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On the Transversal Vibrations of a Conveyor Belt with a Low and Time-Varying Velocity. Part 1: The String-like Case.

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On The Transversal Vibrations of A Conveyor Belt with A Low and Time-Varying Velocity. Part I: The String-like Case.

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Abstract

In this paper initial-boundary value problems for a linear wave (string) equation are considered. These problems can be used as simple models to describe the vertical vibrations of a conveyor belt, for which the velocity is small with respect to the wave speed. In this paper the belt is assumed to move with a time-varying speed. Formal asymptotic approximations of the solutions are constructed to show the complicated dynamical behavior of the conveyor belt. It also will be shown that the truncation method can not be applied to this problem in order to obtain approximations valid on long time scales.

1 Introduction

Investigating transverse vibrations of a belt system is a challenging subject which has been studied for many years (see [1-4] for an overview) and is still of interest today.

The main purpose of studying the dynamic behavior of a belt system is to know the *natural frequencies* of the vibrations. By knowing these natural frequencies, the so-called resonance-free belt system can be designed (see [3]). Resonances that can cause severe vibrations can be initiated by some parts of the belt system, such as the varying belt speed, the roll eccentricities, and other belt imperfections. The occurrence of resonances should be prevented since they can cause operational and maintenance problems including excessive wear of the belt and the support component, and the increase of energy consumption of the system.

Belt vibrations can be classified into two types, i.e. whether it is of a string-like type or of beam-like type, depending on the bending stiffness of the belt. If the bending stiffness can be neglected then the system is classified as string (wave)-like, otherwise it is classified as beam-like. The transverse vibrations of the belt system may be described as:

*TU Delft etc.

- string-like by
$$u_{tt} + 2Vu_{xt} + V_t u_x + (\kappa V^2 - c^2)u_{xx} = 0, \text{and} \tag{1}$$
- beam-like (with a string effect) by
$$u_{tt} + 2Vu_{xt} + V_t u_x + (\kappa V^2 - c^2)u_{xx} + \frac{E_b I_y}{\rho A} u_{xxxx} = 0, \tag{2}$$

where:

$u(x,t)$	: the displacement of the belt in the y (vertical) direction,
V	: the time-varying belt speed,
c	: the wave speed,
E_b	: Young's modulus,
I_y	: the moment of inertia with respect to the x (horizontal) axis,
ρ	: the mass density of the belt,
A	: the area of the cross section of the belt,
κ	: a constant representing the relative stiffness of the belt. Its value is in $[0,1]$,
x	: coordinate in horizontal direction, and
t	: time.

The beam-like system with a low time-varying speed will be considered in the forth coming paper [5]. In this paper we will study the string-like case where the belt velocity $V(t)$ is given by
$$V(t) = \epsilon(V_0 + \alpha \sin(\Omega t)), \tag{3}$$
where ϵ is a small parameter with $0 < \epsilon \ll 1$,and V_0 and α are constants with $V_0 > 0$ and $V_0 > |\alpha|$. The velocity variation frequency of the belt is given by Ω. In fact the small parameter ϵ indicates that the belt speed $V(t)$ is small compared to the wave speed c. The condition $V_0 > |\alpha|$ guarantees that the belt will always move forward in one direction. It will turn out that certain values of Ω can lead to complicated internal resonances of the belt system.

While for more accurate results, a non-linear model is required, it is not meaningless to investigate first a linear model. Knowledge about linear models is important in order to understand results found in non-linear models, especially for those cases which are weakly non-linear. For non-linear models describing the dynamic behavior of belts, we refer the readers to [4], [6], and [7]. In [7] the role played by the external frequency of the non-constant belt velocity and the bending stiffness is studied. It is found that, as the bending stiffness tends to zero, the system behaves more like a string and its dynamics becomes more complicated than the beam-like system.

Most belt studies involve mainly belts moving with a constant velocity. Recently in a series of papers [8-11] several authors considered the vibrations of belts moving with time-dependent velocities and the vibrations of tensioned pipes conveying fluid with time-dependent velocities. In fact in [8-11] the equations (1) or (2) have been studied, where $V(t)$ as given by (3) belongs to cases that have been studied in [8-11]. To find approximations of the displacement of the belt in vertical direction the authors use in [8-11] the method eigenfunction expansions, the Galerkin truncation method, and the multiple-time-scales perturbation method as for instance described in [12,13]. To apply the method of eigenfunction expansions, special attention has to be paid to terms involving u_x and u_{xt} in (1) or (2). To apply the truncation method the internal resonances between the vibrations modes have to be studied. In [8-11] the terms in (1) or (2) involving u_x and u_{xt} are not treated correctly, and it is assumed in [8-11] that truncation to one mode (or a few modes) is allowed. In this paper we will show that truncation is not allowed. In [8,10] no instabilities of the belt system (as described by (1)) were found using the truncation method when the velocity variation frequency Ω is equal to or close to the difference of two natural frequencies of the constant velocity system. In this paper it will be shown that also instabilities can occur when Ω is equal to or close to the difference of two natural frequencies of the constant velocity system. In [4] and in [14-18] several remarks can be found on how and when truncation is allowed. In those papers weakly nonlinear problems for wave and for beam equations have been studied.

In this paper we consider the vibrations of a belt modeled by a string moving with a non-constant velocity $V(t) = \epsilon(V_0 + \alpha \sin \Omega t)$, where V_0, α, and Ω are constants with $V_0 > |\alpha|$. The velocity $V(t)$ can be considered as a periodically changing velocity such that the belt still moves in one direction. This variation in $V(t)$ can be considered as some kind of an excitation. In relation to excitations, some results in this area have been obtained in [19] and in [20]. In [19] problems for a string moving with a constant velocity are considered for which one of its ends (i.e. $x = L$) is subjected to an harmonic excitation. In [21], the vibrations of the string at $x = L$ is forced to be $v(x,t) = v_0 \cos \Omega t$. In [21] the author also studied the case where one end of the moving string is subjected to an harmonic excitation to represent the case of a belt traveling from an eccentric pulley to a smooth pulley. Whereas the case where both ends of the string are excited is studied in [22]. In that paper a moving string model is used to study the transverse vibrations of power transmission chains. In all of these papers[19-22], the belt velocity is assumed to be constant.

This paper is organized as follows. In section 2, an equation to describe the transversal vibrations of a belt (which is modeled as a string) is derived. Here we assume that the belt moves with an arbitrary low velocity which is varied harmonically, i.e. $V(t) = \epsilon(V_0 + \alpha \sin \Omega t)$. In section 3 we study the energy and the boundedness of the solution of the problem as derived in section 2. In section 4 we discuss the application of the two time-scales perturbation method to solve the equation. It turns out that there are infinitely many values of Ω that can cause internal resonances. In this paper we only investigate the resonance case $\Omega = \frac{c\pi}{L}$. All other resonance cases can be studied similarly. In this section it will also be shown that the truncation method can not be applied to this problem due to the distribution of energy among all vibration modes. In the last part of section 4 we also study a detuning case for the value $\Omega = \frac{c\pi}{L}$. Finally, in section 5 some remarks will be made and some conclusions will be drawn.

2 A string model

In this section the dynamic behavior of a conveyor belt which is modeled by a moving string is studied. Since the belt is assumed to move with a speed $V(t)$ (which explicitly

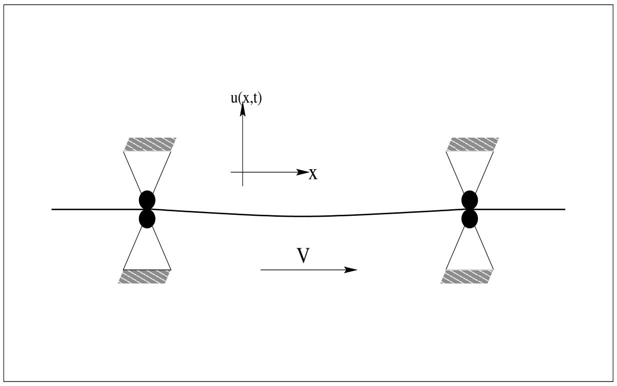

Figure 1: Conveyor belt system

depends on t) we obtain for the time-derivative of the transversal displacement $u(x,t)$ of the belt

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = \frac{\partial u}{\partial t} + V(t)\frac{\partial u}{\partial x}, \tag{4}$$

and for the second order derivative with respect to time

$$\frac{D^2 u}{Dt^2} = u_{tt} + 2V u_{xt} + V^2 u_{xx} + V_t u_x. \tag{5}$$

Accordingly, we have the following equation of motion

$$\begin{aligned} T_0 u_{xx} &= \rho \frac{D^2 u}{Dt^2}, \\ c^2 u_{xx} &= u_{tt} + 2V u_{xt} + V^2 u_{xx} + V_t u_x, \end{aligned} \tag{6}$$

where $c = \sqrt{\frac{T_0}{\rho}}$, in which T_0 and ρ are assumed to be the constant tension and the constant mass-density of the beam, respectively. At $x = 0$ and $x = L$ we assume that the string is fixed in vertical direction, where L is the distance between the pulleys.

For $V(t)$ we use $V(t) = \epsilon(V_0 + \alpha \sin \Omega t)$ with $V_0 > 0$ and $V_0 > |\alpha|$. This low velocity should be interpreted as low compared to the wave speed c of the belt. The condition $V_0 > |\alpha|$ guarantees that the belt will always move forward in one direction. Consequently (6) becomes:

$$\begin{aligned} c^2 u_{xx} - u_{tt} &= \epsilon\left[\alpha\Omega\cos(\Omega t)u_x + 2(V_0 + \alpha\sin(\Omega t))u_{xt}\right] + \\ &\quad \epsilon^2[V_0 + \alpha\sin(\Omega t)]^2 u_{xx}, \end{aligned} \tag{7}$$

where the boundary and initial conditions are given by

$$\begin{aligned} u(0,t;\epsilon) &= u(L,t;\epsilon)=0, \\ u(x,0;\epsilon) &= f(x) \text{ and } u_t(x,0;\epsilon)=g(x), \end{aligned} \tag{8}$$

where $f(x)$ and $g(x)$ represent the initial displacement and the initial velocity of the belt, respectively. Throughout this paper it is assumed that f and g are sufficiently smooth such that a two times continuously differentiable solution for the initial-boundary value problem (7) - (8) exists. Moreover, it is assumed that all series representations for the solution u (and its derivatives), and for the functions f and g are convergent.

To satisfy the boundary conditions all functions should be expanded in Fourier- sin-series. So the solution is of the form $u(x,t;\epsilon)=\sum_{n=1}^{\infty}u_n(t;\epsilon)\sin(\frac{n\pi x}{L})$. This is an odd function in x, both with regard to $x=0$ and $x=L$. All functions in the right hand side of (7) should be extended properly to make them odd with respect to $x=0$ and $x=L$, and periodic with period $2L$ thereof. Note that this extention or expansion process is not applied in [8-10] causing the occurence of incorrect results in the critical values of Ω.

To make the right hand side of (7) odd, terms which are not already in Fourier-sin-series form in x are multiplied with (see also [14,17]):

$$\mathcal{H}(x)=\left\{\begin{array}{rccccc} 1 & if & 0 & < & x & < & L \\ -1 & if & -L & < & x & < & 0 \end{array}\right. =\sum_{j=0}^{\infty}\frac{4}{(2j+1)\pi}\sin\Big(\frac{(2j+1)\pi x}{L}\Big). \tag{9}$$

Substituting (9) into (7) results in

$$\begin{aligned} c^2u_{xx}-u_{tt} &= \epsilon\sum_{j=0}^{\infty}\frac{4}{(2j+1)\pi}\sin\Big(\frac{(2j+1)\pi x}{L}\Big)\,[\alpha\Omega\cos(\Omega t)u_x+ \\ &\qquad 2(V_0+\alpha\sin(\Omega t))u_{xt}]+\epsilon^2(V_0+\alpha\sin(\Omega t))^2u_{xx}. \end{aligned} \tag{10}$$

Substitution of $u(x,t)=\sum_{n=1}^{\infty}u_n(t;\epsilon)\sin(\frac{n\pi x}{L})$ into (10) results in:

$$\begin{aligned} &\sum_{n=1}^{\infty}\left(-\left(\frac{cn\pi}{L}\right)^2u_n-\ddot{u}_n\right)\sin\Big(\frac{n\pi x}{L}\Big)=\epsilon\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\frac{4}{(2j+1)\pi}\sin\Big(\frac{(2j+1)\pi x}{L}\Big) \\ &\quad\left(\alpha\Omega\cos(\Omega t)\frac{n\pi}{L}u_n\cos\Big(\frac{n\pi x}{L}\Big)+2\left(V_0+\alpha\sin(\Omega t)\right)\frac{n\pi}{L}\dot{u}_n\cos\Big(\frac{n\pi x}{L}\Big)\right)- \\ &\quad\epsilon^2\sum_{n=1}^{\infty}(V_0+\alpha\sin\Omega t)^2\left(\frac{n\pi}{L}\right)^2u_n\sin\Big(\frac{n\pi x}{L}\Big). \end{aligned} \tag{11}$$

By multiplying (11) with $\sin(\frac{k\pi x}{L})$, and by integrating the so-obtained equation with respect to x from $x=-L$ to $x=L$, we obtain:

$$\begin{aligned} \ddot{u}_k+\left(\frac{ck\pi}{L}\right)^2u_k &= \epsilon\left[\sum\nolimits_1-\sum\nolimits_2-\sum\nolimits_3\right]\frac{2n}{(2j+1)L}\,[\alpha\Omega\cos(\Omega t)u_n+ \\ &\quad 2(V_0+\alpha\sin(\Omega t))\dot{u}_n]+\epsilon^2(V_0+\alpha\sin(\Omega t))^2\left(\frac{k\pi}{L}\right)^2u_k, \end{aligned} \tag{12}$$

where $\sum_1=\sum_{k=n-(2j+1)}, \sum_2=\sum_{k=2j+1+n}$, and $\sum_3=\sum_{k=2j+1-n}$. Equation (12) will be studied further in section 4.

3 Energy and boundedness of the solution

We are going to use the concept of energy in many parts of the next sections. In this section we shall derive the energy of the moving string as modeled by the wave equation

$$c^2 u_{xx} = u_{tt} + 2V u_{xt} + V^2 u_{xx} + V_t u_x. \tag{13}$$

By multiplying (13) with $(u_t + V u_x)$ we obtain after some elementary calculations

$$\begin{aligned}&(\frac{1}{2}u_t^2 + u_t V u_x + \frac{1}{2}c^2 u_x^2 + \frac{1}{2}V^2 u_x^2)_t + \\ &\quad(-c^2 u_x u_t - \frac{1}{2}c^2 V u_x^2 + V u_t^2 + V^2 u_x u_t + \frac{1}{2}V^3 u_x^2 - \frac{1}{2}V u_t)_x = 0.\end{aligned} \tag{14}$$

Integrating (14) with respect to x from $x = 0$ to $x = L$, and then integrating the so-obtained equation with respect to t from $t = 0$ to t, we obtain:

$$\int_0^L (\frac{1}{2}u_t^2 + V u_t u_x + \frac{1}{2}(c^2 + V^2)u_x^2)|_{t=0}^t dx = \frac{1}{2}\int_0^t (c^2 - V^2) V u_x^2|_{x=0}^L dt. \tag{15}$$

The energy $E(t)$ of the moving string is now defined to be:

$$E(t) = \frac{1}{2}\int_0^L ((u_t + V u_x)^2 + c^2 u_x^2) dx. \tag{16}$$

So, (15) can be written as

$$\begin{aligned} E(t) - E(0) &= \frac{1}{2}\int_0^t (c^2 - V^2) V u_x^2|_{x=0}^L dt \\ \Leftrightarrow \frac{dE}{dt} &= \frac{1}{2}(c^2 - V^2) V \left(u_x^2(L,t) - u_x^2(0,t)\right), \\ &\leq MV, \end{aligned} \tag{17}$$

where M is the maximum of $\frac{1}{2}(c^2 - V^2)\left(u_x^2(L,t) - u_x^2(0,t)\right)$, where we have assumed that $u(x,t)$ is two times continuously differentiable on $0 \leq x \leq L$ and $0 \leq t \leq T\epsilon^{-1}$ for some positive constant $T < \infty$. It follows from (17) that $\frac{dE}{dt} \leq \mathcal{O}(\epsilon)$ on $0 \leq t \leq T\epsilon^{-1}$ since V is $\mathcal{O}(\epsilon)$. And so, $E(t) - E(0) \leq \mathcal{O}(\epsilon t)$ on $0 \leq t \leq T\epsilon^{-1}$. The following estimate on $u(x,t)$ then also holds

$$\begin{aligned} |u(x,t)| &= |\int_0^x u_x(x,t) dx| \leq \int_0^x |u_x(x,t)| dx \\ &\leq \int_0^L |u_x(x,t)| dx \\ &\leq \sqrt{\int_0^L 1^2 dx}\sqrt{\int_0^L 2 \cdot \frac{1}{2}(c^2 u_x^2 + (u_t + V u_x)^2) dx} \\ &= \sqrt{L}\sqrt{2E(t)}, \end{aligned} \tag{18}$$

on $0 \leq t \leq T\epsilon^{-1}$. We refer to [23] for more detailed descriptions of energetics of translating continua.

4 Application of the two time-scales perturbation method

Consider again equation (12). The application of a straight-forward expansion method to solve (12) will result in the occurrence of so-called secular terms which causes the approximations to become unbounded on long time-scales. To remove those secular terms, we introduce two *time-scales* $t_0 = t$ and $t_1 = \epsilon t$. The introduction of these two time-scales defines the following transformations:

$$\begin{aligned} u_k(t;\epsilon) &= w_k(t_0, t_1; \epsilon), \\ \frac{du_k(t;\epsilon)}{dt} &= \frac{\partial w_k}{\partial t_0} + \epsilon \frac{\partial w_k}{\partial t_1}, \\ \frac{d^2 u_k(t;\epsilon)}{dt^2} &= \frac{\partial^2 w_k}{\partial t_0^2} + 2\epsilon \frac{\partial^2 w_k}{\partial t_0 \partial t_1} + \epsilon^2 \frac{\partial^2 w_k}{\partial t_1^2}. \end{aligned} \tag{19}$$

By substituting (19) into (12) we obtain:

$$\begin{aligned} &\frac{\partial^2 w_k}{\partial t_0^2} + 2\epsilon \frac{\partial^2 w_k}{\partial t_0 \partial t_1} + \epsilon^2 \frac{\partial^2 w_k}{\partial t_1^2} + \left(\frac{ck\pi}{L}\right)^2 w_k = \\ &\quad \epsilon \left[\sum\nolimits_1 - \sum\nolimits_2 - \sum\nolimits_3\right] \frac{2n}{(2j+1)L} (\alpha\Omega \cos(\Omega t) w_n + 2[V_0 + \alpha \sin(\Omega t) \frac{\partial w_n}{\partial t_0}]) + \\ &\qquad \mathcal{O}(\epsilon^2). \end{aligned} \tag{20}$$

Assuming that $w_k(t_0, t_1; \epsilon) = w_{k0}(t_0, t_1) + \epsilon w_{k1}(t_0, t_1) + \epsilon^2 w_{k2}(t_0, t_1) + \ldots$, then in order to remove the secular terms up to $\mathcal{O}(\epsilon)$, we have to solve the following problems:

$$\begin{aligned} \mathcal{O}(1) &: \frac{\partial^2 w_{k0}}{\partial t_0^2} + \left(\frac{ck\pi}{L}\right)^2 w_{k0} = 0, \\ \mathcal{O}(\epsilon) &: \frac{\partial^2 w_{k1}}{\partial t_0^2} + \left(\frac{ck\pi}{L}\right)^2 w_{k1} = -2\frac{\partial^2 w_{k0}}{\partial t_0 \partial t_1} + \left[\sum\nolimits_1 - \sum\nolimits_2 - \sum\nolimits_3\right] \frac{2n}{(2j+1)L} \\ &\qquad \left(\alpha\Omega \cos(\Omega t) w_{n0} + 2(V_0 + \alpha \sin(\Omega t)) \frac{\partial w_{n0}}{\partial t_0}\right). \end{aligned}$$

The $\mathcal{O}(1)$ problem has as solution

$$w_{k0}(t_0, t_1) = A_{k0}(t_1) \cos\left(\frac{ck\pi t_0}{L}\right) + B_{k0}(t_1) \sin\left(\frac{ck\pi t_0}{L}\right), \tag{21}$$

where A_{k0} and B_{k0} are still arbitrary functions that can be used to avoid secular terms in the solution of the $\mathcal{O}(\epsilon)$-problem.

From the $\mathcal{O}(\epsilon)$ problem it can readily be seen that there are infinitely many values of Ω that can cause internal resonance. In fact these values are $(n+k)\frac{c\pi}{L}, (n-k)\frac{c\pi}{L}, (k-n)\frac{c\pi}{L}$, and $-(n+k)\frac{c\pi}{L}$, where $k = n - 2j - 1$, or $k = 2j + 1 - n$, or $k = n + 2j + 1$ (see also the summations in (12)). It is also easy to see that these values for Ω are always odd multiples of $\frac{c\pi}{L}$ (or are in an $\mathcal{O}(\epsilon)$-neighbourhood of these odd multiples). In [8] and [10] the critical values of Ω are found to be even multiples of the natural frequency. These incorrect results in [8] and [10] are due to the fact that certain terms in the PDE (that is, terms involving u_x and u_{xt} in (7)) are not extended or expanded correctly.

To show how the secular terms can be eliminated we will consider three cases: $\Omega = \frac{c\pi}{L}, \Omega = \frac{c\pi}{L} + \epsilon\delta$, and the case that Ω is not in a neighborhood of an odd multiple of $\Omega = \frac{c\pi}{L}$.

4.1 Case 1: $\Omega = \frac{c\pi}{L}$.

In appendix 1 it has been shown for $\Omega = \frac{c\pi}{L}$ what equations $A_{k0}(t_1)$ and $B_{k0}(t_1)$ have to satisfy such that the approximations of the solution of the problem do not contain secular terms. It turns out that A_{k0} and B_{k0} have to satisfy:

$$\begin{aligned} \frac{dB_{k0}}{d\bar{t}_1} &= -(k+1)A_{(k+1)0} - (k-1)A_{(k-1)0}, \\ \frac{dA_{k0}}{d\bar{t}_1} &= (k+1)B_{(k+1)0} + (k-1)B_{(k-1)0}, \end{aligned} \tag{22}$$

where $\bar{t}_1 = \frac{\alpha}{L}t_1$, and $k = 1, 2, 3, \ldots$. For $\Omega = m\frac{c\pi}{L}$ where m is odd the same analysis as presented in appendix 1 can be followed. It then follows that A_{k0} and B_{k0} have to satisfy ($k = 1, 2, 3, \ldots$):

$$\begin{aligned} \frac{dA_{k0}}{d\bar{t}_1} &= \frac{(k+m)(2k+2m-1)}{m(2k+m)}B_{(k+m)0} + \frac{(k-m)(2k-2m+1)}{m(2k-m)}B_{(k-m)0}, \\ \frac{dB_{k0}}{d\bar{t}_1} &= -\frac{(k+m)(2k+2m-1)}{m(2k+m)}A_{(k+m)0} - \frac{(k-m)(2k-2m+1)}{m(2k-m)}A_{(k-m)0}. \end{aligned}$$

It should be noticed that for $m = 1$ this system of ordinary differential equations is reduced to system (22). In this section we will study system (22), which is a coupled system of infinitely many ordinary differential equations.

4.1.1 Application of the truncation method

First we will try to find an approximation of the solution of system (22) by using Galerkin's truncation method. So, we will use just some first few modes and neglect the higher order modes. For example, in the case we consider the first 3 modes, we obtain from (22):

$$\dot{X} = AX, \tag{23}$$

where: $X = \begin{pmatrix} B_{10} \\ A_{10} \\ B_{20} \\ A_{20} \\ B_{30} \\ A_{30} \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -3 \\ 1 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}$,

and where $\dot{X}$ represents the derivative of X with respect to $\bar{t}_1$. This system has eigenvalues $2\sqrt{2}i, -2\sqrt{2}i$, and 0, all with multiplicity 2. Their associated eigenvectors are: $(0, 1, \sqrt{2}i, 0, 0, 1), (1, 0, 0, -\sqrt{2}i, 1, 0), (1, 0, 0, \sqrt{2}i, 1, 0), (0, 1, -\sqrt{2}i, 0, 0, 1), (-3, 0, 0, 0, 1, 0)$ and $(0, -3, 0, 0, 0, 1)$, respectively. The solution of (23) is then given by:

$$
\begin{aligned}
B_{10}(t_1) &= C_3 \cos(2\sqrt{2}t_1) + C_4 \sin(2\sqrt{2}t_1) - 3C_5, \\
A_{10}(t_1) &= C_1 \cos(2\sqrt{2}t_1) + C_2 \sin(2\sqrt{2}t_1) - 3C_6, \\
B_{20}(t_1) &= -\sqrt{2}C_1 \sin(2\sqrt{2}t_1) + \sqrt{2}C_2 \cos(2\sqrt{2}t_1) - \sqrt{2}C_4 \cos(2\sqrt{2}t_1), \\
A_{20}(t_1) &= \sqrt{2}C_3 \sin(2\sqrt{2}t_1) - \sqrt{2}C_4 \cos(2\sqrt{2}t_1), \\
B_{30}(t_1) &= C_3 \cos(2\sqrt{2}t_1) + C_4 \sin(2\sqrt{2}t_1) + C_5, \\
A_{30}(t_1) &= C_1 \cos(2\sqrt{2}t_1) + C_2 \sin(2\sqrt{2}t_1) + C_6,
\end{aligned}
\tag{24}
$$

where $C_1, C_2, \ldots, C_6$ are all constants of integration. Note that we have dropped all the bars in (24).

From the initial conditions (8), that is, $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$ it follows that

$$
\begin{aligned}
& f(x) = \sum_{k=1}^{\infty} u_k(0; \epsilon) \sin\Big(\frac{k\pi x}{L}\Big) \\
\Leftrightarrow \quad & u_k(0; \epsilon) = \frac{2}{L} \int_0^L f(x) \sin\Big(\frac{k\pi x}{L}\Big) dx, \\
& g(x) = \sum_{k=1}^{\infty} \dot{u}_k(0; \epsilon) \sin\Big(\frac{k\pi x}{L}\Big) \\
\Leftrightarrow \quad & \dot{u}_k(0; \epsilon) = \frac{2}{L} \int_0^L g(x) \sin\Big(\frac{k\pi x}{L}\Big) dx.
\end{aligned}
\tag{25}
$$

Moreover, since $u_k(0; \epsilon) = w_k(0, 0; \epsilon) = w_{k0}(0, 0) + \epsilon w_{k1}(0, 0) + \ldots$ and $\dot{u}_k(0; \epsilon) = \dot{w}_k(0, 0; \epsilon) = \dot{w}_{k0}(0, 0) + \epsilon \dot{w}_{k1}(0, 0) + \ldots$ it follows that

$$
\begin{aligned}
w_{k0}(0, 0) &= \frac{2}{L} \int_0^L f(x) \sin\Big(\frac{k\pi x}{L}\Big) dx, \\
\dot{w}_{k0}(0, 0) &= \frac{2}{L} \int_0^L g(x) \sin\Big(\frac{k\pi x}{L}\Big) dx.
\end{aligned}
\tag{26}
$$

From (21) and (26) we then obtain

$$
A_{k0}(0) = \frac{2}{L} \int_0^L f(x) \sin\Big(\frac{k\pi x}{L}\Big) dx, \text{and}
$$

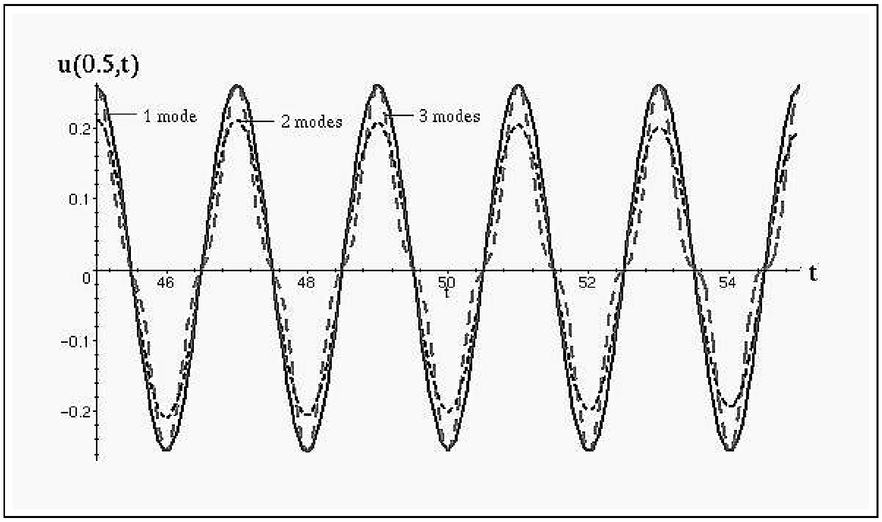

Figure 2: Approximations for $u(x,t)$ with initial displacement $f(x) = \frac{-8}{\pi^3}\sin(\pi x)$ and initial velocity $g(x) = 0$. The graphs are given for $x = 0.5, t \in [45, 55]$, and $\epsilon = 0.01$.

$$B_{k0}(0) = \frac{2}{ck\pi}\int_0^L g(x)\sin\Big(\frac{k\pi x}{L}\Big)dx. \tag{27}$$

Equation (27) can be used to calculate the constants in (24).

In summary, after all constants in (24) have been calculated, $w_{k0}(t_0, t_1)$ can be determined using (21). Then $u(x,t;\epsilon)$ can be approximated by $\sum_{k=1}^{3} u_k(t;\epsilon)$ $\sin(\frac{k\pi x}{L})$.

For example, using 1, 2, or 3 modes, respectively, with $f(x) = \frac{-8}{\pi^3}\sin(\pi x)$, $g(x) = 0, c = L = 1$ we find as approximations for $u(x,t;\epsilon)$:

$$\begin{aligned}
u(x,t;\epsilon) &\approx \frac{-8}{\pi^3}\cos(\pi t_0)\sin(\pi x), \\
u(x,t;\epsilon) &\approx \frac{-8}{\pi^3}\cos(\sqrt{2}t_1)\cos(\pi t_0)\sin(\pi x) + \frac{4\sqrt{2}}{\pi^3}\sin(\sqrt{2}t_1)\sin(2\pi t_0)\sin(2\pi x), \\
u(x,t;\epsilon) &\approx (-\frac{2}{\pi^3}\cos(2\sqrt{2}t_1) - \frac{6}{\pi^3})\cos(\pi t_0)\sin(\pi x) + \frac{2\sqrt{2}}{\pi^3}\sin(2\sqrt{2}t_1)\sin(2\pi t_0) \\
&\qquad \sin(2\pi x) + (\frac{-2}{\pi^3}\cos(2\sqrt{2}t_1) + \frac{2}{\pi^3})\cos(3\pi t_0)\sin(3\pi x).
\end{aligned} \tag{28}$$

The graphs of these approximations for $u(x,t)$ for $x = 0.5$ and $\epsilon = 0.01$ are depicted in Figure 2.

For more than three modes, eigenvalues and eigenvectors become more and more difficult to compute by just using pencil and paper. Using the computer software package Maple , the eigenvalues of system (22) have been computed up to 20 modes and are listed in Table 1. From the table, it can be seen that the eigenvalues of the truncated system are always purely imaginary, each has multiplicity two, and for an odd numbers of modes we get an additional pair of zero eigenvalues. From the approximations (28) and from table

1 it can readily be seen that the truncation method will not give accurate results on long time-scales, that is, on time-scales of order ϵ^{-1}.

4.1.2 Analysis of the infinite dimensional system (22)

In the previous subsection we found that if system (22) is truncated then the eigenvalues of the truncated system are always purely imaginary or zero. In this section we shall show that the results obtained by applying the truncation method are not valid on time-scales of order ϵ^{-1}.

By putting $kB_{k0}(t_1) = Y_{k0}(t_1)$ and $kA_{k0}(t_1) = X_{k0}(t_1)$, system (22) becomes:

$$\begin{aligned} \frac{dY_{k0}}{dt_1} &= k[-X_{(k+1)0} - X_{(k-1)0}], \\ \frac{dX_{k0}}{dt_1} &= k[Y_{(k+1)0} + Y_{(k-1)0}], \end{aligned} \tag{29}$$

for $k = 1, 2, 3, \ldots$, and $X_{00} = Y_{00} = 0$.

Accordingly we also have:

$$\begin{aligned} Y_{k0}\dot{Y}_{k0} &= -k[Y_{k0}X_{(k+1)0} + Y_{k0}X_{(k-1)0}], \\ X_{k0}\dot{X}_{k0} &= k[X_{k0}Y_{(k+1)0} + X_{k0}Y_{(k-1)0}]. \end{aligned} \tag{30}$$

By adding both equations in (30), and then by taking the sum from $k = 1$ to ∞ we obtain:

$$\frac{1}{2}\sum_{k=1}^{\infty}\frac{d}{dt_1}(Y_{k0}^2 + X_{k0}^2) = \sum_{k=1}^{\infty}[X_{(k+1)0}Y_{k0} - Y_{(k+1)0}X_{k0}]. \tag{31}$$

By differentiating (31) with respect to t_1 we find (see also appendix 2)

$$\frac{1}{2}\sum_{k=1}^{\infty}\frac{d^2}{dt_1^2}(Y_{k0}^2 + X_{k0}^2) = 2\sum_{k=1}^{\infty}(X_{k0}^2 + Y_{k0}^2), \tag{32}$$

and so, by putting $\sum_{k=1}^{\infty}(X_{k0}^2 + Y_{k0}^2) = W(t_1)$ we finally obtain:

$$\frac{d^2W(t_1)}{dt_1^2} - 4W(t_1) = 0. \tag{33}$$

The solution of (33) is $W(t_1) = K_1e^{2t_1} + K_2e^{-2t_1}$, where K_1 and K_2 are constants. Note that $W(t_1)$ is a first integral of system (22). K_1 and K_2 are both positive numbers as is shown in the following calculation. From $W(t_1) = \sum_{k=1}^{\infty}[X_{k0}^2 + Y_{k0}^2]$ it follows that

$$W(0) = \sum_{k=1}^{\infty}[X_{k0}^2(0) + Y_{k0}^2(0)] \geq 0 \Rightarrow K_1 + K_2 \geq 0 \tag{34}$$

Differentiating $W(t_1)$ with respect to t_1 and then putting $t_1 = 0$ we get:

$$K_1 - K_2 = \sum_{k=1}^{\infty}[Y_{k0}(0)X_{(k+1)0}(0) - X_{k0}(0)Y_{(k+1)0}(0)]. \tag{35}$$

From (34) and (35) it then follows that

$$
\begin{aligned}
2K_1 &= \sum_{k=1}^{\infty}[X_{k0}^2(0) + Y_{k0}^2(0) + Y_{k0}(0)X_{(k+1)0}(0) - X_{k0}(0)Y_{(k+1)0}(0)] \\
&= \frac{1}{2}X_{10}^2(0) + \frac{1}{2}Y_{10}^2(0) + \frac{1}{2}(X_{10}(0) - Y_{20}(0))^2 + \frac{1}{2}(Y_{10}(0) + X_{20}(0))^2 + \\
&\quad \frac{1}{2}(X_{20}(0) - Y_{30}(0))^2 + \frac{1}{2}(Y_{20}(0) + X_{30}(0))^2 + \ldots + \\
&\qquad \frac{1}{2}(X_{n0}(0) - Y_{(n+1)0}(0))^2 + \frac{1}{2}(Y_{n0}(0) + X_{(n+1)0}(0))^2 + \ldots \\
&\geq 0. \qquad (36)
\end{aligned}
$$

So, $K_1 \geq 0$ and 0 if and only if $X_{k0}(0) = Y_{k0}(0) = 0$ for each $k = 1, 2, 3, \ldots$. Using a similar method, K_2 also can be shown to be a non-negative number. Consequently, $W(t_1)$ is, in general, non-negative and increases as t_1 increases. This behavior is different from the behavior of $A_{k0}(t_1)$ and $B_{k0}(t_1)$ as obtained by applying the truncation method. If we apply the truncation method, we merely obtain sin and cos functions for A_{k0} and B_{k0} while the energy (see next subsection) is described by exponential functions. This means that the approximations obtained by applying the truncation method to system (22) are not accurate on long time-scales, that is, on time-scales of order ϵ^{-1}.

4.1.3 The energy

The energy $E(t)$ of the conveyor belt system can also be approximated using the function $W(t_1)$. Since

$$
\begin{aligned}
u(x,t) &= \sum_{k=1}^{\infty} u_k(t)\sin\left(\frac{k\pi x}{L}\right) \\
&= \sum_{k=1}^{\infty}\left[A_{k0}(t_1)\cos\left(\frac{ck\pi t}{L}\right) + B_{k0}(t_1)\sin\left(\frac{ck\pi t}{L}\right)\right]\sin\left(\frac{k\pi x}{L}\right) + \mathcal{O}(\epsilon) \qquad (37)
\end{aligned}
$$

it follows that the energy $E(t)$ satisfies

$$
\begin{aligned}
E(t) &= \frac{1}{2}\int_0^L \left[(u_t + vu_x)^2 + c^2u_x^2\right]dx \\
&= \frac{c^2\pi^2}{4L}\sum_{k=1}^{\infty} k^2\left[\left(-A_{k0}\sin\left(\frac{k\pi t}{L}\right) + B_{k0}\cos\left(\frac{k\pi t}{L}\right)\right)^2 + \right. \\
&\quad \left.\left(A_{k0}\cos\left(\frac{ck\pi t}{L}\right) + B_{k0}\sin\left(\frac{ck\pi t}{L}\right)\right)^2\right] + \mathcal{O}(\epsilon) \\
&= \frac{c^2\pi^2}{4L}\sum_{k=1}^{\infty}\left[(kA_{k0})^2 + (kB_{k0})^2\right] + \mathcal{O}(\epsilon) \\
&= \frac{c^2\pi^2}{4L}\sum_{k=1}^{\infty}[X_{k0}^2 + Y_{k0}^2] + \mathcal{O}(\epsilon)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{c^2\pi^2}{4L}W(t_1)+\mathcal{O}(\epsilon) && (38)\\
&= \frac{c^2\pi^2}{4L}(K_1e^{2t_1}+K_2e^{-2t_1})+\mathcal{O}(\epsilon). && (39)
\end{aligned}
$$

So, the energy increases, although it is bounded on a time-scale of order $\frac{1}{\epsilon}$.

4.2 Case 2: $\Omega = \frac{c\pi}{L} + \epsilon\delta$

In this section we will consider the detuning from $\Omega = \frac{c\pi}{L}$, that is we will study the case $\Omega = \frac{c\pi}{L} + \epsilon\delta$ where $\delta = \mathcal{O}(1)$. In order to avoid secular terms in the approximation, it can be shown (the calculation are similar to those in section 4.1) that $A_{k0}(t_1)$ and $B_{k0}(t_1)$ have to satisfy:

$$
\begin{aligned}
\frac{dA_{k0}}{d\bar{t}_1} &= (k+1)[B_{(k+1)0}\cos(\delta\bar{t}_1)+A_{(k+1)0}\sin(\delta\bar{t}_1)]+(k-1)[B_{(k-1)0}\cos(\delta\bar{t}_1)-\\
&\quad A_{(k-1)0}\sin(\delta\bar{t}_1)],\\
\frac{dB_{k0}}{d\bar{t}_1} &= -(k+1)[A_{(k+1)0}\cos(\delta\bar{t}_1)-B_{(k+1)0}\sin(\delta\bar{t}_1)]-(k-1)[A_{(k-1)0}\cos(\delta\bar{t}_1)+\\
&\quad B_{(k-1)0}\sin(\delta\bar{t}_1)], \qquad (40)
\end{aligned}
$$

for $k = 1, 2, 3, \ldots$. It should be noticed that for $\delta = 0$ we obtain again system (22). For convenience, we will drop the bar from $\bar{t}_1$.

The calculations as given in section 4.1.2 can be followed again, and we obtain:

$$
\frac{d^2W(t_1)}{dt_1^2}+(\delta^2-4)W(t_1)=D_1\delta^2, \qquad (41)
$$

where $W(t_1)$ is defined as in section 4.1.2, and $D_1 = W(0)$. Elementary calculations then yield:

$$
\begin{aligned}
\text{for } |\delta|<2: W(t_1) &= \frac{D_1}{4-\delta^2}[4\cosh(t_1\sqrt{4-\delta^2})-\delta^2]+\frac{D_2}{\sqrt{4-\delta^2}}\sinh(t_1\sqrt{4-\delta^2}),\\
\text{for } |\delta|=2: W(t_1) &= D_1+D_2t_1+\frac{1}{2}D_1\delta^2t_1^2,\\
\text{for } |\delta|>2: W(t_1) &= \frac{D_1}{\delta^2-4}[\delta^2-4\cos(t_1\sqrt{\delta^2-4})]+\frac{D_2}{\sqrt{\delta^2-4}}\sin(t_1\sqrt{\delta^2-4}),
\end{aligned}
$$

where $D_2 = \frac{dW(0)}{dt_1}$. The interesting features of these solutions are, that for $|\delta| < 2$, $W(t_1)$ (and so the energy) increases exponentially. For $|\delta| = 2, W(t_1)$ increases polynomally, and finally for $|\delta| > 2, W(t_1)$ is bounded due to the trigonometric functions.

4.3 Case 3: The non-resonant case

If Ω is not within an order ϵ-neighborhood of the frequencies that cause internal resonance, that is, not within an order $\epsilon-$neighborhood of $m\frac{c\pi}{L}$ (with m odd) then $A_{k0}(t_1)$ and $B_{k0}(t_1)$ have to satisfy

$$\frac{dA_{k0}}{dt_1} = 0, \qquad \frac{dB_{k0}}{dt_1} = 0, \tag{42}$$

in order to avoid secular terms. Consequently, $A_{k0}(t_1)$ and $B_{k0}(t_1)$ are constants, say $K1_{k0}$ and $K2_{k0}$. So, we have $u_{k0}(t_0,t_1) = K1_{k0}\cos(\frac{ck\pi t_0}{L}) + K2_{k0}\sin(\frac{ck\pi t_0}{L})$. Since $u(x,t) = \sum_{k=1}^{\infty} u_k(t)\sin(\frac{k\pi x}{L})$, where $u_k(t)$ is approximated by $w_{k0}(t_0,t_1)$, it follows from the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ that

$$\begin{aligned} K1_{k0} &= \frac{2}{L}\int_0^L f(x)\sin\Big(\frac{k\pi x}{L}\Big)dx, \quad \text{and} \\ K2_{k0} &= \frac{2}{ck\pi}\int_0^L g(x)\sin\Big(\frac{k\pi x}{L}\Big)dx. \end{aligned} \tag{43}$$

The energy $E(t)$ of the conveyor belt system for this case can be approximated from:

$$u(x,t) \approx \sum_{k=1}^{\infty}\Bigg(K1_{k0}\cos\Big(\frac{ck\pi t_0}{L}\Big) + K2_{k0}\sin\Big(\frac{ck\pi t_0}{L}\Big)\Bigg)\sin\Big(\frac{k\pi x}{L}\Big) + \mathcal{O}(\epsilon), \tag{44}$$

where $K1_{k0}$ and $K2_{k0}$ are given by (43). Then,

$$\begin{aligned} E(t) &= \int_0^L (u_t^2 + c^2u_x^2)dx + \mathcal{O}(\epsilon), \\ &= \sum_{k=1}^{\infty}\frac{(ck\pi)^2}{2L}\Big(K1_{k0}^2 + K2_{k0}^2\Big) + \mathcal{O}(\epsilon), \\ &= \frac{c^2\pi^2}{2L}\sum_{k=1}^{\infty}k^2(K1_{k0}^2 + K2_{k0}^2) + \mathcal{O}(\epsilon). \end{aligned} \tag{45}$$

Using (43), we finally obtain:

$$\begin{aligned} E(t) &= \frac{2c^2L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\Bigg[\int_0^L f''\sin\Big(\frac{k\pi x}{L}\Big)dx\Bigg]^2 + \\ &\qquad \frac{2L^3}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{k^4}\Bigg[\int_0^L g''\sin\Big(\frac{k\pi x}{L}\Big)dx\Bigg]^2 + \mathcal{O}(\epsilon) \\ &= \text{constant} + \mathcal{O}(\epsilon). \end{aligned} \tag{46}$$

5 Conclusions

In this paper we studied initial-boundary value problems which can be used as models to describe transversal vibrations of belt systems. The belt is assumed to move with a non-constant velocity $V(t)$, that is, $V(t) = \epsilon(V_0 + \alpha\sin(\Omega t))$, where $0 < \epsilon \ll 1$ and V_0, α, Ω are constants. Formal approximations of the solution of the initial-boundary value problem have been constructed. Also explicit approximations of the energy of the belt system are given. It turns out that there are infinitely many values of Ω giving rise to internal resonances in the belt system. These values for Ω are $m\frac{c\pi}{L}+\epsilon\delta$ where m is an arbitrary odd integer, $\frac{c\pi}{L}$ is the lowest natural frequency of the constant velocity system, and δ is a detuning parameter of $\mathcal{O}(1)$. For $\Omega=\frac{c\pi}{L}+\epsilon\delta$ (that is, $m=1$) the problem has been studied completely. The following interesting results have been found: for $|\delta|<2$ the energy of the belt system increases exponentially, for $|\delta|=2$ the energy increases polynomally, and for $|\delta|>2$ the energy is bounded and varies trigonometrically. When Ω is not in an order $\epsilon-$neighborhood of $m\frac{c\pi}{L}$ (with m odd) the energy of the belt system is constant up to order ϵ. All the results found are valid on long time-scales, that is, on time-scales of order ϵ^{-1}.

One major conclusion in this paper is that the truncation method can not be applied to obtain asymptotic results on long time-scales (that is, on time-scales of order ϵ^{-1}) when Ω is in an order $\epsilon-$neighborhood of an odd multiple of the lowest natural frequency of the constant velocity system. Moreover, in this paper we improve the (incorrect) results and applied methods as for instance given and used in [8-11].

Appendix 1

To avoid secular terms in the approximation for $u(x,t;\epsilon)$ we will show in this appendix that the function $A_{k0}(t_1)$ and $B_{k0}(t_1)$ have to satisfy:

$$\begin{aligned}\frac{dA_{k0}(t_1)}{dt_1} &= (k+1)B_{(k+1)0}(t_1)+(k-1)B_{(k-1)0}(t_1),\\ \frac{dB_{k0}(t_1)}{dt_1} &= -(k+1)A_{(k+1)0}(t_1)-(k-1)A_{(k-1)0}(t_1)\end{aligned}\tag{A-1}$$

for $k=1,2,3,\ldots$. This can be derived as follows. After introducing a slow and a fast time in section 4, we obtain:

$$\begin{aligned}\mathcal{O}(1) &: \frac{\partial^2 u_{k0}}{\partial t_0^2}+(\frac{ck\pi}{L})^2u_{k0}=0,\\ \mathcal{O}(\epsilon) &: \frac{\partial^2 u_{k1}}{\partial t_0^2}+(\frac{ck\pi}{L})^2u_{k1}=-2\frac{\partial^2 u_{k0}}{\partial t_0\partial t_1}+\left[\sum\nolimits_1-\sum\nolimits_2-\sum\nolimits_3\right]\frac{2n}{(2j+1)L}\\ &\qquad(\alpha\Omega\cos(\Omega t)u_{n0}+2(V_0+\alpha\sin(\Omega t))\frac{\partial u_{n0}}{\partial t_0},\end{aligned}$$

where $\sum_1=\sum_{k=n-(2j+1)}, \sum_2=\sum_{k=n+2j+1}$, and $\sum_3=\sum_{k=2j+1-n}$, and where $\Omega=\frac{c\pi}{L}$. The solution of the $\mathcal{O}(1)$ problem is $u_{k0}(t_0,t_1)=A_{k0}(t_1)\cos(\frac{ck\pi t_0}{L})+B_{k0}(t_1)\sin(\frac{ck\pi t_0}{L})$, where A_{k0} and B_{k0} can be determined from the $\mathcal{O}(\epsilon)$ equation by removing terms in the right hand side of this equation that cause secular terms in $u_{k1}(t_0,t_1)$.

The first term in the right hand side of the $\mathcal{O}(\epsilon)$ equation causing secular terms is $-2\frac{\partial^2 u_{k0}}{\partial t_0\partial t_1}=2\frac{ck\pi}{L}[\frac{dA_{k0}}{dt_1}\sin(\frac{ck\pi t_0}{L})+\frac{dB_{k0}}{dt_1}\cos(\frac{ck\pi t_0}{L})]$.

Taking apart those terms in the second term of the right hand side the $\mathcal{O}(\epsilon)$ equation that cause secular terms, we find:

$$\Big[\sum\nolimits_1 - \sum\nolimits_2 - \sum\nolimits_3\Big] \frac{2n\alpha\Omega}{(2j+1)L}\cos(\Omega t_0)u_{n0} =$$
$$\Big[\sum\nolimits_1 - \sum\nolimits_2 - \sum\nolimits_3\Big] \frac{2n\alpha\Omega}{(2j+1)L}\cos(\Omega t_0)[A_{n0}(t_1)\cos(\frac{cn\pi t_0}{L}) + B_{n0}(t_1)\sin(\frac{cn\pi t_0}{L})]$$
$$= \frac{\alpha c\pi}{L^2}\cos\Big(\frac{ck\pi t_0}{L}\Big)\Big[(k+1)A_{(k+1)0} - (k-1)A_{(k-1)0} - \frac{k+1}{2k+1}A_{(k+1)0} -$$
$$\frac{k-1}{2k-1}A_{(k-1)0}\Big] + \frac{\alpha c\pi}{L^2}\sin\Big(\frac{ck\pi t_0}{L}\Big)\Big[(k+1)B_{(k+1)0} - (k-1)B_{(k-1)0} -$$
$$\frac{k+1}{2k+1}B_{(k+1)0} - \frac{k-1}{2k-1}B_{(k-1)0}\Big] + \text{"terms not giving rise to secular terms in } u_{k1}\text{"}$$

Similarly we find for the third term:

$$\Big[\sum\nolimits_1 - \sum\nolimits_2 - \sum\nolimits_3\Big] \frac{4n}{(2j+1)L}(V_0 + \alpha\sin(\Omega t_0))\frac{\partial u_{n0}}{\partial t_0} =$$
$$\Big[\sum\nolimits_1 - \sum\nolimits_2 - \sum\nolimits_3\Big] \frac{4n}{(2j+1)L}(V_0 + \alpha\sin(\Omega t_0))\frac{cn\pi}{L}\Big[B_{n0}\cos\Big(\frac{cn\pi t_0}{L}\Big) -$$
$$A_{n0}\sin\Big(\frac{cn\pi t_0}{L}\Big)\Big] = \frac{\alpha c\pi}{L^2}\cos\Big(\frac{ck\pi t_0}{L}\Big)\Big[-2(k+1)^2A_{(k+1)0} - 2(k-1)^2A_{(k-1)0} +$$
$$\frac{2(k+1)^2}{2k+1}A_{(k+1)0} - \frac{2(k-1)^2}{2k-1}A_{(k-1)0}\Big] + \frac{\alpha c\pi}{L^2}\sin\Big(\frac{ck\pi t_0}{L}\Big)$$
$$\Big[-2(k+1)^2B_{(k+1)0} - 2(k-1)^2B_{(k-1)0} + \frac{2(k+1)^2}{2k+1}B_{(k+1)0} -$$
$$\frac{2(k-1)^2}{2k-1}B_{(k-1)0}\Big] + \text{"terms not giving rise to secular terms in } u_{k1}\text{"}.$$

Collecting all terms in the right hand side of the $\mathcal{O}(\epsilon)$ equation containing $\cos(\frac{ck\pi t_0}{L})$ and all terms containing $\sin(\frac{ck\pi t_0}{L})$ and then setting their coefficients equal to 0 in order to remove the secular terms, we obtain (A-1).

Appendix 2

In this appendix we will show that:

$$\frac{1}{2}\sum_{k=1}^{\infty}\frac{d^2}{dt_1^2}(Y_{k0}^2 + X_{k0}^2) = 2\sum_{k=1}^{\infty}(X_{k0}^2 + Y_{k0}^2). \tag{A-2}$$

From (4.12) and (4.13) it follows that

$$\begin{aligned}\frac{1}{2}\sum_{k=1}^{\infty}\frac{d}{dt_1}(Y_{k0}^2+X_{k0}^2) &= \sum_{k=1}^{\infty}[Y_{k0}\dot{Y}_{k0}+X_{k0}\dot{X}_{k0}] \\ &= \sum_{k=1}^{\infty}[X_{(k+1)0}Y_{k0}-Y_{(k+1)0}X_{k0}].\end{aligned}$$

Differentiating this expression with respect to t_1, and using (4.11) we find:

$$\begin{aligned}\frac{1}{2}\sum_{k=1}^{\infty}\frac{d^2}{dt_1^2}(Y_{k0}^2+X_{k0}^2) &= \sum_{k=1}^{\infty}[\dot{X}_{(k+1)0}Y_{k0}+X_{(k+1)0}\dot{Y}_{k0}-\dot{Y}_{(k+1)0}X_{k0}-Y_{(k+1)0}\dot{X}_{k0}] \\ &= \sum_{k=1}^{\infty}(k+1)[X_{k0}^2+Y_{k0}^2]-\sum_{m=2}^{\infty}(m-1)[X_{m0}^2+Y_{m0}^2] \\ &= 2(X_{10}^2+Y_{10}^2)+\sum_{k=2}^{\infty}(k+1)[X_{k0}^2+Y_{k0}^2]- \\ &\quad \sum_{m=2}^{\infty}(m-1)[X_{m0}^2+Y_{m0}^2] \\ &= 2(X_{10}^2+Y_{10}^2)+\sum_{k=2}^{\infty}[(k+1)-(k-1)][X_{k0}^2+Y_{k0}^2] \\ &= 2(X_{10}^2+Y_{10}^2)+\sum_{k=2}^{\infty}2[X_{k0}^2+Y_{k0}^2] \\ &= 2\sum_{k=1}^{\infty}[X_{k0}^2+Y_{k0}^2].\end{aligned}$$

And so, (A-2) has been proved.

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No. of modes	Eigenvalues of matrix A (all multiplicity 2)	Dimension eigenspace of A
1	0	2
2	$\pm\sqrt{2}i$	4
3	$0, \pm 2\sqrt{2}i$	6
4	$\pm 1.13i, \pm 4.33i$	8
5	$0, \pm 2.30i, \pm 5.89i$	10
6	$\pm 7.50i, \pm 1.00i, \pm 3.56i$	12
7	$0, \pm 9.15i, \pm 2.05i, \pm 4.90i$	14
8	$\pm 10.83i, \pm 0.93i, \pm 3.18i, \pm 6.30i,$	16
9	$0, \pm 12.54i, \pm 1.89i, \pm 4.38i, \pm 7.74i$	18
10	$\pm 14.26i, \pm 0.87i, \pm 5.65i, \pm 9.23i, \pm 2.93i$	20
11	$0, \pm 16.01i, \pm 1.78i, \pm 4.05i, \pm 6.97i, \pm 10.76i$	22
12	$\pm 17.76i, \pm 0.83i, \pm 2.76i, \pm 5.22i, \pm 8.33i, \pm 12.31i$	24
13	$0, \pm 19.53i, \pm 1.70i, \pm 3.81i, \pm 6.45i, \pm 9.73i, \pm 13.88i, \pm 19.53i$	26
14	$\pm 21.31i, \pm 15.48i, \pm 0.80i, \pm 2.63i, \pm 4.92i, \pm 7.72i, \pm 11.16i$	28
15	$0, \pm 23.11i, \pm 17.10i, \pm 1.64i, \pm 3.63i, \pm 6.07i, \pm 9.03i, \pm 12.63i$	30
16	$\pm 24.91i, \pm 18.73i, \pm 0.78i, \pm 2.53i, \pm 4.68i, \pm 7.28i, \pm 10.38i, \pm 14.11i$	32
17	$0, \pm 26.71i, \pm 20.38i, \pm 1.58i, \pm 3.49i, \pm 5.79i, \pm 8.52i, \pm 11.75i, \pm 15.62i$	34
18	$\pm 28.53i, \pm 22.05i, \pm 0.75i, \pm 2.45i, \pm 4.50i, \pm 6.93i, \pm 9.79i, \pm 13.16i, \pm 17.15i$	36
19	$0, \pm 30.35i, \pm 23.72i, \pm 1.54i, \pm 3.37i, \pm 5.55i, \pm 8.12i, \pm 11.10i, \pm 14.58i, \pm 18.70i$	38
20	$\pm 32.18i, \pm 25.41i, \pm 0.73i, \pm 2.38i, \pm 4.34i, \pm 6.65i, \pm 9.33i, \pm 12.43i, \pm 16.03i, \pm 20.27i$	40

Table 1: **Approximations of the eigenvalues of the truncated system (22).**