} = -\frac{\partial}{\partial\theta_2}\left(\mu_1 f_1 + \mu_2 f_2 + \mu_3 f_3 + \mu_4 f_4\right). \end{cases} \tag{3.4}$$

Expanding μ_1, μ_2, μ_3, and μ_4 in powers series in ϵ, that is, $\mu_i(r_1,r_2,\theta_1,\theta_2,t) = \mu_{i,0}(r_1,r_2,\theta_1,\theta_2,t)+\epsilon\mu_{i,1}(r_1,r_2,\theta_1,\theta_2,t)+\ldots$ (for i=1,2,3, and 4), substituting f_1, f_2, f_3, f_4, and the expansions for μ_1, μ_2, μ_3, and μ_4 into (3.4), and by taking together terms of equal powers in ϵ, we finally obtain the $\mathcal{O}(\epsilon^0)$-problem

$$\begin{cases} \frac{\partial\mu_{1,0}}{\partial\theta_1} = \frac{\partial\mu_{2,0}}{\partial r_1}, \\ \frac{\partial\mu_{1,0}}{\partial r_2} = \frac{\partial\mu_{3,0}}{\partial r_1},\ \frac{\partial\mu_{2,0}}{\partial r_2} = \frac{\partial\mu_{3,0}}{\partial\theta_1}, \\ \frac{\partial\mu_{1,0}}{\partial\theta_2} = \frac{\partial\mu_{4,0}}{\partial r_1},\ \frac{\partial\mu_{2,0}}{\partial\theta_2} = \frac{\partial\mu_{4,0}}{\partial\theta_1},\ \frac{\partial\mu_{3,0}}{\partial\theta_2} = \frac{\partial\mu_{4,0}}{\partial r_2}, \\ \frac{\partial\mu_{1,0}}{\partial t} = -\frac{\partial}{\partial r_1}\left(-\mu_{2,0} - \mu_{4,0}\right), \\ \frac{\partial\mu_{2,0}}{\partial t} = -\frac{\partial}{\partial\theta_1}\left(-\mu_{2,0} - \mu_{4,0}\right), \\ \frac{\partial\mu_{3,0}}{\partial t} = -\frac{\partial}{\partial r_2}\left(-\mu_{2,0} - \mu_{4,0}\right), \\ \frac{\partial\mu_{4,0}}{\partial t} = -\frac{\partial}{\partial\theta_2}\left(-\mu_{2,0} - \mu_{4,0}\right), \end{cases} \tag{3.5}$$

the $\mathcal{O}(\epsilon^1)$-problem

$$(3.6)\quad \begin{cases} \frac{\partial\mu_{1,1}}{\partial\theta_1} = \frac{\partial\mu_{2,1}}{\partial r_1}, \\ \frac{\partial\mu_{1,1}}{\partial r_2} = \frac{\partial\mu_{3,1}}{\partial r_1},\ \frac{\partial\mu_{2,1}}{\partial r_2} = \frac{\partial\mu_{3,1}}{\partial\theta_1}, \\ \frac{\partial\mu_{1,1}}{\partial\theta_2} = \frac{\partial\mu_{4,1}}{\partial r_1},\ \frac{\partial\mu_{2,1}}{\partial\theta_2} = \frac{\partial\mu_{4,1}}{\partial\theta_1},\ \frac{\partial\mu_{3,1}}{\partial\theta_2} = \frac{\partial\mu_{4,1}}{\partial r_2}, \\ \frac{\partial\mu_{1,1}}{\partial t} = -\frac{\partial}{\partial r_1}\left(\mu_{1,0}g_1 + \mu_{2,0}g_2 - \mu_{2,1} + \mu_{3,0}g_3 + \mu_{4,0}g_4 - \mu_{4,1}\right), \\ \frac{\partial\mu_{2,1}}{\partial t} = -\frac{\partial}{\partial\theta_1}\left(\mu_{1,0}g_1 + \mu_{2,0}g_2 - \mu_{2,1} + \mu_{3,0}g_3 + \mu_{4,0}g_4 - \mu_{4,1}\right), \\ \frac{\partial\mu_{3,1}}{\partial t} = -\frac{\partial}{\partial r_2}\left(\mu_{1,0}g_1 + \mu_{2,0}g_2 - \mu_{2,1} + \mu_{3,0}g_3 + \mu_{4,0}g_4 - \mu_{4,1}\right), \\ \frac{\partial\mu_{4,1}}{\partial t} = -\frac{\partial}{\partial\theta_2}\left(\mu_{1,0}g_1 + \mu_{2,0}g_2 - \mu_{2,1} + \mu_{3,0}g_3 + \mu_{4,0}g_4 - \mu_{4,1}\right), \end{cases}$$

and for $n \geq 2$ the $\mathcal{O}(\epsilon^n)$-problems

$$(3.7)\quad \begin{cases} \frac{\partial\mu_{1,n}}{\partial\theta_1} = \frac{\partial\mu_{2,n}}{\partial r_1}, \\ \frac{\partial\mu_{1,n}}{\partial r_2} = \frac{\partial\mu_{3,n}}{\partial r_1},\ \frac{\partial\mu_{2,n}}{\partial r_2} = \frac{\partial\mu_{3,n}}{\partial\theta_1}, \\ \frac{\partial\mu_{1,n}}{\partial\theta_2} = \frac{\partial\mu_{4,n}}{\partial r_1},\ \frac{\partial\mu_{2,n}}{\partial\theta_2} = \frac{\partial\mu_{4,n}}{\partial\theta_1},\ \frac{\partial\mu_{3,n}}{\partial\theta_2} = \frac{\partial\mu_{4,n}}{\partial r_2}, \\ \frac{\partial\mu_{1,n}}{\partial t} = -\frac{\partial}{\partial r_1}\left(\mu_{1,n-1}g_1 + \mu_{2,n-1}g_2 - \mu_{2,n} + \mu_{3,n-1}g_3 + \mu_{4,n-1}g_4 - \mu_{4,n}\right), \\ \frac{\partial\mu_{2,n}}{\partial t} = -\frac{\partial}{\partial\theta_1}\left(\mu_{1,n-1}g_1 + \mu_{2,n-1}g_2 - \mu_{2,n} + \mu_{3,n-1}g_3 + \mu_{4,n-1}g_4 - \mu_{4,n}\right), \\ \frac{\partial\mu_{3,n}}{\partial t} = -\frac{\partial}{\partial r_2}\left(\mu_{1,n-1}g_1 + \mu_{2,n-1}g_2 - \mu_{2,n} + \mu_{3,n-1}g_3 + \mu_{4,n-1}g_4 - \mu_{4,n}\right), \\ \frac{\partial\mu_{4,n}}{\partial t} = -\frac{\partial}{\partial\theta_2}\left(\mu_{1,n-1}g_1 + \mu_{2,n-1}g_2 - \mu_{2,n} + \mu_{3,n-1}g_3 + \mu_{4,n-1}g_4 - \mu_{4,n}\right). \end{cases}$$

The $\mathcal{O}(\epsilon^0)$-problem (3.5) can easily be solved, yielding $\mu_{1,0} = h_{1,0}(r_1, r_2, \theta_1 + t, \theta_2 + t)$, $\mu_{2,0} = h_{2,0}(r_1, r_2, \theta_1 + t, \theta_2 + t)$, $\mu_{3,0} = h_{3,0}(r_1, r_2, \theta_1 + t, \theta_2 + t)$, $\mu_{4,0} = h_{4,0}(r_1, r_2, \theta_1 + t, \theta_2 + t)$ with $\frac{\partial h_{1,0}}{\partial\theta_1} = \frac{\partial h_{2,0}}{\partial r_1}$, $\frac{\partial h_{1,0}}{\partial r_2} = \frac{\partial h_{3,0}}{\partial r_1}$, $\frac{\partial h_{1,0}}{\partial\theta_2} = \frac{\partial h_{4,0}}{\partial r_1}$, $\frac{\partial h_{2,0}}{\partial r_2} = \frac{\partial h_{3,0}}{\partial\theta_1}$, $\frac{\partial h_{2,0}}{\partial\theta_2} = \frac{\partial h_{4,0}}{\partial\theta_1}$, $\frac{\partial h_{3,0}}{\partial\theta_2} = \frac{\partial h_{4,0}}{\partial r_2}$. The functions $h_{1,0}$, $h_{2,0}$, $h_{3,0}$, and $h_{4,0}$ are still arbitrary and will now be chosen as simple as possible. First we choose $h_{1,0} = 1$, and $h_{2,0} = h_{3,0} = h_{4,0} = 0$ or equivalently $(\mu_{1,0}, \mu_{2,0}, \mu_{3,0}, \mu_{4,0}) = (1, 0, 0, 0)$. The $\mathcal{O}(\epsilon^1)$-problem (3.6) can also readily be solved, yielding

$$\begin{aligned} \mu_{1,1} = {} & \tfrac{1}{2}a_{1,0}t + \tfrac{1}{4}\delta_1\cos(2\theta_1) + \tfrac{1}{4}a_{1,0}\sin(2\theta_1) - \tfrac{3}{2}a_{2,0}r_1\cos(\theta_1) + \tfrac{1}{6}a_{2,0}r_1\cos(3\theta_1) \\ & + \tfrac{1}{2}a_{1,1}r_2\cos(3\theta_2) - \tfrac{3}{20}a_{1,1}r_2\cos(3\theta_2 + 2\theta_1) - \tfrac{3}{4}a_{1,1}r_2\cos(3\theta_2 - 2\theta_1) \\ & + h_{1,1}(r_1, r_2, \theta_1 + t, \theta_2 + t), \end{aligned}$$

$$
\begin{aligned}
\mu_{2,1} = \ & -\tfrac{1}{2}\delta_1 r_1 \sin(2\theta_1) + \tfrac{1}{2}a_{1,0}\cos(2\theta_1) - \tfrac{3}{4}a_{0,1}r_2\cos(-\theta_1+3\theta_2) \\
& -\tfrac{3}{8}a_{0,1}r_2\cos(\theta_1+3\theta_2) + \tfrac{3}{4}a_{2,0}r_1^2\sin(\theta_1) - \tfrac{1}{4}a_{2,0}r_1^2\sin(3\theta_1) \\
& +\tfrac{3}{10}a_{1,1}r_1r_2\sin(3\theta_2+2\theta_1) - \tfrac{3}{2}a_{1,1}r_1r_2\sin(3\theta_2-2\theta_1) + \tfrac{9}{2}a_{0,2}r_2^2\sin(\theta_1) \\
& -\tfrac{9}{28}a_{0,2}r_2^2\sin(\theta_1+6\theta_2) - \tfrac{9}{20}a_{0,2}r_2^2\sin(-\theta_1+6\theta_2) + \tfrac{81}{16}a_{0,3}r_2^3\cos(-\theta_1+3\theta_2) \\
& +\tfrac{81}{32}a_{0,3}r_2^3\cos(\theta_1+3\theta_2) - \tfrac{27}{64}a_{0,3}r_2^3\cos(9\theta_2-\theta_1) - \tfrac{27}{80}a_{0,3}r_2^3\cos(9\theta_2+\theta_1) \\
& +h_{2,1}(r_1,r_2,\theta_1+t,\theta_2+t), \\
\mu_{3,1} = \ & \tfrac{3}{4}a_{0,1}\sin(-\theta_1+3\theta_2) - \tfrac{3}{8}a_{0,1}\sin(\theta_1+3\theta_2) + \tfrac{1}{2}a_{1,1}r_1\cos(3\theta_2) \\
& -\tfrac{3}{20}a_{1,1}r_1\cos(3\theta_2+2\theta_1) - \tfrac{3}{4}a_{1,1}r_1\cos(3\theta_2-2\theta_1) - 9a_{0,2}r_2\cos(\theta_1) \\
& +\tfrac{9}{14}a_{0,2}r_2\cos(\theta_1+6\theta_2) - \tfrac{9}{10}a_{0,2}r_2\cos(\theta_1-6\theta_2) - \tfrac{243}{16}a_{0,3}r_2^2\sin(-\theta_1+3\theta_2) \\
& +\tfrac{243}{32}a_{0,3}r_2^2\sin(\theta_1+3\theta_2) + \tfrac{81}{64}a_{0,3}r_2^2\sin(9\theta_2-\theta_1) - \tfrac{81}{80}a_{0,3}r_2^2\sin(9\theta_2+\theta_1) \\
& +h_{3,1}(r_1,r_2,\theta_1+t,\theta_2+t), \\
\mu_{4,1} = \ & \tfrac{9}{4}a_{0,1}r_2\cos(-\theta_1+3\theta_3) - \tfrac{9}{8}a_{0,1}r_2\cos(\theta_1+3\theta_3) - \tfrac{3}{2}a_{1,1}r_1r_2\sin(3\theta_2) \\
& +\tfrac{9}{20}a_{1,1}r_1r_2\sin(3\theta_2+2\theta_1) + \tfrac{9}{4}a_{1,1}r_1r_2\sin(3\theta_2-2\theta_1) - \tfrac{27}{14}a_{0,2}r_2^2\sin(\theta_1+6\theta_2) \\
& +\tfrac{27}{10}a_{0,2}r_2^2\sin(-\theta_1+6\theta_2) - \tfrac{243}{16}a_{0,3}r_2^3\cos(-\theta_1+3\theta_2) + \tfrac{243}{32}a_{0,3}r_2^3\cos(\theta_1+3\theta_2) \\
& +\tfrac{243}{64}a_{0,3}r_2^3\cos(9\theta_2-\theta_1) - \tfrac{243}{80}a_{0,3}r_2^3\cos(9\theta_2+\theta_1) + h_{4,1}(r_1,r_2,\theta_1+t,\theta_2+t),
\end{aligned}
\tag{3.8}
$$

where $h_{1,1}$, $h_{2,1}$, $h_{3,1}$, and $h_{4,1}$ have to satisfy $\frac{\partial h_{1,1}}{\partial \theta_1} = \frac{\partial h_{2,1}}{\partial r_1}$, $\frac{\partial h_{1,1}}{\partial r_2} = \frac{\partial h_{3,1}}{\partial r_1}$, $\frac{\partial h_{1,1}}{\partial \theta_2} = \frac{\partial h_{4,1}}{\partial r_1}$, $\frac{\partial h_{2,1}}{\partial r_2} = \frac{\partial h_{3,1}}{\partial \theta_1}$, $\frac{\partial h_{2,1}}{\partial \theta_2} = \frac{\partial h_{4,1}}{\partial \theta_1}$, $\frac{\partial h_{3,1}}{\partial \theta_2} = \frac{\partial h_{4,1}}{\partial r_2}$. The functions $h_{1,1}$, $h_{2,1}$, $h_{3,1}$, and $h_{4,1}$ are still arbitrary and will now be chosen as simple as possible: $h_{1,1} = h_{2,1} = h_{3,0} = h_{4,0} = 0$. The $\mathcal{O}(\epsilon^n)$-problems with $n \geq 2$ can also be solved. By using (2.3) and the approximation $(1+\epsilon\mu_{1,1}, \epsilon\mu_{2,1}, \epsilon\mu_{3,1}, \epsilon\mu_{4,1})$ for the integrating vector $(\mu_1, \mu_2, \mu_3, \mu_4)$ we can construct an approximation F_1 of a first integral $F = \mathit{constant}$, yielding

$$F_1 = r_1 + \epsilon F_{1,1}, \tag{3.9}$$

where

$$
\begin{aligned}
F_{1,1} \ = \ & \frac{1}{2}a_{1,0}r_1 t + \frac{1}{4}\delta_1 r_1\cos(2\theta_1) + \frac{1}{4}a_{1,0}r_1\sin(2\theta_1) - \frac{3}{4}a_{2,0}r_1^2\cos(\theta_1) \\
& +\frac{1}{12}a_{2,0}r_1^2\cos(3\theta_1) + \frac{1}{2}a_{1,1}r_1r_2\cos(3\theta_2) - \frac{3}{20}a_{1,1}r_1r_2\cos(3\theta_2+2\theta_1) \\
& -\frac{3}{4}a_{1,1}r_1r_2\cos(-3\theta_2+2\theta_1) - \frac{3}{4}a_{0,1}r_2\sin(\theta_1-3\theta_2) - \frac{3}{8}a_{0,1}r_2\sin(\theta_1+3\theta_2)
\end{aligned}
$$

$$-\frac{9}{2}a_{0,2}r_2^2\,\cos(\theta_1) + \frac{9}{28}\,a_{0,2}r_2^2\cos(\theta_1 + 6\,\theta_2) - \frac{9}{20}\,a_{0,2}r_2^2\cos(\theta_1 - 6\,\theta_2)$$
$$+\frac{81}{16}\,a_{0,3}r_2^3\sin(\theta_1 - 3\,\theta_2) + \frac{81}{32}\,a_{0,3}r_2^3\sin(\theta_1 + 3\,\theta_2)$$
$$-\frac{27}{64}\,a_{0,3}r_2^3\sin(-9\,\theta_2 + \theta_1) - \frac{27}{80}\,a_{0,3}r_2^3\sin(9\,\theta_2 + \theta_1). \tag{3.10}$$

How well F_1 approximates F in a first integral $F = constant$ follows from (2.4). In this case we have

$$\begin{aligned}\frac{dF_1}{dt} &= [f_1 + \epsilon\mu_{1,1}f_1 + \epsilon\mu_{2,1}f_2 + \epsilon\mu_{3,1}f_3 + \epsilon\mu_{4,1}f_4]_{**}\\ &= \epsilon^2 R_{2,1}(r_1,\theta_1,r_2,\theta_2),\end{aligned} \tag{3.11}$$

where f_1 , f_2 , f_3, f_4 are given by (3.3) and $\mu_{1,1}$, $\mu_{2,1}$, $\mu_{3,1}$, $\mu_{4,1}$ are given by (3.8) respectively. From the existence and uniqueness theorems for ODEs we know that an initial-value problem for (3.2) is well-posed on a time-scale of order $\frac{1}{\epsilon}$. This implies that also an initial-value problem for system (3.3) is well-posed on this time-scale. From (3.3) it then follows on this time-scale that if $r_1(0)$, $r_2(0)$ are bounded then $r_1(t)$, $r_2(t)$ are bounded, and $\theta_1(t)$, $\theta_2(t)$ are bounded by constants plus t. Since $|R_{2,1}| \leq c_0 + c_1 t$ on a time scale of order $\frac{1}{\epsilon}$, where c_0, c_1 are constants, it follows that

$$\begin{aligned}F_1(r_1(t),\theta_1(t),r_2(t),\theta_2(t),t;\epsilon) &= constant + \mathcal{O}(\epsilon^2) \text{ for } 0 \leq t \leq T_1 < \infty, \text{ and}\\ F_1(r_1(t),\theta_1(t),r_2(t),\theta_2(t),t;\epsilon) &= constant + \mathcal{O}(\epsilon), \text{ for } 0 \leq t \leq \frac{L_1}{\sqrt{\epsilon}},\end{aligned} \tag{3.12}$$

where T_1 and L_1 are ϵ-independent constants. By putting $(\mu_{1,0},\mu_{2,0},\mu_{3,0},\mu_{4,0}) = (0,1,0,0)$, or $(0,0,1,0)$, or $(0,0,0,1)$ we can construct a second, a third, and a fourth (functionally independent) approximation F_2, F_3, and F_4 of a first integral $F = constant$. After some elementary calculations we then obtain

$$\begin{cases} F_2 = (\theta_1 + t) + \epsilon F_{2,1},\\ F_3 = r_2 + \epsilon F_{3,1},\\ F_4 = (\theta_2 + t) + \epsilon F_{4,1},\end{cases} \tag{3.13}$$

where $F_{2,1}$, $F_{3,1}$, and $F_{4,1}$ are given in Appendix B formula (B.3). How well F_2, F_3, and F_4 (as given by (3.13)) approximate F in a first integral $F = constant$ can be determined similar to (3.11)-(3.12). It can be shown that

$$\begin{aligned}F_2(r_1(t),\theta_1(t),r_2(t),\theta_2(t),t;\epsilon) &= constant + \mathcal{O}(\epsilon^2) \text{ for } 0 \leq t \leq T_2 < \infty, \text{ and}\\ F_2(r_1(t),\theta_1(t),r_2(t),\theta_2(t),t;\epsilon) &= constant + \mathcal{O}(\epsilon), \text{ for } 0 \leq t \leq \frac{L_2}{\sqrt{\epsilon}},\end{aligned} \tag{3.14}$$

where T_2 and L_2 are ϵ-independent constants,

$$F_3(r_1(t),\theta_1(t),r_2(t),\theta_2(t),t;\epsilon) = constant + \mathcal{O}(\epsilon^2) \text{ for } 0 \leq t \leq T_3 < \infty, \text{ and}$$

$$F_3(r_1(t),\theta_1(t),r_2(t),\theta_2(t),t;\epsilon) = constant + \mathcal{O}(\epsilon), \text{ for } 0 \le t \le \frac{L_3}{\sqrt{\epsilon}}, \tag{3.15}$$

where T_3 and L_3 are ϵ-independent constants, and

$$\begin{aligned} F_4(r_1(t),\theta_1(t),r_2(t),\theta_2(t),t;\epsilon) &= constant + \mathcal{O}(\epsilon^2) \text{ for } 0 \le t \le T_4 < \infty, \text{ and} \\ F_4(r_1(t),\theta_1(t),r_2(t),\theta_2(t),t;\epsilon) &= constant + \mathcal{O}(\epsilon), \text{ for } 0 \le t \le \frac{L_4}{\sqrt{\epsilon}}, \end{aligned} \tag{3.16}$$

where T_4 and L_4 are ϵ-independent constants.

3.2. The 3:1 internal resonance case. We consider in this subsection

$$\begin{cases} \ddot{X} + (9+\epsilon\delta_1)\,X = \epsilon\left[-a_{1,0}\dot{X} + a_{0,1}\dot{Y} + a_{2,0}\dot{X}^2 - a_{1,1}\dot{X}\dot{Y} + a_{0,2}\dot{Y}^2 - a_{0,3}\dot{Y}^3\right], \\ \ddot{Y} + (1+\epsilon\delta_2)\,Y = \epsilon\left[-b_{1,0}\dot{X} + b_{0,1}\dot{Y} + b_{2,0}\dot{X}^2 - b_{1,1}\dot{X}\dot{Y} - b_{0,2}\dot{Y}^2 - b_{0,3}\dot{Y}^3\right]. \end{cases} \tag{3.17}$$

Putting $X = X_1, \dot{X} = X_2, Y = X_3$,and $\dot{Y} = X_4$, it follows from (3.17) that

$$\begin{cases} \dot{X}_1 = X_2, \\ \dot{X}_2 = -9X_1 + \epsilon\,[-\delta_1 X_1 - a_{1,0}X_2 + a_{0,1}X_4 \\ \qquad + a_{2,0}X_2^2 - a_{1,1}X_2X_4 + a_{0,2}X_4^2 - a_{0,3}X_4^3], \\ \dot{X}_3 = X_4, \\ \dot{X}_4 = -X_3 + \epsilon\,[-\delta_2 X_3 - b_{1,0}X_2 + b_{0,1}X_4 \\ \qquad + b_{2,0}X_2^2 - b_{1,1}X_2X_4 - b_{0,2}X_4^2 - b_{0,3}X_4^3]. \end{cases} \tag{3.18}$$

By using the transformation $X_1 = r_1\cos(3\theta_1), X_2 = 3r_1\sin(3\theta_1), X_3 = r_2\cos(\theta_2)$ and $X_4 = r_2\sin(\theta_2)$ system (3.18) then becomes

$$\begin{cases} \frac{dr_1}{dt} = \epsilon h_1(r_1,r_2,\theta_1,\theta_2), \\ \frac{d\theta_1}{dt} = -1 + \epsilon h_2(r_1,r_2,\theta_1,\theta_2), \\ \frac{dr_2}{dt} = \epsilon h_3(r_1,r_2,\theta_1,\theta_2), \\ \frac{d\theta_2}{dt} = -1 + \epsilon h_4(r_1,r_2,\theta_1,\theta_2), \end{cases} \tag{3.19}$$

where h_1, h_2, h_3, and h_4 are given in Appendix C formula (C.2). In a similar way as in subsection (3.1) we can obtain after some elementary calculations approximations

G_1, G_2, G_3, and G_4 of first integrals for system (3.19):

$$\begin{cases} G_1 &= r_1 + \epsilon G_{1,1}, \\ G_2 &= (\theta_1 + t) + \epsilon G_{2,1}, \\ G_3 &= r_2 + \epsilon G_{3,1}, \\ G_4 &= (\theta_2 + t) + \epsilon G_{4,1}, \end{cases} \tag{3.20}$$

where G_1, G_2, G_3, and G_4 are given explicitly in Appendix C formula (C.3). How well $G_i(i = 1, 2, 3, \text{or}\, 4)$ approximates G in a first integral $G = \textit{constant}$ follows from (2.4) (see also (3.11)-(3.12)). It can be shown that for $i = 1, 2, 3, \text{and}\, 4$

$$\begin{aligned} G_i(r_1(t), \theta_1(t), r_2(t), \theta_2(t), t; \epsilon) &= \textit{constant} + \mathcal{O}(\epsilon^2) \text{ for } 0 \le t \le T_i < \infty, \text{ and} \\ G_i(r_1(t), \theta_1(t), r_2(t), \theta_2(t), t; \epsilon) &= \textit{constant} + \mathcal{O}(\epsilon), \text{ for } 0 \le t \le \frac{L_i}{\sqrt{\epsilon}}, \end{aligned} \tag{3.21}$$

where T_i and L_i for $i = 1, 2, 3, \text{and}\, 4$ are ϵ-independent constants.

4. Approximations for time-periodic solutions and analysis of bifurcations.

In section 3 we constructed asymptotic approximations of first integrals. In this section we will show how the existence, the stability, and the approximations of non-trivial, time-periodic solutions can be determined from these asymptotic approximations of first integrals. We will also give some bifurcation diagrams for a set of values of the parameters.

4.1. The 1:3 internal resonance case.

The asymptotic approximations (3.13) for the first integrals of the weakly nonlinear, coupled harmonic oscillators with a 1:3 internal resonance can be used to determine the existence and stability of time-periodic solutions. Let $T < \infty$ be the period of a periodic solution and let c_1 be a constant in the first integrals $F(r_1,\, \theta_1,\, r_2,\, \theta_2,\, t\,;\, \epsilon) = \textit{constant}$ for which a periodic solution exists. Consider $F = c_1$ for $t = nT$ and $t = (n-1)T$ with $n \in N^+$, then

$$\begin{cases} F\left(r_1(nT), \theta_1(nT), r_2(nT), \theta_2(nT), nT; \epsilon\right) &= c_1, \\ F\left(r_1\left((n-1)T\right), \theta_1((n-1)T), r_2\left((n-1)T\right), \theta_2((n-1)T), (n-1)T; \epsilon\right) &= c_1. \end{cases} \tag{4.1}$$

For the autonomous system (3.1) we may assume that $\theta_1(0) = \alpha$ and $\theta_2(0) = \beta$, where α and β are arbitrary constants. From (3.3) it follows that

$$\begin{cases} r_1(nT) &= r_1\left((n-1)T\right) + \mathcal{O}(\epsilon), \\ \theta_1(nT) &= \theta_1\left((n-1)T\right) - T + \mathcal{O}(\epsilon), \\ r_2(nT) &= r_2\left((n-1)T\right) + \mathcal{O}(\epsilon), \\ \theta_2(nT) &= \theta_2\left((n-1)T\right) - T + \mathcal{O}(\epsilon). \end{cases} \tag{4.2}$$

Approximating F by F_1 (given by (3.9)), eliminating c_1 from (4.1) by subtraction, and using (4.2), we obtain

$$r_1(nT) = \left(1 - \frac{1}{2}\epsilon T a_{1,0}\right) r_1\left((n-1)T\right) + \mathcal{O}(\epsilon^2 t), \tag{4.3}$$

on a time scale of order $\frac{1}{\epsilon}$. Since $a_{1,0} > 0$ we can see from (4.3) that $r_1(nT)$ decreases for increasing n. Hence, the only possible candidate for a periodic solution is $r_1 \equiv 0$. However, for $r_1 = 0$ the approximation F_2 in (3.13) is not valid. From (3.1), it can readily be seen that the only resonance term in the right hand side of the equation for X is $-\epsilon a_{1,0}\dot{X}$, and this is a damping term. So, if $X(0)$ is $\mathcal{O}(\epsilon)$ then $X(t)$ will be at most $\mathcal{O}(\epsilon)$ for $t > 0$. So we only have to study

$$\ddot{Y} + (9 + \epsilon\delta_2)\, Y = \epsilon \left[b_{0,1}\dot{Y} - b_{0,2}\dot{Y}^2 - b_{0,3}\dot{Y}^3 \right] \tag{4.4}$$

when we are interested in periodic solutions. Approximations of first integrals can be obtained by taking $r_1 = 0$ in F_3 and F_4 (see (3.13)), yielding

$$\begin{aligned} F_3 &= r_2 + \epsilon \left[\tfrac{1}{36}\, \delta_2 r_2 \cos(6\,\theta_2) - \tfrac{1}{2}\, b_{0,1} r_2 t + \tfrac{27}{8}\, b_{0,3} r_2^3 t - \tfrac{1}{12}\, b_{0,1} r_2 \sin(6\,\theta_2) \right. \\ &\quad \left. + \tfrac{3}{4}\, \cos(3\,\theta_2) b_{0,2} r_2^2 - \tfrac{1}{12}\, b_{0,2} r_2^2 \cos(9\,\theta_2) + \tfrac{3}{4}\, b_{0,3} r_2^3 \sin(6\,\theta_2) - \tfrac{3}{32}\, b_{0,3} r_2^3 \sin(12\,\theta_2) \right], \\ F_4 &= \theta_2 + t + \epsilon \left[\tfrac{1}{18}\, \delta_2 t - \tfrac{1}{12}\, b_{0,2} r_2 \sin(3\,\theta_2) + \tfrac{1}{36}\, b_{0,2} r_2 \sin(9\,\theta_2) + \tfrac{1}{8}\, b_{0,3} r_2^2 \cos(6\,\theta_2) \right. \\ &\quad \left. - \tfrac{1}{32}\, b_{0,3} r_2^2 \cos(12\,\theta_2) - \tfrac{1}{108}\, \delta_2 \sin(6\,\theta_2) - \tfrac{1}{36}\, b_{0,1} \cos(6\,\theta_2) \right]. \end{aligned} \tag{4.5}$$

Let again $T < \infty$ be the period of a periodic solution and let c_2 be a constant in a first integral $F(r_1, \theta_1, r_2, \theta_2, t; \epsilon) =$ *contant* for which a periodic solution exists. Consider $F = c_2$ for $t = nT$ and $t = (n-1)T$ with $n \in N^+$, then

$$\begin{cases} F\left(r_1(nT), \theta_1(nT), r_2(nT), \theta_2(nT), nT; \epsilon\right) = c_2, \\ F\left(r_1\left((n-1)T\right), \theta_1((n-1)T), r_2\left((n-1)T\right), \theta_2((n-1)T), (n-1)T; \epsilon\right) = c_2. \end{cases} \tag{4.6}$$

Approximating F by F_3 (given by (4.5)), eliminating c_2 from (4.6) by subtraction, and using (4.2), we obtain

$$r_2(nT) = r_2\left((n-1)T\right) + \epsilon T \left(\tfrac{1}{2} b_{0,1} r_2((n-1)T) - \tfrac{27}{8} b_{0,3} r_2((n-1)T)^3 \right) + \mathcal{O}(\epsilon^2 t), \tag{4.7}$$

on a time scale of order $\frac{1}{\epsilon}$. In fact (4.7) defines a map $Q : r_2 \to Q(r_2) \Leftrightarrow r_{2n} = Q(r_{2n-1})$ with $r_{2n} = r_2(nT)$. We define a new map P by neglecting the term of $O(\epsilon^2 t)$ in (4.7). That is, $P : \tilde{r_2} \to P(\tilde{r_2}) \Leftrightarrow \tilde{r_2}_n = P(\tilde{r_2}_{n-1})$ with $\tilde{r_2}_n = \tilde{r_2}(nT)$. It will be shown that for $r_2 > 0$:

(i) If $|r_{20} - \tilde{r_2}_0| = \mathcal{O}(\epsilon)$ for $\epsilon \downarrow 0$ then $|r_{2n} - \tilde{r_2}_n| = \mathcal{O}(\epsilon)$ for $n = \mathcal{O}\left(\frac{1}{\sqrt{\epsilon}}\right)$, that is, for $n \sim \frac{1}{\sqrt{\epsilon}}$ and $\epsilon \downarrow 0$, r_{2n} and $\tilde{r_2}_n$ remain "ϵ-close".

(ii) The map P has a unique, hyperbolic fixed point $\tilde{r_2} = \frac{2}{3}\sqrt{\frac{b_{0,1}}{3b_{0,3}}}$, which is asymptotically stable.

(iii) There exists an $\epsilon_0 > 0$ such that for all $0 < \epsilon \le \epsilon_0$ the map Q has a unique hyperbolic fixed point $r_2 = \frac{2}{3}\sqrt{\frac{b_{0,1}}{3b_{0,3}}} + \mathcal{O}(\epsilon)$ with the same stability property as the fixed point $\tilde{r_2} = \frac{2}{3}\sqrt{\frac{b_{0,1}}{3b_{0,3}}}$ of the map P.

Proof of (i): From $|r_{20} - \tilde{r_2}_0| = \mathcal{O}(\epsilon)$ for $\epsilon \downarrow 0$ it follows that there exists a positive constant M_0 such that $|r_{20} - \tilde{r_2}_0| = M_0\epsilon$. We have

$$|r_{2n} - \tilde{r_2}_n| = |P(r_{2n-1}) - P(\tilde{r_2}_{n-1}) + \mathcal{O}(\epsilon^2 n)|$$

$$
\begin{aligned}
&\leq |P(r_{2n-1}) - P(\tilde{r_{2n-1}})| + M_1\epsilon^2 n \\
&\leq L|r_{2n-1} - \tilde{r_{2n-1}}| + M_1\epsilon^2 n, \qquad (4.8)
\end{aligned}
$$

where M_1 and L are positive constants, with $L = 1+\epsilon^2 M_2$ and M_2 a positive constant. So, we have

$$
\begin{aligned}
|r_{2n} - \tilde{r_{2n}}| &\leq (1+\epsilon M_2)|r_{2n-1} - \tilde{r_{2n-1}}| + M_1\epsilon^2 n \leq \ldots \\
&\leq \epsilon(M_0 + \epsilon n^2 M_1)e^{\epsilon n M_2}, \qquad (4.9)
\end{aligned}
$$

and so for $n = \mathcal{O}(\frac{1}{\sqrt{\epsilon}})$ we conclude that $|r_{2n} - \tilde{r_{2n}}| = \mathcal{O}(\epsilon)$.

Proof of (ii): The fixed points of the map P follow from $\tilde{r_{2n}} = P(\tilde{r_{2n-1}})$ for $n \to \infty$ or equivalent from $\tilde{r_2} = \tilde{r_2} + \epsilon T\left(\frac{1}{2}b_{0,1}\tilde{r_2} - \frac{27}{8}b_{0,3}\tilde{r_2}^3\right) \Leftrightarrow \frac{1}{2}\tilde{r}(b_{0,1} - \frac{27}{4}b_{0,3}\tilde{r}^2) = 0$. For $\tilde{r_2} > 0$ we have a unique fixed point $\tilde{r_2} = \frac{2}{3}\sqrt{\frac{b_{0,1}}{3b_{0,3}}}$. The fixed point of the map P is hyperbolic if the linearized map around this fixed point has no eigenvalues of unit modulus. Let DP be this linearized map, then $DP = 1 - \epsilon T b_{0,1}$. Since $0 < \epsilon << 1$ and $b_{0,1} > 0$, we have $|\lambda| < 1$, and so the fixed point is hyperbolic and stable.

Proof of (iii): For the proof of (iii) we refer to [15] for a similar proof.

So far we can conclude that there exists an asymptotically stable, nontrivial, T-periodic solution for system (3.1). We can conclude that the periodic solution for system (3.1) is a combination of a trivial periodic solution in X-direction (that is, $r_1 \equiv 0$) and a nontrivial periodic solution in Y-direction. It has been shown that the nontrivial time-periodic solution of the weakly nonlinear, coupled harmonic oscillators with a 1:3 internal resonance can be determined from the first integrals (3.9) and (3.13), yielding $X(t) \equiv 0$, and $Y(t) = A_2 \cos(3\theta_2(t))$, where $A_2 = \frac{2}{3}\sqrt{\frac{b_{0,1}}{3b_{0,3}}}$ and where $\theta_2(t)$ can be approximated from (3.3) or (3.16) by $\theta_2(0) - \left(1 + \epsilon\frac{\delta_2}{18}\right)t$.

4.2. The 3:1 internal resonance case. The four functionally independent asymptotic approximations (3.20) for first integrals of system (3.17) can be used to determine the existence and stability of non-trivial time-periodic solutions for this system. Let $T < \infty$ be the period of a periodic solution and let c_i(for $i = 1, 2, 3,$ and 4) be constants in the first integrals $G(r_1, \theta_1, r_2, \theta_2, t; \epsilon) = \textit{constant}$ for which a periodic solution exists. Approximate G by G_i (as given by (3.20)) and consider $G_i + \mathcal{O}(\epsilon^2 t) = c_i$ $i = 1, 2, 3, 4$ for $t = nT$ and $t = (n-1)T$ with $n \in N^+$, then for $i = 1, 2, 3,$ and 4 we have

$$
\begin{cases}
G_i\left(r_1(nT), \theta_1(nT), r_2(nT), \theta_2(nT), nT; \epsilon\right) + \mathcal{O}(\epsilon^2 t) = c_i, \\
G_i\left(r_1\left((n-1)T\right), \theta_1((n-1)T), r_2\left((n-1)T\right), \theta_2((n-1)T), (n-1)T; \epsilon\right) \\
\qquad + \mathcal{O}(\epsilon^2 t) = c_i,
\end{cases}
\qquad (4.10)
$$

where G_i for $i = 1, 2, 3,$ and 4 are given explicitly in Appendix C by formula (C.3). By eliminating the constants c_i for $i = 1, 2, 3,$ and 4 from (4.10) by simple subtractions we obtain

$$
(4.11)\quad \begin{cases} r_1(nT) = & r_1\left((n-1)T\right) + \epsilon T\left(-\frac{1}{2}a_{1,0}r_1\left((n-1)T\right)\right. \\ & \left. +\frac{1}{24}a_{0,3}r_2((n-1)T)^3\cos(3\theta_1\left((n-1)T\right) - 3\theta_2\left((n-1)T\right))\right) + \mathcal{O}(\epsilon^2 t), \\ \theta_1(nT) = & \theta_1\left((n-1)T\right) - T + \epsilon\left(-\frac{1}{18}\delta_1 T\right. \\ & \left. -\frac{1}{72}a_{0,3}T\frac{r_2((n-1)T)^3}{r_1((n-1)T)}\sin(3\theta_1\left((n-1)T\right) - 3\theta_2\left((n-1)T\right))\right) + \mathcal{O}(\epsilon^2 t), \\ r_2(nT) = & r_2\left((n-1)T\right) + \epsilon T\left(\frac{1}{2}b_{0,1}r_2((n-1)T) - \frac{3}{8}b_{0,3}r_2((n-1)T)^3\right) + \mathcal{O}(\epsilon^2 t), \\ \theta_2(nT) = & \theta_2\left((n-1)T\right) - \epsilon\frac{1}{2}\delta_2 T + \mathcal{O}(\epsilon^2 t). \end{cases}
$$

By letting $\psi = \theta_1 - \theta_2$, we then obtain

$$
(4.12)\quad \begin{cases} r_1(nT) &= r_1\left((n-1)T\right) + \epsilon T\left(-\frac{1}{2}a_{1,0}r_1\left((n-1)T\right)\right. \\ & \left. +\frac{1}{24}a_{0,3}r_2((n-1)T)^3\cos(3\psi\left((n-1)T\right))\right) + \mathcal{O}(\epsilon^2 t), \\ r_2(nT) &= r_2\left((n-1)T\right) + \epsilon T\left(\frac{1}{2}b_{0,1}r_2((n-1)T)\right. \\ & \left. -\frac{3}{8}b_{0,3}r_2((n-1)T)^3\right) + \mathcal{O}(\epsilon^2 t), \\ \psi(nT) &= \psi\left((n-1)T\right) + \epsilon T\left(\left(\frac{1}{2}\delta_2 - \frac{1}{18}\delta_1\right)\right. \\ & \left. -\frac{1}{72}a_{0,3}\frac{r_2((n-1)T)^3}{r_1((n-1)T)}\sin(3\psi\left((n-1)T\right))\right) + \mathcal{O}(\epsilon^2 t). \end{cases}
$$

In fact (4.12) defines a map which we will use to determine the nontrivial periodic solution(s) of system (3.17). First it should be remarked that the trivial periodic solution of system (3.17) (that is, $X(t) \equiv 0$ and $Y(t) \equiv 0$) is unstable. This can readily be deduced from (3.17) or (3.18) by linearizing the system around the trivial solution. Since the only resonant term in the equation for X is $-\epsilon a_{0,3}\dot{Y}^3$ (see (3.17)) it is obvious that there can not be a nontrivial periodic solution for which $X(t) \equiv 0$ unless $a_{0,3} = 0$. As in section 4.1 (see also (4.7)) we can now study the map as defined by (4.12). A completely similar analysis as given in section 4.1 then yields that the map as defined by (4.12) has a unique, stable, nontrivial, hyperbolic fixed point (r_1, r_2, ψ) which up to $\mathcal{O}(\epsilon)$ is equal to (A_1, A_2, θ_0), where $A_1 = \frac{2b_{0,1}a_{0,3}}{27b_{0,3}^2}\sqrt{\frac{3b_{0,3}b_{0,1}}{36\delta^2+a_{1,0}^2}}$, $A_2 = \frac{2}{3}\sqrt{\frac{3b_{0,1}}{b_{0,3}}}$, $\delta = \frac{1}{2}\delta_2 - \frac{1}{18}\delta_1$, and where θ_0 is given by $\cos(3\theta_0) = \frac{12a_{1,0}A_1}{a_{0,3}A_2^3}$ and $\sin(3\theta_0) = \frac{72\delta A_1}{a_{0,3}A_2^3}$. From this it follows that the system of weakly nonlinear, coupled harmonic oscillators with a 3:1 internal resonance has a nontrivial periodic solution $X(t) = A_1\cos(3\theta_1(t))$ and $Y(t) = A_2\cos(\theta_2(t))$, where $\theta_1(t)$ and $\theta_2(t)$ can be approximated from (3.19) or (3.20) by $\theta_1(0) - (1 + \epsilon(\delta + \frac{1}{18}\delta_1))t$ and $\theta_2(0) - (1 + \epsilon(\frac{1}{2}\delta_2))t$ respectively with $\theta_1(0) - \theta_2(0) = \theta_0$.

4.3. Analysis of Bifurcations. In the weakly nonlinear system (1.1) the coefficients $a_{i,j}$ and $b_{i,j}$ depend on the quasi-static drag and lift forces acting on a conductor in uniform windfield. These quasi-static forces $C_D(\alpha)$ and $C_L(\alpha)$, where α is the angle between the virtual windvelocity and the axis of symmetry of the conductor, can be measured in a wind-tunnel. According to the Den Hartog criterion (a linear instability criterion for the equilibrium position) galloping may set in if $C_D(\alpha) + \frac{\partial}{\partial\alpha}C_L(\alpha) < 0$ for $\alpha = \alpha_s$, where α_s is the static angle of attack, that is, α_s is the angle between the direction of the uniform windflow (in this case the X-direction) and the axis of symmetry of the conductor. Typical results from windtunnel measurements for a certain range of values α are:(see also [3, 4, 13])

$$\begin{aligned} C_D(\alpha) &= C_{D,0}, \\ C_L(\alpha) &= C_{L,1}(\alpha - \alpha_0) + C_{L,3}(\alpha - \alpha_0)^3, \end{aligned}$$

where $C_{D,0}$, $C_{L,1}$, and $C_{L,3}$ are constants and where α_0 is usually the angle of attack for which galloping sets in and for which $C_D(\alpha) + \frac{\partial}{\partial\alpha}C_L(\alpha)$ is as negative as possible. In this paper we will use the parameter values $C_{D,0} = \frac{1}{2}$, $C_{L,1} = -3$, and $C_{L,3} = 6$ (see also [4]). Furthermore, we define $\bar{\alpha}_s = \alpha_s - \alpha_0$. How the coefficients $a_{i,j}$ and $b_{i,j}$ depend on $\bar{\alpha}_s$ is given in [4] and in appendix A. Generally it is assumed that for $\bar{\alpha}_s = 0$ the largest oscillation amplitudes due to galloping occur. In this section we will show how this assumption can be verified. It should be remarked that in order to have galloping $b_{0,1} = -(C_{D,0} + C_{L,1} + 3\,C_{L,3}\bar{\alpha}_s^2)$ has to be positive, implying that $\bar{\alpha}_s$ should satisfy in this case : $|\bar{\alpha}_s| < \frac{1}{6}\sqrt{5}$.

4.3.1. The 1:3 internal resonance case. Since $a_{1,0}$ is independent of $\bar{\alpha}_s$ it follows from (4.3) that $X(t)$ tends to zero for increasing time. And from (4.7) it follows that $r_2 = 0$ is unstable for $|\bar{\alpha}_s| < \frac{1}{6}\sqrt{5}$. So, the trivial periodic solution is unstable for $|\bar{\alpha}_s| < \frac{1}{6}\sqrt{5}$. For the nontrivial periodic solution $X(t) = 0$, and $Y(t) = A_2\cos(3\theta_2(t))$ it follows easily from proof (ii) in section 4.1 that this nontrivial periodic solution is stable for $|\bar{\alpha}_s| < \frac{1}{6}\sqrt{5}$. A plot of the amplitude A_2 as function of $\bar{\alpha}_s$ is given in figure (4.1). This plot confirms the assumption that for $\bar{\alpha}_s = 0$ the largest vibration

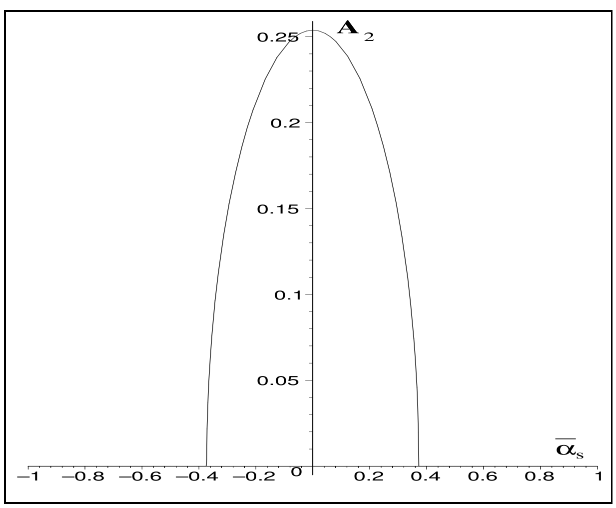


FIG. 4.1. *Plot of amplitude A_2 as fuction of $\bar{\alpha}_s$.*

amplitudes occur. In figure (4.2) the oscillations for $t \to \infty$ in (X, Y)-plane are given.

4.3.2. The 3:1 internal resonance case. As in the previous subsection 4.3.1 it can easily be shown that the trivial periodic solution is unstable for $|\bar{\alpha}_s| < \frac{1}{6}\sqrt{5}$.

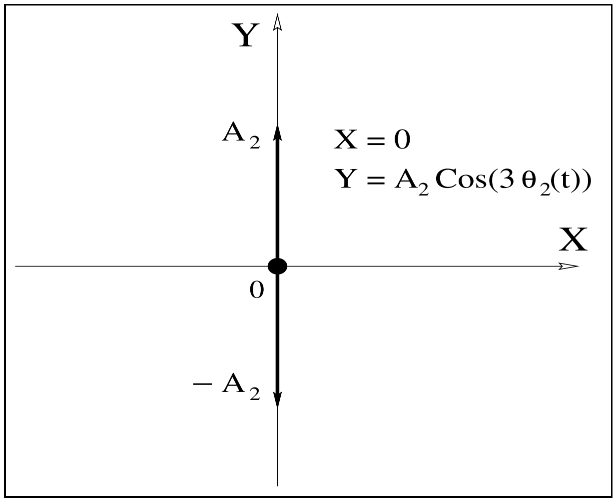


FIG. 4.2. *Plot of the stable periodic solution for the 1:3 internal resonance case.*

For the nontrivial periodic solution $X(t) = A_1 \cos(3\theta_1(t))$, and $Y(t) = A_2 \cos(\theta_2(t))$ it follows from the (reduced) linearized map (4.12) around this periodic solution that the eigenvalues of this map of (4.12) are : $-b_{0,1}$, $-\frac{1}{2}a_{1,0} + 3i\left(-\frac{1}{18}\delta_1 + \frac{1}{2}\delta_2\right)$, and $-\frac{1}{2}a_{1,0} - 3i\left(-\frac{1}{18}\delta_1 + \frac{1}{2}\delta_2\right)$. Since $b_{0,1}$ and $a_{1,0}$ are positive for $|\bar{\alpha}_s| < \frac{1}{6}\sqrt{5}$ it follows that the nontrivial periodic solution is stable for $|\bar{\alpha}_s| < \frac{1}{6}\sqrt{5}$. In figure (4.3) and in figure (4.4) plots are given of the amplitudes A_1 and A_2 of the stable, periodic solutions as functions of $\bar{\alpha}_s$ and the detuning parameter $\delta = -\frac{1}{18}\delta_1 + \frac{1}{2}\delta_2$. In

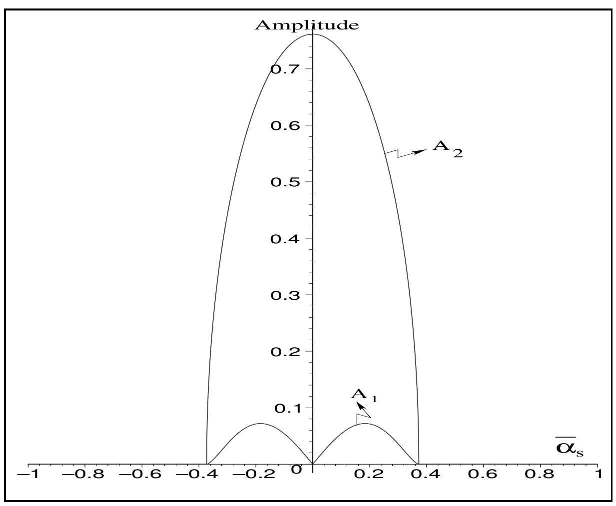
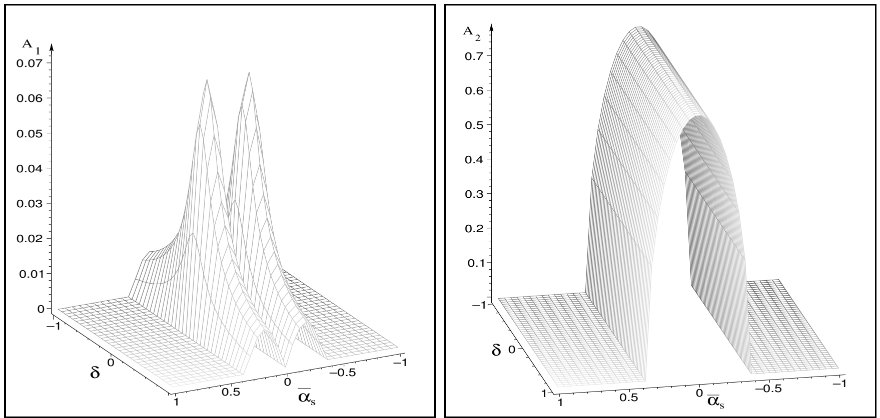


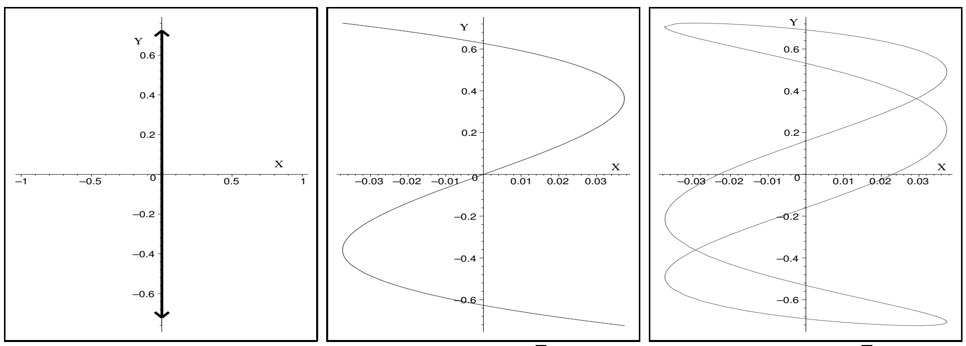
FIG. 4.3. *Plot of amplitudes A_1 and A_2 as function of $\bar{\alpha}_s$ for $\delta = 0$.*

figure (4.5) plots in the (X, Y)-plane are given for the stable periodic solutions of the oscillator for different values of the phase difference $\theta_0 = \theta_1(0) - \theta_2(0)$. It should be remarked that from figure (4.3) it follows that the largest vibration amplitudes in X-direction do not occur for $\bar{\alpha}_s = 0$.



(a) A_1 as function of $\bar{\alpha}_s$ and δ. (b) A_2 as function of $\bar{\alpha}_s$ and δ.

FIG. 4.4. *Plots of amplitudes as functions of $\bar{\alpha}_s$ and δ.*



(a) $\theta_0 = 0$, $\bar{\alpha}_s = 0$. (b) $\theta_0 = 0$, $\bar{\alpha}_s = \frac{\sqrt{5}}{20}$, $\delta = \frac{1}{2}$. (c) $\theta_0 = \frac{2}{9}$, $\bar{\alpha}_s = \frac{\sqrt{5}}{20}$, $\delta = \frac{1}{2}$.

FIG. 4.5. *Plots of the stable periodic solution for the 3:1 internal resonance case.*

5. Conclusions and remarks. In this paper it has been shown that the perturbation method based on integrating factors can be used efficiently to approximate first integrals for a system of weakly nonlinear, coupled harmonic oscillators. In section 2 (and 3) of this paper a justification of the presented perturbation method has been given. It has also been shown how the existence and stability of time-periodic solutions can be deduced from the approximations of the first integrals for a system of weakly nonlinear, coupled harmonic oscillators with a 1:3 or a 3:1 internal resonance. The presented perturbation methods can easily be applied to other systems of weakly nonlinear, coupled harmonic oscillators. In [4] system (1.1) has been studied for the 1:1, 1:2, and 2:1 internal resonance cases. First order normal form techniques and averaging techniques have been used in [4] to determine the existence and stability of nontrivial, periodic solutions. In this paper we used the recently developed perturbation method based on integrating vectors to study system (1.1) with a 1:3 and a 3:1 internal resonance. This paper in fact completes the study of system (1.1). It should be remarked that system (1.1) is a model that describes galloping of conductor lines (in a windfield) on which ice has accreted. As is well-known galloping is an almost purely vertical oscillation of conductor lines. Our results imply that the system of oscillators will eventually oscillate in an almost purely vertical direction (that is, in

Y-direction).

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Appendix

A. An oscillator with two degrees of freedom in a uniform windfield. In [4] a model has been developed to describe the dynamics of an oscillator with two degrees of freedom in a uniform windfield. The following system of a weakly nonlinear, coupled harmonic oscillators has been derived

$$\begin{cases} \ddot{X} + \left(\omega_1^2 + \epsilon\delta_1\right) X = \epsilon \left[-a_{1,0}\dot{X} + a_{0,1}\dot{Y} + a_{2,0}\dot{X}^2 - a_{1,1}\dot{X}\dot{Y} + a_{0,2}\dot{Y}^2 - a_{0,3}\dot{Y}^3\right], \\ \ddot{Y} + \left(\omega_2^2 + \epsilon\delta_2\right) Y = \epsilon \left[-b_{1,0}\dot{X} + b_{0,1}\dot{Y} + b_{2,0}\dot{X}^2 - b_{1,1}\dot{X}\dot{Y} - b_{0,2}\dot{Y}^2 - b_{0,3}\dot{Y}^3\right], \end{cases}$$
(A.1)

where $X = X(t), Y = Y(t), \dot{} = \frac{d}{dt}$, and where ϵ is a small parameter with $0 < \epsilon << 1$, and where the coefficients $a_{i,j}$ and $b_{i,j} \in \mathbb{R}$ are given by

$$\begin{array}{ll} a_{1,0} = 2C_{D,0} > 0, & b_{1,0} = 2\left(C_{L,1}\bar{\alpha}_s + C_{L,3}\bar{\alpha}_s^3\right), \\ a_{0,1} = C_{L,1}\bar{\alpha}_s + C_{L,3}\bar{\alpha}_s^3, & b_{0,1} = -(C_{D,0} + C_{L,1} + 3\,C_{L,3}\bar{\alpha}_s^2) > 0, \\ a_{2,0} = C_{D,0} > 0, & b_{2,0} = C_{L,1}\bar{\alpha}_s + C_{L,3}\bar{\alpha}_s^3, \\ a_{1,1} = C_{L,1}\bar{\alpha}_s + C_{L,3}\bar{\alpha}_s^3, & b_{1,1} = -C_{D,0} - C_{L,1} - 3\,C_{L,3}\bar{\alpha}_s^3 > 0, \\ a_{0,2} = \frac{1}{2}C_{D,0} - C_{L,1} + 3C_{L,3}\bar{\alpha}_s^2 > 0, & b_{0,2} = -3C_{L,3}\bar{\alpha}_s - \frac{C_{L,3}}{2}\bar{\alpha}_s^3 - \frac{C_{L,1}}{2}\bar{\alpha}_s, \\ a_{0,3} = -\frac{1}{2}\bar{\alpha}_s C_{L,1} - 3C_{L,3}\bar{\alpha}_s - C_{L,3}\frac{\bar{\alpha}_s^3}{2}, & b_{0,3} = \frac{C_{D,0}}{2} + \frac{C_{L,1}}{6} + (1 + \frac{1}{2}\bar{\alpha}_s^2)C_{L,3} > 0. \end{array}$$

The quasi-static drag and lift forces $C_D(\alpha)$ and $C_L(\alpha)$ acting on a cylinder with ridge can be obtained from wind-tunnel experiments. The coefficients $C_{D,0}$, $C_{L,1}$, and $C_{L,3}$ can be derived from these forces (see also section 4.3). The angles α, α_s, and $\bar{\alpha}_s$ are defined in section 4.3. In figure (A.1) a sketch of the oscillator is presented. The oscillator consists of a cylinder with a small ridge. In figure (A.2) the drag and lift forces acting on the cylinder are given. For more (and complete) details we refer to [3, 4, 13].

B. The 1:3 internal Resonance. In polar coordinates, we can rewrite system (3.2) as follows:

$$\text{(B.1)} \qquad \begin{cases} \frac{dr_1}{dt} = \epsilon\, g_1(r_1, r_2, \theta_1, \theta_2), \\ \frac{d\theta_1}{dt} = -1 + \epsilon\, g_2(r_1, r_2, \theta_1, \theta_2), \\ \frac{dr_2}{dt} = \epsilon\, g_3(r_1, r_2, \theta_1, \theta_2), \\ \frac{d\theta_2}{dt} = -1 + \epsilon\, g_4(r_1, r_2, \theta_1, \theta_2), \end{cases}$$

where

$$g_1 = \frac{1}{2}\,\delta_1 r_1 \sin(2\,\theta_1) - \frac{1}{2}\,a_{1,0} r_1 + \frac{1}{2}\,a_{1,0} r_1 \cos(2\,\theta_1) + \frac{3}{2}\,a_{0,1} r_2 \cos(\theta_1 - 3\,\theta_2)$$

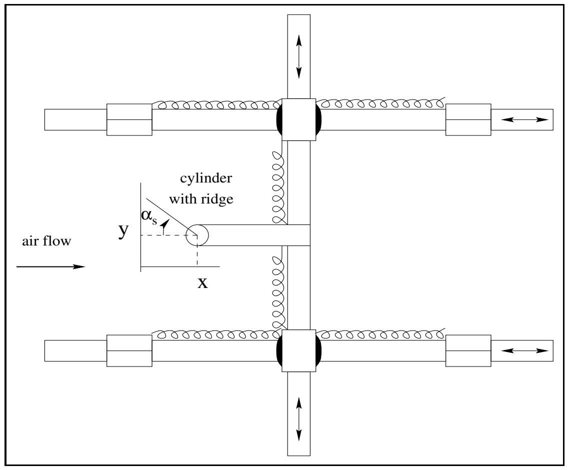

FIG. A.1. *The aeroelastic oscillator as viewed from above.*

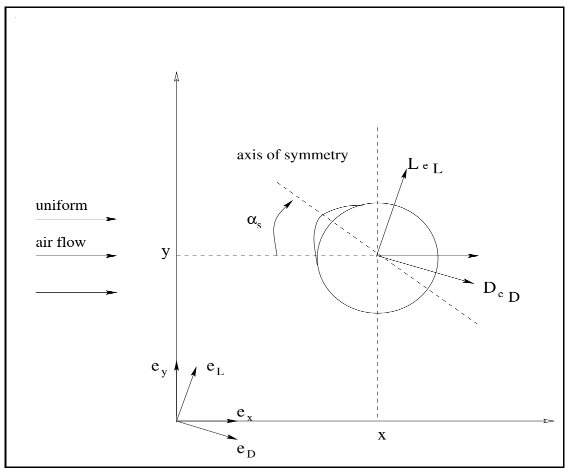

FIG. A.2. *Windvelocities and aerodynamic forces acting on the cross section of the cylinder with ridge.*

$$
\begin{aligned}
&-\frac{3}{2}\, a_{0,1} r_2 \cos(\theta_1 + 3\,\theta_2) + \frac{3}{4}\, a_{2,0} {r_1}^2 \sin(\theta_1) - \frac{1}{4}\, a_{2,0} {r_1}^2 \sin(3\,\theta_1) \\
&-\frac{3}{2}\, a_{1,1} r_1 r_2 \sin(3\,\theta_2) + \frac{3}{4}\, a_{1,1} r_1 r_2 \sin(3\,\theta_2 + 2\,\theta_1) - \frac{3}{4}\, a_{1,1} r_1 r_2 \sin(-3\,\theta_2 + 2\,\theta_1) \\
&+\frac{9}{2} a_{0,2} {r_2}^2 \sin(\theta_1) - \frac{9}{4}\, a_{0,2} {r_2}^2 \sin(\theta_1 + 6\,\theta_2) - \frac{9}{4}\, a_{0,2} {r_2}^2 \sin(\theta_1 - 6\,\theta_2) \\
&-\frac{81}{8}\, a_{0,3} {r_2}^3 \cos(\theta_1 - 3\,\theta_2) + \frac{81}{8}\, a_{0,3} {r_2}^3 \cos(\theta_1 + 3\,\theta_2) + \frac{27}{8}\, a_{0,3} {r_2}^3 \cos(-9\,\theta_2 + \theta_1) \\
&-\frac{27}{8}\, a_{0,3} {r_2}^3 \cos(9\,\theta_2 + \theta_1),
\end{aligned}
$$

$$
\begin{aligned}
g_2 &= -\frac{1}{2}\,\delta_1\cos(2\,\theta_1)-\frac{1}{2}\,\delta_1-\frac{1}{2}\,a_{1,0}\sin(2\,\theta_1)+\frac{3}{2}\,a_{0,1}\frac{r_2}{r_1}\sin(\theta_1+3\,\theta_2)\\
&-\frac{3}{2}\,a_{0,1}\frac{r_2}{r_1}\sin(\theta_1-3\,\theta_2)+\frac{1}{4}\,a_{2,0}r_1\cos(\theta_1)-\frac{1}{4}\,a_{2,0}r_1\cos(3\,\theta_1)\\
&-\frac{3}{4}\,a_{1,1}r_2\cos(-3\,\theta_2+2\,\theta_1)+\frac{3}{4}\,a_{1,1}r_2\cos(3\,\theta_2+2\,\theta_1)+\frac{9}{2}a_{0,2}\frac{{r_2}^2}{r_1}\cos(\theta_1)\\
&-\frac{9}{4}\,a_{0,2}\frac{{r_2}^2}{r_1}\cos(\theta_1-6\,\theta_2)-\frac{9}{4}\,a_{0,2}\frac{{r_2}^2}{r_1}\cos(\theta_1+6\,\theta_2)-\frac{81}{8}\,a_{0,3}\frac{{r_2}^3}{r_1}\sin(\theta_1+3\,\theta_2)\\
&+\frac{81}{8}\,a_{0,3}\frac{{r_2}^3}{r_1}\sin(\theta_1-3\,\theta_2)+\frac{27}{8}\,a_{0,3}\frac{{r_2}^3}{r_1}\sin(9\,\theta_2+\theta_1)-\frac{27}{8}\,a_{0,3}\frac{{r_2}^3}{r_1}\sin(-9\,\theta_2+\theta_1),\\
g_3 &= -\frac{1}{6}\,\delta_2r_2\sin(6\,\theta_2)-\frac{1}{6}\,b_{1,0}r_1\cos(\theta_1-3\,\theta_2)+\frac{1}{6}\,b_{1,0}r_1\cos(\theta_1+3\,\theta_2)\\
&+\frac{1}{2}\,b_{0,1}r_2-\frac{1}{2}\,b_{0,1}r_2\cos(6\,\theta_2)+\frac{1}{6}b_{2,0}{r_1}^2\sin(3\,\theta_2)-\frac{1}{12}\,b_{2,0}{r_1}^2\sin(3\,\theta_2+2\,\theta_1)\\
&+\frac{1}{12}\,b_{2,0}{r_1}^2\sin(-3\,\theta_2+2\,\theta_1)-\frac{1}{2}\,b_{1,1}r_1r_2\sin(\theta_1)+\frac{1}{4}\,b_{1,1}r_1r_2\sin(\theta_1+6\,\theta_2)\\
&+\frac{1}{4}\,b_{1,1}r_1r_2\sin(\theta_1-6\,\theta_2)-\frac{9}{4}\,b_{0,2}{r_2}^2\sin(3\,\theta_2)+\frac{3}{4}\,b_{0,2}{r_2}^2\sin(9\,\theta_2)\\
&-\frac{27}{8}\,b_{0,3}{r_2}^3+\frac{9}{2}\,b_{0,3}{r_2}^3\cos(6\,\theta_2)-\frac{9}{8}\,b_{0,3}{r_2}^3\cos(12\,\theta_2),\\
g_4 &= -\frac{1}{18}\delta_2\cos(6\,\theta_2)-\frac{1}{18}\delta_2-\frac{1}{18}\,b_{1,0}\frac{r_1}{r_2}\sin(\theta_1+3\,\theta_2)-\frac{1}{18}\,b_{1,0}\frac{r_1}{r_2}\sin(\theta_1-3\,\theta_2)\\
&+\frac{1}{6}\,b_{0,1}\sin(6\,\theta_2)+\frac{1}{18}\,b_{2,0}\frac{{r_1}^2}{r_2}\cos(3\,\theta_2)-\frac{1}{36}\,b_{2,0}{r_1}^2\cos(-3\,\theta_2+2\,\theta_1)\\
&-\frac{1}{36}\,b_{2,0}{r_1}^2\cos(3\,\theta_2+2\,\theta_1)-\frac{1}{12}\,b_{1,1}r_1\cos(\theta_1-6\,\theta_2)+\frac{1}{12}\,b_{1,1}r_1\cos(\theta_1+6\,\theta_2)\\
&-\frac{1}{4}\,b_{0,2}r_2\cos(3\,\theta_2)+\frac{1}{4}\,b_{0,2}r_2\cos(9\,\theta_2)-\frac{3}{4}\,b_{0,3}{r_2}^2\sin(6\,\theta_2)+\frac{3}{8}\,b_{0,3}{r_2}^2\sin(12\,\theta_2).
\end{aligned}
$$

(B.2)

The approximations of first integrals in the 1:3 internal resonance case are

$$
\begin{aligned}
F_1 &= r_1+\epsilon\quad\frac{1}{2}\,a_{1,0}r_1t+\frac{1}{4}\,\delta_1r_1\cos(2\,\theta_1)+\frac{1}{4}\,a_{1,0}r_1\sin(2\,\theta_1)-\frac{3}{4}a_{2,0}r_1^2\cos(\theta_1)\\
&+\frac{1}{12}\,a_{2,0}r_1^2\cos(3\,\theta_1)+\frac{1}{2}\,a_{1,1}r_1r_2\cos(3\,\theta_2)-\frac{3}{20}\,a_{1,1}r_1r_2\cos(3\,\theta_2+2\,\theta_1)\\
&-\frac{3}{4}\,a_{1,1}r_1r_2\cos(-3\,\theta_2+2\,\theta_1)-\frac{3}{4}\,a_{0,1}r_2\sin(\theta_1-3\,\theta_2)-\frac{3}{8}\,a_{0,1}r_2\sin(\theta_1+3\,\theta_2)\\
&-\frac{9}{2}a_{0,2}r_2^2\cos(\theta_1)+\frac{9}{28}\,a_{0,2}r_2^2\cos(\theta_1+6\,\theta_2)-\frac{9}{20}\,a_{0,2}r_2^2\cos(\theta_1-6\,\theta_2)\\
&+\frac{81}{16}\,a_{0,3}r_2^3\sin(\theta_1-3\,\theta_2)+\frac{81}{32}\,a_{0,3}r_2^3\sin(\theta_1+3\,\theta_2)-\frac{27}{64}\,a_{0,3}r_2^3\sin(-9\,\theta_2+\theta_1)
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{27}{80}\, a_{0,3} r_2^3 \sin(9\,\theta_2+\theta_1) \quad , \\
F_2 \;=\;& \theta_1 + t + \epsilon \quad \frac{\delta_1 t}{2} - \frac{\delta_1 \sin(2\,\theta_1)}{4} + \frac{a_{1,0}}{4}\cos(2\,\theta_1) + \frac{a_{2,0} r_1}{4}\, \sin(\theta_1) - \frac{a_{2,0} r_1}{12}\, \sin(3\,\theta_1) \\
& -\frac{3}{8}\, \frac{a_{0,1} r_2 \cos(\theta_1+3\,\theta_2)}{r_1} - \frac{3}{4}\, \frac{a_{0,1} r_2 \cos(\theta_1-3\,\theta_2)}{r_1} + \frac{9}{2}\, \frac{a_{0,2} r_2^2 \sin(\theta_1)}{r_1} \\
& +\frac{9}{20}\, \frac{a_{0,2} r_2^2 \sin(\theta_1-6\,\theta_2)}{r_1} - \frac{9}{28}\, \frac{a_{0,2} r_2^2 \sin(\theta_1+6\,\theta_2)}{r_1} + \frac{81}{32}\, \frac{a_{0,3} r_2^3 \cos(\theta_1+3\,\theta_2)}{r_1} \\
& +\frac{81}{16}\, \frac{a_{0,3} r_2^3 \cos(\theta_1-3\,\theta_2)}{r_1} - \frac{27}{80}\, \frac{a_{0,3} r_2^3 \cos(9\,\theta_2+\theta_1)}{r_1} - \frac{27}{64}\, \frac{a_{0,3} r_2^3 \cos(-9\,\theta_2+\theta_1)}{r_1} \\
& +\frac{3 a_{1,1} r_2}{4}\, \sin(-3\,\theta_2+2\,\theta_1) + \frac{3 a_{1,1} r_2}{20}\, \sin(3\,\theta_2+2\,\theta_1) \quad , \\
F_3 \;=\;& r_2 + \epsilon \quad \frac{1}{12}\, b_{1,0} r_1 \sin(\theta_1-3\,\theta_2) + \frac{1}{24}\, b_{1,0} r_1 \sin(\theta_1+3\,\theta_2) - \frac{1}{18} b_{2,0} r_1^2\, \cos(3\,\theta_2) \\
& +\frac{1}{60}\, b_{2,0} r_1^2 \cos(3\,\theta_2+2\,\theta_1) + \frac{1}{12}\, b_{2,0} r_1^2 \cos(-3\,\theta_2+2\,\theta_1) + \frac{1}{2}\, b_{1,1} r_1 r_2 \cos(\theta_1) \\
& -\frac{1}{28}\, b_{1,1} r_1 r_2 \cos(\theta_1+6\,\theta_2) + \frac{1}{20}\, b_{1,1} r_1 r_2 \cos(\theta_1-6\,\theta_2) + \frac{1}{36}\, \delta_2 r_2 \cos(6\,\theta_2) \\
& -\frac{1}{2}\, b_{0,1} r_2 t + \frac{27}{8}\, b_{0,3} r_2^3 t - \frac{1}{12}\, b_{0,1} r_2 \sin(6\,\theta_2) + \frac{3}{4} b_{0,2} r_2^2\, \cos(3\,\theta_2) \\
& -\frac{1}{12}\, b_{0,2} r_2^2 \cos(9\,\theta_2) + \frac{3}{4}\, b_{0,3} r_2^3 \sin(6\,\theta_2) - \frac{3}{32}\, b_{0,3} r_2^3 \sin(12\,\theta_2) \quad , \\
F_4 \;=\;& \theta_2 + t + \epsilon \quad \frac{1}{18}\, \delta_2 t + \frac{1}{72}\, \frac{b_{1,0} r_1 \cos(\theta_1+3\,\theta_2)}{r_2} - \frac{1}{36}\, \frac{b_{1,0} r_1 \cos(\theta_1-3\,\theta_2)}{r_2} \\
& +\frac{1}{54}\, \frac{b_{2,0} r_1^2 \sin(3\,\theta_2)}{r_2} + \frac{1}{36}\, \frac{b_{2,0} r_1^2 \sin(-3\,\theta_2+2\,\theta_1)}{r_2} - \frac{1}{180}\, \frac{b_{2,0} r_1^2 \sin(3\,\theta_2+2\,\theta_1)}{r_2} \\
& +\frac{1}{60}\, b_{1,1} r_1 \sin(\theta_1-6\,\theta_2) + \frac{1}{84}\, b_{1,1} r_1 \sin(\theta_1+6\,\theta_2) - \frac{1}{12}\, b_{0,2} r_2 \sin(3\,\theta_2) + \frac{1}{36}\, b_{0,2} r_2 \sin(9\,\theta_2) \\
& +\frac{1}{8}\, b_{0,3} r_2^2 \cos(6\,\theta_2) - \frac{1}{32}\, b_{0,3} r_2^2 \cos(12\,\theta_2) - \frac{1}{108}\, \delta_2 \sin(6\,\theta_2) - \frac{1}{36}\, b_{0,1} \cos(6\,\theta_2) \quad .
\end{aligned}
$$

(B.3)

C. The 3:1 internal resonance case. In polar coordinates, system (3.18) becomes

$$\text{(C.1)} \qquad \begin{cases} \frac{dr_1}{dt} = \epsilon h_1(r_1, r_2, \theta_1, \theta_2), \\ \frac{d\theta_1}{dt} = -1 + \epsilon h_2(r_1, r_2, \theta_1, \theta_2), \\ \frac{dr_2}{dt} = \epsilon h_3(r_1, r_2, \theta_1, \theta_2), \\ \frac{d\theta_2}{dt} = -1 + \epsilon h_4(r_1, r_2, \theta_1, \theta_2), \end{cases}$$

where

$$\begin{aligned}
h_1 \;=\;& -\frac{\delta_1 r_1}{6}\sin(6\theta_1) - \frac{a_{1,0}r_1}{2} + \frac{a_{1,0}r_1}{2}\sin(6\theta_1) + \frac{a_{0,1}r_2}{6}\cos(3\theta_1-\theta_2) - \frac{a_{0,1}r_2}{6}\cos(3\theta_1+\theta_2) \\
&+\frac{9}{4}a_{2,0}r_1^2\sin(3\theta_1) - \frac{3}{4}a_{2,0}r_1^2\sin(9\theta_1) - \frac{1}{2}a_{1,1}r_1r_2\sin(\theta_2) + \frac{1}{4}a_{1,1}r_1r_2\sin(\theta_2+6\theta_1) \\
&-\frac{1}{4}a_{1,1}r_1r_2\sin(-\theta_2+6\theta_1) + \frac{1}{6}a_{0,2}r_2^2\sin(3\theta_1) - \frac{1}{12}a_{0,2}r_2^2\sin(3\theta_1+2\theta_2) \\
&-\frac{1}{12}a_{0,2}r_2^2\sin(3\theta_1-2\theta_2) - \frac{1}{12}a_{0,3}r_2^3\cos(3\theta_1-\theta_2) + \frac{1}{8}a_{0,3}r_2^3\cos(3\theta_1+\theta_2) \\
&+\frac{1}{24}a_{0,3}r_2^3\cos(-3\theta_1+3\theta_2) - \frac{1}{24}a_{0,3}r_2^3\cos(3\theta_1+3\theta_2) - \frac{1}{24}a_{0,3}r_2^3\cos(\theta_2-3\theta_1), \\
h_2 \;=\;& -\frac{1}{18}\delta_1\cos(6\theta_1) - \frac{1}{18}\delta_1 - \frac{1}{6}a_{1,0}\sin(6\theta_1) + \frac{1}{18}a_{0,1}\frac{r_2}{r_1}\sin(3\theta_1+\theta_2) - \frac{1}{18}a_{0,1}\frac{r_2}{r_1}\sin(3\theta_1-\theta_2) \\
&+\frac{1}{4}a_{2,0}r_1\cos(3\theta_1) - \frac{1}{4}a_{2,0}r_1\cos(9\theta_1) - \frac{1}{12}a_{1,1}r_2\cos(-\theta_2+6\theta_1) + \frac{1}{12}a_{1,1}r_2\cos(\theta_2+6\theta_1) \\
&+\frac{1}{18}a_{0,2}\frac{r_2^2}{r_1}\cos(3\theta_1) - \frac{1}{36}a_{0,2}\frac{r_2^2}{r_1}\cos(3\theta_1-2\theta_2) - \frac{1}{36}a_{0,2}\frac{r_2^2}{r_1}\cos(3\theta_1+2\theta_2) \\
&-\frac{1}{24}a_{0,3}\frac{r_2^3}{r_1}\sin(3\theta_1+\theta_2) + \frac{1}{24}a_{0,3}\frac{r_2^3}{r_1}\sin(3\theta_1-\theta_2) + \frac{1}{72}a_{0,3}\frac{r_2^3}{r_1}\sin(3\theta_1+3\theta_2) \\
&-\frac{1}{72}a_{0,3}\frac{r_2^3}{r_1}\sin(3\theta_1-3\theta_2), \\
h_3 \;=\;& -\frac{1}{2}\delta_2 r_2\sin(2\theta_2) - \frac{3}{2}b_{1,0}r_1\cos(3\theta_1-\theta_2) + \frac{3}{2}b_{1,0}r_1\cos(3\theta_1+\theta_2) + \frac{1}{2}b_{0,1}r_2 \\
&-\frac{1}{2}r_2\cos(2\theta_2) + \frac{9}{2}b_{2,0}r_1^2\sin(\theta_2) - \frac{9}{4}b_{2,0}r_1^2\sin(\theta_2+6\theta_1) + \frac{9}{4}b_{2,0}r_1^2\sin(-\theta_2+6\theta_1) \\
&-\frac{3}{2}b_{1,1}r_1r_2\sin(3\theta_1) + \frac{3}{4}b_{1,1}r_1r_2\sin(3\theta_1+2\theta_2) + \frac{3}{4}b_{1,1}r_1r_2\sin(3\theta_1-2\theta_2) \\
&-\frac{3}{4}b_{0,2}r_2^2\sin(\theta_2) + \frac{1}{4}b_{0,2}r_2^2\sin(3\theta_2) - \frac{3}{8}b_{0,3}r_2^3 + \frac{1}{2}b_{0,3}r_2^3\cos(2\theta_2) - \frac{1}{8}b_{0,3}r_2^3\cos(4\theta_2), \\
h_4 \;=\;& -\frac{1}{2}\delta_2\cos(2\theta_2) - \frac{1}{2}\delta_2 - \frac{3}{2}b_{1,0}\frac{r_1}{r_2}\sin(3\theta_1+\theta_2) - \frac{3}{2}b_{1,0}\frac{r_1}{r_2}\sin(3\theta_1-\theta_2)
\end{aligned}$$

$$+\frac{1}{2}b_{0,1}\sin(2\theta_2)+\frac{9}{2}b_{2,0}\frac{r_1^2}{r_2}\cos(\theta_2)-\frac{9}{4}b_{2,0}\frac{r_1^2}{r_2}\cos(-\theta_2+6\theta_1)-\frac{9}{4}b_{2,0}\frac{r_1^2}{r_2}\cos(\theta_2+6\theta_1)$$

$$-\frac{3}{4}b_{1,1}r_1\cos(3\theta_1-2\theta_2)+\frac{3}{4}b_{1,1}r_1\cos(3\theta_1+2\theta_2)-\frac{1}{4}b_{0,2}r_2\cos(\theta_2)+\frac{1}{4}b_{0,2}r_2\cos(3\theta_2)$$

$$-\frac{1}{4}b_{0,3}r_2^2\sin(2\theta_2)+\frac{1}{8}b_{0,3}r_2^2\sin(4\theta_2). \tag{C.2}$$

The approximations of first integrals in the 3:1 internal resonance case are

$$G_1 = r_1+\epsilon\ \frac{1}{36}\,\delta_1 r_1\cos(6\,\theta_1)+\frac{1}{2}\,a_{1,0}tr_1+\frac{1}{12}\,a_{1,0}r_1\sin(6\,\theta_1)-\frac{3}{4}a_{2,0}r_1^2\,\cos(3\,\theta_1)$$

$$+\frac{1}{12}\,a_{2,0}r_1^2\cos(9\,\theta_1)+\frac{1}{2}\,a_{1,1}r_1r_2\cos(\theta_2)-\frac{1}{28}\,a_{1,1}r_1r_2\cos(\theta_2+6\,\theta_1)$$

$$+\frac{1}{20}\,a_{1,1}r_1r_2\cos(-\theta_2+6\,\theta_1)+\frac{1}{12}\,a_{0,1}r_2\sin(3\,\theta_1-\theta_2)-\frac{1}{24}\,a_{0,1}r_2\sin(3\,\theta_1+\theta_2)$$

$$-\frac{1}{18}a_{0,2}r_2^2\,\cos(3\,\theta_1)+\frac{1}{60}\,a_{0,2}r_2^2\cos(3\,\theta_1+2\,\theta_2)+\frac{1}{12}\,a_{0,2}r_2^2\cos(3\,\theta_1-2\,\theta_2)$$

$$-\frac{1}{16}\,a_{0,3}r_2^3\sin(3\,\theta_1-\theta_2)+\frac{1}{32}\,a_{0,3}r_2^3\sin(3\,\theta_1+\theta_2)-\frac{1}{24}\,a_{0,3}r_2^3t\cos(-3\,\theta_2+3\,\theta_1)$$

$$-\frac{1}{144}\,a_{0,3}r_2^3\sin(3\,\theta_2+3\,\theta_1)\ ,$$

$$G_2 = \theta_1+t+\epsilon\ \frac{1}{18}\,\delta_1 t+\frac{1}{12}\,a_{2,0}r_1\sin(3\,\theta_1)-\frac{1}{36}\,a_{2,0}r_1\sin(9\,\theta_1)-\frac{1}{72}\,\frac{a_{0,1}r_2\cos(3\,\theta_1+\theta_2)}{r_1}$$

$$+\frac{1}{36}\,\frac{a_{0,1}r_2\cos(3\,\theta_1-\theta_2)}{r_1}+\frac{1}{54}\,\frac{a_{0,2}r_2^2\sin(3\,\theta_1)}{r_1}-\frac{1}{36}\,\frac{a_{0,2}r_2^2\sin(3\,\theta_1-2\,\theta_2)}{r_1}$$

$$-\frac{1}{180}\,\frac{a_{0,2}r_2^2\sin(3\,\theta_1+2\,\theta_2)}{r_1}+\frac{1}{96}\,\frac{a_{0,3}r_2^3\cos(3\,\theta_1+\theta_2)}{r_1}-\frac{1}{48}\,\frac{a_{0,3}r_2^3\cos(3\,\theta_1-\theta_2)}{r_1}$$

$$-\frac{1}{432}\,\frac{a_{0,3}r_2^3\cos(3\,\theta_2+3\,\theta_1)}{r_1}+\frac{1}{72}\,\frac{ta_{0,3}r_2^3\sin(-3\,\theta_2+3\,\theta_1)}{r_1}-\frac{1}{108}\,\delta_1\sin(6\,\theta_1)$$

$$+\frac{1}{36}\,a_{1,0}\cos(6\,\theta_1)-\frac{1}{60}\,a_{1,1}r_2\sin(-\theta_2+6\,\theta_1)+\frac{1}{84}\,a_{1,1}r_2\sin(\theta_2+6\,\theta_1)\ ,$$

$$G_3 = r_2+\epsilon\ -\frac{3}{4}\,b_{1,0}r_1\sin(3\,\theta_1-\theta_2)+\frac{3}{8}\,b_{1,0}r_1\sin(3\,\theta_1+\theta_2)-\frac{9}{2}b_{2,0}r_1^2\,\cos(\theta_2)$$

$$+\frac{9}{28}\,b_{2,0}r_1^2\cos(\theta_2+6\,\theta_1)-\frac{9}{20}\,b_{2,0}r_1^2\cos(-\theta_2+6\,\theta_1)+\frac{1}{2}\,b_{1,1}r_1r_2\cos(3\,\theta_1)$$

$$-\frac{3}{20}\,b_{1,1}r_1r_2\cos(3\,\theta_1+2\,\theta_2)-\frac{3b_{1,1}r_1r_2}{4}\,\cos(3\,\theta_1-2\,\theta_2)+\frac{\delta_2 r_2}{4}\,\cos(2\,\theta_2)-\frac{b_{0,1}tr_2}{2}$$

$$-\frac{1}{4}\,b_{0,1}r_2\sin(2\,\theta_2)+\frac{3}{4}\,b_{0,2}r_2^2\cos(\theta_2)-\frac{1}{12}\,b_{0,2}r_2^2\cos(3\,\theta_2)+\frac{3}{8}\,b_{0,3}r_2^3t$$

$$+\frac{1}{4}\,b_{0,3}r_2^3\sin(2\,\theta_2)-\frac{1}{32}\,b_{0,3}r_2^3\sin(4\,\theta_2)\ ,$$

$$
\begin{aligned}
G_4 \quad = \quad & \theta_2 + t + \epsilon \quad \frac{1}{2}\,\delta_2 t + \frac{3}{8}\,\frac{b_{1,0} r_1 \cos(3\,\theta_1 + \theta_2)}{r_2} + \frac{3}{4}\,\frac{b_{1,0} r_1 \cos(3\,\theta_1 - \theta_2)}{r_2} + \frac{9}{2}\,\frac{b_{2,0} r_1^2 \sin(\theta_2)}{r_2} \\
& - \frac{9}{20}\,\frac{b_{2,0} r_1^2 \sin(-\theta_2 + 6\,\theta_1)}{r_2} - \frac{9}{28}\,\frac{b_{2,0} r_1^2 \sin(\theta_2 + 6\,\theta_1)}{r_2} - \frac{3}{4}\,b_{1,1} r_1 \sin(3\,\theta_1 - 2\,\theta_2) \\
& + \frac{3}{20}\,b_{1,1} r_1 \sin(3\,\theta_1 + 2\,\theta_2) - \frac{1}{4}\,b_{0,2} r_2 \sin(\theta_2) + \frac{1}{12}\,b_{0,2} r_2 \sin(3\,\theta_2) + \frac{1}{8}\,b_{0,3} r_2^2 \cos(2\,\theta_2) \\
& - \frac{1}{32}\,b_{0,3} r_2^2 \cos(4\,\theta_2) - \frac{1}{4}\,\delta_2 \sin(2\,\theta_2) - \frac{1}{4}\,b_{0,1} \cos(2\,\theta_2) \quad .
\end{aligned}
$$

(C.3)

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