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Abstract

In this paper the weakly nonlinear, transversal vibrations of a conveyor belt will be considered. The belt is assumed to move with a low and time-varying speed. Using Kirchhoff's approach a single equation of motion will be derived from a coupled system of partial differential equations describing the longitudinal and transversal vibrations of the belt. A two time-scales perturbation method is then applied to approximate the solutions of the problem. It will turn out that the frequencies of the belt speed fluctuations play an important role in the dynamic behaviour of the belt. It is well-known in linear systems that instabilities can occur if the frequency of the belt speed fluctuations is the sum of two natural frequencies. However, in the weakly nonlinear case as considered in this paper this is no longer true. It turns out that the weak nonlinearity stabilizes the system.

1 Introduction

Axially moving systems are present in a wide class of engineering problems which arise in industrial, civil, aerospatial, mechanical, electronic and automotive applications. Aerial cables, tram-ways, oil pipelines, magnetic tapes, power transmission belts, paper sheet and web processes, fiber winding and band saw blades are examples of cases where an axial transport of mass can be associated with transverse vibrations.

Investigating transverse vibrations of a belt system is a challenging subject which has been studied for many years (see [1] - [4] for a recent overview) and is still of interest today. In general, the studies about the dynamical behaviour of belt systems have been restricted to belts moving with a constant speed (see for instance [1] - [5]). Recently there are some studies about the transversal vibrations of belt systems moving with a non-constant speed (see for instance [6] - [12]). The vibrations of a belt system moving with a low non-constant velocity have been studied in [6], [7] and [8]. In [6] the belt vibrations have been modeled using a linear string-like equation while in [7] the vibrations have been modeled using a linear beam-like equation. The transversal vibrations of a belt system moving with an $\mathcal{O}(1)$ time-dependent speed have been studied in [9] and [10], while the associated nonlinear vibrations have been studied in [11] and [12]. A major drawback in the papers [9] - [12] which has been observed in [6] and [7], is the use of the truncation method (specifically the use of only one term). It has been pointed out in [1], [6] and [7] that a strong reduction in the phase space can lead to a poor description of the dynamic phenomena and in particular the use of only an one degree-of-freedom approximation can lead to errors in the spatial description and in the forecasting of the time evolution of the system. In [6] and [7] it has been shown that the truncation method as applied in [9] - [12] indeed leads to incorrect results for low speed belt systems on long timescales.

In this paper the weakly nonlinear transversal vibrations of a moving belt will be studied. These vibrations are described by a single weakly nonlinear beam equation. Kirchhoff's approach has been used to obtain this single governing equation from the original coupled system of partial differential equations which describe the longitudinal and transversal vibrations of the belt. The belt speed is considered to be time-varying and to be small compared to the wave speed. It is assumed that the speed is $V(t) = \tilde{\epsilon}(V_0 + \alpha \sin(\Omega t))$, where $\tilde{\epsilon}, V_0, \alpha$, and Ω are all constants with $0 < \tilde{\epsilon} \ll 1$ and $V_0 > |\alpha|$. It should be observed that the velocity changes periodically such that the belt moves in one direction. In fact the small parameter $\tilde{\epsilon}$ indicates that the belt speed $V(t)$ is small compared to the wave speed. The variation in $V(t)$ may be due to the pulleys imperfection or some other sources of imperfection and it can be considered as some kind of excitation. In this paper it is assumed that the displacement of the belt in the longitudinal and in the transversal directions are small.

In relation to excitations, some results in this area have been obtained by Sack [13] and Archibald and Emslie [14]. Sack considered the problem of a string moving with a constant velocity at which one of its end (i.e. $x = L$) is subjected to an harmonic excitation. In [13] the vibrations of the string at $x = L$ is forced to be $v(x,t) = v_0 \cos(\Omega t)$. Archibald and Emslie also studied the case where one end of the moving string is subjected to a harmonic excitation to represent the case of a belt traveling from an eccentric pulley to a smooth pulley. Whereas the case where both ends of the string are excited is studied by Mahalingam in [15]. A moving string model has been used in [15] to study the transverse vibrations of power transmission chains. In all of these works, the belt movement is assumed to be constant.

This paper is organized as follows. In section 2 the coupled equations describing the motion of the belt system in longitudinal and in transversal direction are derived. These coupled partial differential equations are then reduced in section 3 to a single partial differential equation by applying Kirchhoff's approximation. In section 4 a two time-scales perturbation analysis of the equation as obtained in section 3 will be carried out. Some specific values of Ω, the frequency of the belt speed fluctuations, are used to demonstrate what kind of resonances can occur. Finally, in the last section some conclusions will be drawn and some remarks will be made.

2 Equations of motion

The equations of motion for a belt system with constant axial velocity have been derived in [16] using Hamilton's principle. For a time-varying velocity the same approach which has been used in [16] can also be applied with some modifications. A schematic model of a belt system under consideration has been given in Figure 1.

A point particle P on the belt under consideration will have transversal and longitudinal velocities:

$$\begin{aligned} \frac{dU}{d\tau} &= \frac{\partial U}{\partial \tau} + \frac{\partial U}{\partial X}\frac{\partial X}{\partial \tau} \Leftrightarrow \frac{dU}{d\tau} = U_\tau + c(\tau)U_X, \\ \frac{dW}{d\tau} &= c(\tau) + W_\tau + c(\tau)W_X, \end{aligned} \tag{1}$$

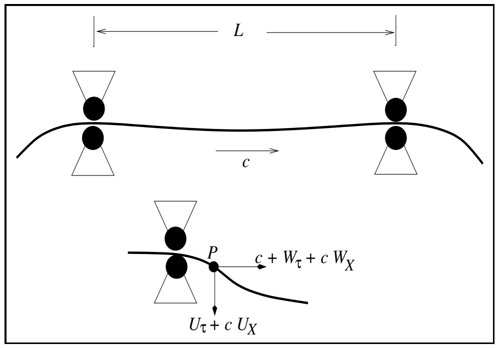


Figure 1: Schematic model of a conveyor belt, and velocity components at a point P on the belt.

respectively. Using these two velocities the kinetic energy of the belt is given by:

$$KE = \frac{1}{2}\rho A \int_0^L \Big\{(U_\tau + cU_X)^2 + [W_\tau + c(1+W_X)]^2\Big\}dx, \tag{2}$$

and the potential energy is given by:

$$PE = \frac{1}{2}\int_0^L \Big(\frac{1}{EA}\{R_0 - EA + EA[(1+W_X)^2 + U_X^2]^{\frac{1}{2}}\}^2 + EIU_{XX}^2\Big)dx, \tag{3}$$

with:

ρ	:	the mass density of the belt,
A	:	the cross-sectional area of the belt,
$c(\tau)$	:	the belt velocity,
E	:	the modulus of elasticity,
R_0	:	the constant tension in a dynamic equilibrium,
I	:	the second moment of area with respect to the horizontal axis,
$U(X,\tau)$	:	the transversal displacement of the belt,
$W(X,\tau)$	:	the longitudinal displacement of the belt,
X	:	the position along the horizontal axis,
τ	:	the time, and
L	:	the distance between the pulleys.

The Hamilton function $H(X,\tau,U_X,U_\tau,W_X,W_\tau,U_{XX})$ is defined by

$$\frac{1}{2}\rho A\Big\{(U_\tau + cU_X)^2 + [W_\tau + c(1+W_X)]^2\Big\} - \\ \frac{1}{2}\Big(\frac{1}{EA}\{R_0 - EA + EA[(1+W_X)^2 + U_X^2]^{\frac{1}{2}}\}^2 + EIU_{XX}^2\Big). \tag{4}$$

Then according to Hamilton's principle, the equations of motion can be derived from $\frac{dI(\epsilon)}{d\epsilon} = 0$ with $\epsilon = 0$, where

$$I(\epsilon) = \int_{\tau_1}^{\tau_2}\int_0^L H(X,\tau,\bar{U}_X,\bar{U}_\tau,\bar{W}_X,\bar{W}_\tau,\bar{U}_{XX})dxdt,$$

in which:

$$\bar{W}(X,\tau) = W(X,\tau) + \epsilon\eta(X,\tau), \quad \text{and} \quad \bar{U}(X,\tau) = U(X,\tau) + \epsilon\zeta(X,\tau).$$

The arbitrary functions $\eta(X,\tau)$ and $\zeta(X,\tau)$ have to satisfy:

$$\begin{aligned}&\eta(0,\tau)=\eta(L,\tau)=0=\eta(X,\tau_1)=\eta(X,\tau_2),\text{and}\\&\zeta(0,\tau)=\zeta(L,\tau)=0=\zeta(X,\tau_1)=\zeta(X,\tau_2).\end{aligned}\tag{5}$$

It then follows that

$$\begin{aligned}\frac{dI(\epsilon)}{d\epsilon} &= \int_{\tau_1}^{\tau_2}\int_0^L \frac{d}{d\epsilon}H(X,\tau,\bar{U}_X,\bar{U}_\tau,\bar{W}_X,\bar{W}_\tau,\bar{U}_{XX})dXd\tau\\ &= \int_{\tau_1}^{\tau_2}\int_0^L \Big\{\frac{\partial H}{\partial \bar{W}_X}\frac{\partial \bar{W}_X}{\partial\epsilon}+\frac{\partial H}{\partial \bar{U}_X}\frac{\partial \bar{U}_X}{\partial\epsilon}+\frac{\partial H}{\partial \bar{W}_\tau}\frac{\partial \bar{W}_\tau}{\partial\epsilon}+\frac{\partial H}{\partial \bar{U}_\tau}\frac{\partial \bar{U}_\tau}{\partial\epsilon}+\\ &\qquad \frac{\partial H}{\partial \bar{U}_{XX}}\frac{\partial \bar{U}_{XX}}{\partial\epsilon}\Big\}dXd\tau,\\ &= \int_{\tau_1}^{\tau_2}\int_0^L \Big\{\frac{\partial H}{\partial \bar{W}_X}\eta_X+\frac{\partial H}{\partial \bar{U}_X}\zeta_X+\frac{\partial H}{\partial \bar{W}_\tau}\eta_\tau+\frac{\partial H}{\partial \bar{U}_\tau}\zeta_\tau+\frac{\partial H}{\partial \bar{U}_{XX}}\zeta_{XX}\Big\}dXd\tau.\end{aligned}\tag{6}$$

So, $\frac{dI(0)}{d\epsilon}=$

$$\int_{\tau_1}^{\tau_2}\int_0^L \Big\{\frac{\partial H}{\partial W_X}\eta_X+\frac{\partial H}{\partial U_X}\zeta_X+\frac{\partial H}{\partial W_\tau}\eta_\tau+\frac{\partial H}{\partial U_\tau}\zeta_\tau+\frac{\partial H}{\partial U_{XX}}\zeta_{XX}\Big\}dXd\tau=0.\tag{7}$$

Integrating (7) by parts and using (5) it then follows that (7) can be rewritten in:

$$\begin{aligned}&\int_{\tau_2}^{\tau_1}\int_0^L \Big\{\eta\Big[\frac{d}{dX}\Big(\frac{\partial H}{\partial W_X}\Big)+\frac{d}{d\tau}\Big(\frac{\partial H}{\partial W_\tau}\Big)\Big]+\\ &\qquad \zeta\Big[\frac{d}{dX}\Big(\frac{\partial H}{\partial U_X}\Big)+\frac{d}{d\tau}\Big(\frac{\partial H}{\partial U_\tau}\Big)-\frac{d^2}{dX^2}\Big(\frac{\partial H}{\partial U_{XX}}\Big)\Big]\Big\}dXd\tau=0.\end{aligned}\tag{8}$$

Since the functions $\eta(X,\tau)$ and $\zeta(X,\tau)$ are arbitrary it follows from (8) that

$$\begin{aligned}&\frac{d}{dX}\Big(\frac{\partial H}{\partial W_X}\Big)+\frac{d}{d\tau}\Big(\frac{\partial H}{\partial W_\tau}\Big)=0,\\ &\frac{d}{dX}\Big(\frac{\partial H}{\partial U_X}\Big)+\frac{d}{d\tau}\Big(\frac{\partial H}{\partial U_\tau}\Big)-\frac{d^2}{dX^2}\Big(\frac{\partial H}{\partial U_{XX}}\Big)=0.\end{aligned}\tag{9}$$

These equations are called the Euler-Lagrange equations. By substituting $H(X,\tau,U_X,U_\tau,W_X,W_\tau,U_{XX})$ as given by (4) into (9), the following equations are obtained:

$$\begin{aligned}&\rho A W_{\tau\tau}+2\rho Ac W_{X\tau}+\rho Ac_\tau(1+W_X)+(\rho Ac^2-EA)W_{XX}=\\ &\qquad (EA-R_0)\frac{(1+W_X)U_XU_{XX}-U_X^2W_{XX}}{[(1+W_X)^2+U_X^2]^{3/2}},\\ &\rho A U_{\tau\tau}+2\rho Ac U_{X\tau}+\rho Ac_\tau U_X+(\rho Ac^2-EA)U_{XX}+EIU_{XXXX}=\\ &\qquad (R_0-EA)\frac{(1+W_X)^2U_{XX}-(1+W_X)U_XW_{XX}}{[(1+W_X)^2+U_X^2]^{3/2}}.\end{aligned}\tag{10}$$

Using a Taylor series, the denumerator in (10) can be approximated by:

$$[(1+W_x)^2+U_X^2]^{-3/2}=1-3W_X+6W_X^2-\frac{3}{2}U_X^2-10W_X^3+\frac{15}{2}W_XU_X^2+\mathcal{O}(4),\tag{11}$$

where $\mathcal{O}(4)$ stands for terms of degree 4 or higher. Assuming that the displacements in the longitudinal direction are much smaller than the displacements in the transversal direction, that is, $\mathcal{O}(W) = \mathcal{O}(U^2)$ it follows from (11) that $[(1+W_X)^2 + U_X^2]^{3/2} \approx 1 - 3W_X - \frac{3}{2}U_X^2$. Substitution of this approximation into (10) gives (approximately)

$$\begin{aligned}
&\rho A W_{\tau\tau} + 2\rho A c W_{X\tau} + \rho A c_\tau (1+W_X) + (\rho A c^2 - EA) W_{XX} = (EA - R_0) U_X U_{XX}, \\
&\rho A U_{\tau\tau} + 2\rho A c U_{X\tau} + \rho A c_\tau U_X + (\rho A c^2 - R_0) U_{XX} + EI U_{XXXX} = \\
&\qquad \Big(EA - R_0\Big)\Big(\frac{3}{2}U_X^2 U_{XX} + W_X U_{XX} + U_X W_{XX}\Big), \;\; \tau > 0, 0 < X < L. \qquad (12)
\end{aligned}$$

To put the equation of motion (12) into a non-dimensional form, the following substitutions are applied:

$$w(x,t) = \frac{W(X,\tau)}{L}, \;\; u(x,t) = \frac{U(X,\tau)}{L}, \;\; x = \frac{X}{L}, \;\; \beta^2 = \frac{T_0}{\rho A}, \;\; t = \frac{\beta\tau}{L}, \;\; V(t) = \frac{c(\tau)}{\beta},$$

$$P_0^2 = \frac{EI}{T_0 L^2}, \;\; \text{and} \;\; P_1^2 = \frac{EA}{T_0},$$

where L is the distance between the two pulleys which are assumed to be two simple supports, and T_0 is the initial tension which is related to R_0 through $R_0 = T_0 + \eta\rho A c^2$ with $0 \leq \eta \leq 1$. Substituting all those non-dimensional variables into (12) and letting $\kappa = 1 - \eta$ the following system of partial differential equations is then obtained:

$$\begin{aligned}
&w_{tt} + 2V w_{xt} + V_t(1+w_x) - (P_1^2 - V^2) w_{xx} = (P_1^2 - 1 - \eta V^2) u_x u_{xx}, \\
&u_{tt} + 2V u_{xt} + V_t u_x + (\kappa V^2 - 1) u_{xx} + P_0^2 u_{xxxx} = \\
&\quad (P_1^2 - 1 - \eta V^2)(\frac{3}{2}u_x^2 u_{xx} + u_x w_{xx} + w_x u_{xx}), \quad t \geq 0, 0 < x < 1. \qquad (13)
\end{aligned}$$

The boundary conditions for the two simple supports are given by:

$$w(0,t) = w(1,t) = 0, \quad \text{and} \quad u(x,t) = u_{xx}(x,t) = 0 \quad \text{for} \quad x = 0, 1, \qquad (14)$$

while the initial displacements and initial velocities are:

$$w(x,0) = w_0(x), \;\; w_t(x,0) = w_1(x), \;\; u(x,0) = u_0(x), \;\; \text{and} \;\; u_t(x,0) = u_1(x). \qquad (15)$$

In (13) it is assumed that $P_0^2 = \mathcal{O}(1)$ and that P_1^2 is much larger than P_0^2. In fact it will be assumed that $P_1^2 = \mathcal{O}(\frac{1}{\tilde{\epsilon}})$ with $0 < \tilde{\epsilon} \ll 1$. Also it will be assumed that the belt speed c is small compared to the wave speed β (or equivalently V is small compared to 1, that is, it will be assumed that $V(t) = \mathcal{O}(\tilde{\epsilon})$).

3 Kirchhoff's approach

In this paper it will be assumed that u and V are $\mathcal{O}(\tilde{\epsilon})$, w is $\mathcal{O}(\tilde{\epsilon}^2)$, P_0^2 is $\mathcal{O}(1)$, and P_1^2 is $\mathcal{O}(\frac{1}{\tilde{\epsilon}})$, where $\tilde{\epsilon}$ is a small parameter with $0 < \tilde{\epsilon} \ll 1$. Using these assumptions and following Kirchhoff's approach it will be shown in this section that the coupled system of PDEs (13) can be reduced to a single PDE for the transversal displacement $u(x,t)$.

Now, it should be observed that the equation for the longitudinal displacements $w(x,t)$ in (13) can be rewritten in:

$$w_{tt} + 2Vw_{xt} + V_t(1 + w_x) + V^2 w_{xx} = P_1^2(w_x + \frac{1}{2}u_x^2)_x - (1 + \eta V^2)u_x u_{xx}. \tag{1}$$

Since u and V are $\mathcal{O}(\tilde{\epsilon}), w = \mathcal{O}(\tilde{\epsilon}^2)$, and $P_1^2 = \mathcal{O}(\frac{1}{\tilde{\epsilon}})$ then (1) up to order $\tilde{\epsilon}$ becomes:

$$P_1^2(w_x + \frac{1}{2}u_x^2)_x = V_t \Rightarrow P_1^2(w_x + \frac{1}{2}u_x^2) = xV_t + f(t) \tag{2}$$

$$\Rightarrow P_1^2 \int_0^1 (w_x + \frac{1}{2}u_x^2)dx = \frac{1}{2}V_t + f(t) \Rightarrow f(t) = \frac{1}{2}\Big(P_1^2 \int_0^1 u_x^2 dx - V_t\Big), \tag{3}$$

where use has been made of the boundary conditions $w(0,t) = w(1,t) = 0$.

Similarly the equation for u in (13) can be rewritten in

$$u_{tt} - u_{xx} + P_0^2 u_{xxxx} = \Big[P_1^2\Big\{u_x(\frac{1}{2}u_x^2 + w_x)_x + u_{xx}(\frac{1}{2}u_x^2 + w_x)\Big\} - 2Vu_{xt} - V_t u_x\Big] + \text{"}h.o.t.\text{"}, \tag{4}$$

where $h.o.t.$ stands for higher order terms. Substituting $w_x + \frac{1}{2}u_x^2$ from (2) and (3) into (4) gives:

$$u_{tt} - u_{xx} + P_0^2 u_{xxxx} = \Big[(x - \frac{1}{2})V_t u_{xx} - 2Vu_{xt} + \frac{1}{2}P_1^2 u_{xx}\int_0^1 u_x^2 dx\Big] + \text{"}h.o.t.\text{"}, \tag{5}$$

where $u(x,t)$ additionally has to satisfy the boundary conditions (14) and the initial conditions (15).

When it is assumed that $P_1^2 \gg \mathcal{O}(\frac{1}{\tilde{\epsilon}})$ (instead of $P_1^2 = \mathcal{O}(\frac{1}{\tilde{\epsilon}})$) it follows from (1) that $(w_x + \frac{1}{2}u_x^2)_x = 0$ approximately. Following the same steps as given in (2) and (3) it then follows that $u(x,t)$ has to satisfy

$$u_{tt} - u_{xx} + P_0^2 u_{xxxx} = \Big[-V_t u_x - 2Vu_{xt} + \frac{1}{2}P_1^2 u_{xx}\int_0^1 u_x^2 dx\Big] + \text{"}h.o.t.\text{"}. \tag{6}$$

An equation similar to (6) has been studied in [12] using Galerkin's truncation method. In [6] and [7] it has been explained that for these type of equations many phenomena which are present in infinite dimensional systems can be lost in its finite dimensional approximations. In this paper a justification of the applicability of the truncation method will be given by explicitly studying all (internal and external) resonances which are present in equation (5).

In (5) u, V, and P_1^2 are now replaced by $\tilde{\epsilon}\tilde{u}, \tilde{\epsilon}\tilde{V}$, and $\frac{1}{\tilde{\epsilon}}\tilde{P_1}^2$ respectively, where $\tilde{u}, \tilde{V}$ and $\tilde{P_1^2}$ are of $\mathcal{O}(1)$. Equation (5) then becomes:

$$\tilde{u}_{tt} - \tilde{u}_{xx} + P_0^2 \tilde{u}_{xxxx} = \tilde{\epsilon}\Big[(x - \frac{1}{2})\tilde{V}_t \tilde{u}_{xx} - 2\tilde{V}\tilde{u}_{xt} + \frac{1}{2}\tilde{P_1}^2 \tilde{u}_{xx}\int_0^1 \tilde{u}_x^2 dx\Big] + \text{"}h.o.t. \text{ in } \tilde{\epsilon}\text{"}, \quad 0 < x < 1, t > 0, \tag{7}$$

where $\tilde{u}(x,t)$ also has to satisfy the following boundary and initial values

$$\tilde{u}(x,t) = \tilde{u}_{xx}(x,t) = 0, \quad \text{for} \quad x = 0 \quad \text{and} \quad x = 1, t \geq 0, \tag{8}$$

$$\tilde{u}(x,0) = \tilde{u}_0(x), \quad \tilde{u}_t(x,0) = \tilde{u}_1(x), \quad \text{for} \ t = 0, 0 < x < 1. \tag{9}$$

4 A perturbation analysis

In this section approximations of the solution $\tilde{u}(x,t)$ of the initial-boundary value problem (7)-(9) will be constructed. As mentioned in the introduction of this paper it is assumed that the velocity $V(t) = \tilde{\epsilon}\tilde{V}(t)$ of the belt is given by

$$V(t) = \tilde{\epsilon}\tilde{V}(t) = \tilde{\epsilon}(V_0 + \alpha\sin(\Omega t)), \tag{1}$$

where $\tilde{\epsilon}, V_0, \alpha,$ and Ω are all constants with $0 < \tilde{\epsilon} \ll 1$ and $V_0 > |\alpha|$. For special values of Ω it will turn out in this section that complicated resonances occur. Some of these cases for Ω will be studied in detail.Based on the boundary conditions (8) for $\tilde{u}(x,t)$ it follows that $\tilde{u}(x,t)$ can be written in the form: $\tilde{u}(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin(n\pi x)$. Since this series is odd and 2-periodic in x each term in (5) should be expanded odd with respect to $x = 0$ and $x = 1$ and 2-periodic in x. This is accomplished by multiplying each term in (7) which is not already odd in x, (i.e. terms like xu_{xx} and u_{xt}) with $\mathcal{H}(x)$ (see also [6], [17], [18]) where

$$\mathcal{H}(x) = \left\{ \begin{array}{rlrcrcr} 1 & \text{for} & 0 & < & x & < & 1, \\ -1 & \text{for} & -1 & < & x & < & 0, \end{array} \right. = \sum_{j=0}^{\infty} \frac{4}{(2j+1)\pi} \sin((2j+1)\pi x), \tag{2}$$

and $\mathcal{H}(x) = \mathcal{H}(x+2)$. So, equation (7) then becomes on $-1 < x < 1$:

$$\begin{aligned} \tilde{u}_{tt} - \tilde{u}_{xx} + P_0^2\tilde{u}_{xxxx} = \tilde{\epsilon}\Big[& \tilde{V}_t\tilde{u}_{xx}(x\mathcal{H}(x) - \frac{1}{2}) - 2\tilde{V}\tilde{u}_{xt}\mathcal{H}(x) + \\ & \frac{1}{2}\tilde{P_1}^2\tilde{u}_{xx}\int_0^1 \tilde{u}_x^2 dx\Big] + \text{"}h.o.t.\ \text{in}\ \epsilon\text{"}. \end{aligned} \tag{3}$$

It can be shown elementarily that the Fourier series of $x\mathcal{H}(x)$ on $-1 < x < 1$ is

$$\frac{1}{2} - \sum_{j=0}^{\infty} \frac{4}{(2j+1)^2\pi^2} \cos((2j+1)\pi x). \tag{4}$$

Substitution of (4) in (3) gives:

$$\begin{aligned} \tilde{u}_{tt} - \tilde{u}_{xx} + P_0^2\tilde{u}_{xxxx} = \tilde{\epsilon}\Big[& -4\sum_{j=0}^{\infty} \frac{\cos((2j+1)\pi x)}{(2j+1)^2\pi^2}\tilde{V}_t\tilde{u}_{xx} - 2\tilde{V}\tilde{u}_{xt}\mathcal{H}(x) + \\ & \frac{1}{2}\tilde{P_1}^2\tilde{u}_{xx}\int_0^1 \tilde{u}_x^2 dx\Big] + \text{"}h.o.t.\ \text{in}\ \epsilon\text{"}. \end{aligned} \tag{5}$$

Now by substituting $\tilde{V}(t)$ as given by (1) and the series $\sum_{n=1}^{\infty} u_n(t)\sin(n\pi x)$ for $\tilde{u}(x,t)$ into (5) and then by using the orthogonality properties of the Fourier sin −series on $-1 < x < 1$ it follows that u_k has to satisfy (for $k = 1, 2, 3, \ldots$)

$$\begin{aligned} \ddot{u}_k + \omega_k^2 u_k = \tilde{\epsilon}\Big[& \sum_{k=2j+1+n} + \sum_{k=n-2j-1} - \sum_{k=2j+1-n}\Big] \frac{2n^2\alpha\Omega\cos(\Omega t)}{(2j+1)^2} u_n - \\ & 4\tilde{\epsilon}(V_0 + \alpha\sin(\Omega t))\Big[\sum_{k=2j+1+n} + \sum_{k=2j+1-n} - \sum_{k=n-2j-1}\Big] \frac{n\dot{u}_n}{(2j+1)} - \\ & \tilde{\epsilon}\frac{k^2\tilde{P}_1^2\pi^4}{4} u_k \Big(\sum_{l=1}^{\infty} l^2 u_l^2\Big) + \mathcal{O}(\tilde{\epsilon}^2), \end{aligned} \tag{6}$$

where $\omega_k^2 = (k\pi)^2 + P_0^2(k\pi)^4$. It should be observed that (6) is also obtained when (after the sin −series for $u(x,t)$ is substituted into (7)) equation (7) is multiplied with $\sin(k\pi x)$ and then integrated with respect to x from $x=0$ to $x=1$.

When a naive perturbation method is used secular terms will occur. To avoid these secular terms a two time-scales perturbation method will be used to solve (6) approximately. The introduction of two time-scales $t_0 = t$ and $t_1 = \tilde{\epsilon}t$ implies that

$$u_k(t) = \bar{u}_k(t_0,t_1), \qquad \dot{u}_k = \frac{\partial \bar{u}_k}{\partial t_0} + \tilde{\epsilon}\frac{\partial \bar{u}_k}{\partial t_1}, \qquad \ddot{u}_k = \frac{\partial^2 \bar{u}_k}{\partial t_0^2} + 2\tilde{\epsilon}\frac{\partial^2 \bar{u}_k}{\partial t_0 \partial t_1} + \tilde{\epsilon}^2\frac{\partial^2 \bar{u}_k}{\partial t_1^2}.$$

For convenience the bar on $\bar{u}_k(t_0,t_1)$ will be dropped in the further analysis. Assuming that $u_k(t_0,t_1)$ can be written in the formal expansion $u_{k0} + \tilde{\epsilon}u_{k1} + \mathcal{O}(\tilde{\epsilon}^2)$ it then follows from the $\mathcal{O}(1)-$terms and the $\mathcal{O}(\tilde{\epsilon})-$terms in (6) that u_{k0} and u_{k1} have to satisfy:

$$\begin{aligned}
\mathcal{O}(1) &: \frac{\partial^2 u_{k0}}{\partial t_0^2} + \omega_k^2 u_{k0} = 0,\\
\mathcal{O}(\tilde{\epsilon}) &: \frac{\partial^2 u_{k1}}{\partial t_0^2} + \omega_k^2 u_{k1} = \\
&-2\frac{\partial^2 u_{k0}}{\partial t_0 \partial t_1} + \Big[\sum_{k=2j+1+n} + \sum_{k=n-2j-1} - \sum_{k=2j+1-n}\Big]\frac{2n^2\alpha\Omega\cos(\Omega t_0)}{(2j+1)^2}u_{n0} - \\
&\Big[\sum_{k=2j+1+n} + \sum_{k=2j+1-n} - \sum_{k=n-2j-1}\Big]\Big(\frac{4n(V_0 + \alpha\sin(\Omega t))}{2j+1}\frac{\partial u_{n0}}{\partial t_0}\Big) - \\
&\frac{\tilde{P_1}^2 k^2\pi^4}{4}u_{k0}\Big(\sum_{l=1}^{\infty} l^2 u_{l0}^2\Big),
\end{aligned}$$

respectively. The solution of the $\mathcal{O}(1)$ problem is given by

$$u_{k0}(t_0,t_1) = A_{k0}(t_1)\sin(\omega_k t_0) + B_{k0}(t_1)\cos(\omega_k t_0), \tag{7}$$

where the functions $A_{k0}(t_1)$ and $B_{k0}(t_1)$ in (7) are still arbitrary and can be used to avoid secular terms in the $\mathcal{O}(\tilde{\epsilon})-$problem for u_{k1}. By substituting $u_{k0}(t_0,t_1)$ into the $\mathcal{O}(\tilde{\epsilon})-$problem it follows that

$$\begin{aligned}
&\frac{\partial^2 u_{k1}}{\partial t_0^2} + \omega_k^2 u_{k1} = -2\omega_k[\dot{A}_{k0}\cos(\omega_k t_0) - \dot{B}_{k0}\sin(\omega_k t_0)] + \\
&\Big[\sum_{k=2j+1+n} + \sum_{k=n-2j-1} - \sum_{k=2j+1-n}\Big]\frac{\alpha\Omega n^2}{(2j+1)^2}\Big[A_{n0}\Big\{\sin((\omega_n+\Omega)t_0) + \\
&\qquad \sin((\omega_n-\Omega)t_0)\Big\} + B_{n0}\Big\{\cos((\omega_n+\Omega)t_0) + \cos((\omega_n-\Omega)t_0)\Big\}\Big] + \\
&\Big[\sum_{k=n-2j-1} - \sum_{k=n+2j+1} - \sum_{k=2j+1-n}\Big]\frac{4n\omega_n V_0}{2j+1}\Big[A_{n0}\cos(\omega_n t_0) - B_{n0}\sin(\omega_n t_0)\Big] + \\
&\Big[\sum_{k=n-2j-1} - \sum_{k=n+2j+1} - \sum_{k=2j+1-n}\Big]\frac{2\alpha n\omega_n}{2j+1}\Big[A_{n0}\Big\{\sin((\omega_n+\Omega)t_0) - \\
&\qquad \sin((\omega_n-\Omega)t_0)\Big\} + B_{n0}\Big\{\cos((\omega_n+\Omega)t_0) - \cos((\omega_n-\Omega)t_0)\Big\}\Big] +
\end{aligned}$$

$$
-\frac{k^2\tilde{P}_1^2\pi^4}{8}\Big[A_{k0}\sin(\omega_k t_0)+B_{k0}\cos(\omega_k t_0)\Big]\sum_{l=1}^{\infty}l^2\Big(A_{l0}^2+B_{l0}^2\Big)+
$$
$$
-\frac{k^2\tilde{P}_1^2\pi^4}{16}\sum_{l=1}^{\infty}l^2\Big(B_{l0}^2-A_{l0}^2\Big)\Big[A_{k0}\Big\{\sin((2\omega_l+\omega_k)t_0)-\sin((2\omega_l-\omega_k)t_0)\Big\}+
$$
$$
B_{k0}\Big\{\cos((2\omega_l+\omega_k)t_0)+\cos((2\omega_l-\omega_k)t_0)\Big\}\Big]+
$$
$$
-\frac{k^2\tilde{P}_1^2\pi^4}{8}\sum_{l=1}^{\infty}l^2A_{l0}B_{l0}\Big[A_{k0}\Big\{\cos((2\omega_l-\omega_k)t_0)-\cos((2\omega_l+\omega_k)t_0)\Big\}+
$$
$$
B_{k0}\Big\{\sin((2\omega_l+\omega_k)t_0)+\sin((2\omega_l-\omega_k)t_0)\Big\}\Big]. \tag{8}
$$

Now it can be seen from the right-hand side of (8) that secular terms (or equivalently resonances) will occur when $\omega_n \pm \Omega = \pm\omega_k$ or when $\omega_l = \omega_k$. In the following subsections, some cases will be studied in which resonances occur. In section 4.1 the case $\Omega \neq \pm\omega_k \pm \omega_n$ will be studied. In this case only internal resonances occur due to the nonlinear term in the PDE (7). In section 4.2 the case $\Omega = \omega_2 - \omega_1 + \tilde{\epsilon}\phi$ will be studied in which ϕ is a detuning parameter. This case is an example in which the frequency of the belt-velocity fluctuations is the difference of two natural frequencies of the constant belt-velocity problem. In section 4.3 and 4.4 the case $\Omega = \omega_2 + \omega_1 + \tilde{\epsilon}\phi$ and $\Omega = \omega_3 + \omega_2 + \tilde{\epsilon}\phi$ respectively will be studied. Again ϕ is a detuning parameter. These cases are examples in which the frequencies of the belt-velocity fluctuations are the sum of two natural frequencies of the constant belt-velocity problem.

4.1 The case where Ω causes no resonances

When $\Omega \neq \pm\omega_k \pm \omega_n$ (or not ϵ-close to these values) only internal resonances will occur due to the nonlinear term in the PDE (7). It can be shown elementarily from (8) that secular terms in u_{k1} can be avoided if A_{k0} and B_{k0} satisfy

$$
\begin{aligned}
\dot{A}_{k0} &= -\frac{k^2\tilde{P}_1{}^2\pi^4}{32\omega_k}B_{k0}\Big[k^2(A_{k0}^2+B_{k0}^2)+2\sum_{l=1}^{\infty}l^2(A_{l0}^2+B_{l0}^2)\Big],\\
\dot{B}_{k0} &= \frac{k^2\tilde{P}_1{}^2\pi^4}{32\omega_k}A_{k0}\Big[k^2(A_{k0}^2+B_{k0}^2)+2\sum_{l=1}^{\infty}l^2(A_{l0}^2+B_{l0}^2)\Big],
\end{aligned} \tag{9}
$$

for $k = 1, 2, 3, \ldots$. System (9) can be solved exactly by introducing polar coordinates, that is, $A_{k0}(t_1) = r_k(t_1)\sin(\phi_k(t_1))$ and $B_{k0}(t_1) = r_k(t_1)\cos(\phi_k(t_1))$. System (9) in polar coordinates then becomes:

$$
\dot{r}_k = 0, \quad \dot{\phi}_k = -\frac{k^2\tilde{P}_1{}^2\pi^4}{32\omega_k}\Big(k^2r_k^2+2\sum_{l=1}^{\infty}l^2r_l^2\Big). \tag{10}
$$

From (10) it follows that $r_k(t_1) = r_k(0)$ and $\phi_k(t_1) = -\frac{k^2\tilde{P}_1{}^2\pi^4}{32\omega_k}\Big(k^2r_k(0)^2+2\sum_{l=1}^{\infty}l^2r_l(0)^2\Big)t_1+\phi_k(0)$ for $k = 1, 2, 3, \ldots$. The constants $r_k(0)$ and $\phi_k(0)$ follow from the initial values $A_{k0}(0)$ and $B_{k0}(0)$.

4.2 The case $\Omega = \omega_2 - \omega_1 + \tilde{\epsilon}\phi$

It has been shown at the end of section 3 that resonances will occur when $\omega_n \pm \Omega = \pm\omega_k$ or when $\omega_l = \omega_k$. In this section the case $\Omega = \omega_2 - \omega_1 + \tilde{\epsilon}\phi$ will be discussed where ϕ is a detuning parameter. By using this special value of Ω additional mode interactions will only occur between mode 1 and mode 2 as has been shown in [7]. Substituting $\Omega = \omega_2 - \omega_1 + \tilde{\epsilon}\phi$ into (8), taking apart terms that cause resonances and setting these terms equal to zero to avoid secular terms, the following set of equations for $A_{k0}(t_1)$ and $B_{k0}(t_1)$ will be obtained:

$$\begin{aligned}
\dot{A}_{10} &= -\frac{4\alpha(4\omega_1-\omega_2)}{9\omega_1}[B_{20}\cos(\phi t_1) - A_{20}\sin(\phi t_1)] - \frac{\tilde{P_1}^2\pi^4}{32\omega_1}B_{10}\Big((A_{10}^2+B_{10}^2) + \\
&\quad 2\sum_{l=1}^{\infty} l^2(A_{l0}^2+B_{l0}^2)\Big), \\
\dot{B}_{10} &= \frac{4\alpha(4\omega_1-\omega_2)}{9\omega_1}[A_{20}\cos(\phi t_1) + B_{20}\sin(\phi t_1)] + \frac{\tilde{P_1}^2\pi^4}{32\omega_1}A_{10}\Big((A_{10}^2+B_{10}^2) + \\
&\quad 2\sum_{l=1}^{\infty} l^2(A_{l0}^2+B_{l0}^2)\Big), \\
\dot{A}_{20} &= -\frac{4\alpha(4\omega_1-\omega_2)}{9\omega_2}[A_{10}\sin(\phi t_1) + B_{10}\cos(\phi t_1)] - \frac{\tilde{P_1}^2\pi^4}{4\omega_2}B_{20}\Big(2(A_{20}^2+B_{20}^2) + \\
&\quad \sum_{l=1}^{\infty} l^2(A_{l0}^2+B_{l0}^2)\Big), \\
\dot{B}_{20} &= \frac{4\alpha(4\omega_1-\omega_2)}{9\omega_2}[A_{10}\cos(\phi t_1) + B_{10}\sin(\phi t_1)] + \frac{\tilde{P_1}^2\pi^4}{4\omega_2}A_{20}\Big(2(A_{20}^2+B_{20}^2) + \\
&\quad \sum_{l=1}^{\infty} l^2(A_{l0}^2+B_{l0}^2)\Big),
\end{aligned} \tag{11}$$

and (9) for $k = 3, 4, 5, \ldots$. By introducing polar coordinates in (9) for $k = 3, 4, 5, \ldots$ and in (11), that is, $A_{k0}(t_1) = r_k(t_1)\sin(\phi_k(t_1))$ and $B_{k0}(t_1) = r_k(t_1)\cos(\phi_k(t_1))$ it follows that

$$\begin{aligned}
\dot{r}_1 &= \frac{4\alpha(4\omega_1-\omega_2)}{9\omega_1} r_2\sin(\phi_2-\phi_1+\phi t_1), \qquad \dot{r}_2 = -\frac{4\alpha(4\omega_1-\omega_2)}{9\omega_2} r_1\sin(\phi_2-\phi_1+\phi t_1), \\
\dot{\phi}_1 &= -\frac{4\alpha(4\omega_1-\omega_2)r_2}{9\omega_1 r_1}\cos(\phi_2-\phi_1+\phi t_1) - \frac{\tilde{P_1}^2\pi^4}{32\omega_1}\Big(r_1^2 + 2\sum_{l=1}^{\infty} l^2 r_l^2\Big), \\
\dot{\phi}_2 &= -\frac{4\alpha(4\omega_1-\omega_2)r_1}{9\omega_2 r_2}\cos(\phi_2-\phi_1+\phi t_1) - \frac{\tilde{P_1}^2\pi^4}{4\omega_2}\Big(2r_2^2 + \sum_{l=1}^{\infty} l^2 r_l^2\Big),
\end{aligned} \tag{12}$$

and $\dot{r}_k = 0$ for $k = 3, 4, 5, \ldots$. To obtain (12) it has been assumed that $r_1 \neq 0$, and $r_2 \neq 0$. From (11) and (12) it can be seen that if there is no initial energy present in the kth mode, $k = 3, 4, 5, \ldots$ then the energy in that mode will be zero up to $\mathcal{O}(\tilde{\epsilon})$ on time-scales of $\mathcal{O}(\frac{1}{\tilde{\epsilon}})$. From (12) it can also be seen that if there is energy of $\mathcal{O}(1)$ present in the first mode then an $\mathcal{O}(1)$ part of this energy will be transferred to the second mode, and vice versa. This energy transport will take place on time-scales of $\mathcal{O}(\frac{1}{\tilde{\epsilon}})$. In what follows it is assumed that there is energy present in each mode of vibration at $t = 0$. Since $\dot{r}_k = 0$ for $k = 3, 4, 5, \ldots$ it then follows that $r_k(t_1) = r_k(0)$ for $t_1 > 0$. From the first two equations in (12) it follows that $\omega_1 r_1 \dot{r}_1 + \omega_2 r_2 \dot{r}_2 = 0$. This implies that $\omega_1 r_1^2 + \omega_2 r_2^2 = C$, where C is a constant of integration. In fact $r_k(t_1) = r_k(0)$ for $k = 3, 4, 5, \ldots$, and $\omega_1 r_1^2 + \omega_2 r_2^2 = C$ are first integrals of the infinite dimensional system of ODEs (12). Now let $\Phi(t_1) = \phi_2(t_1) - \phi_1(t_1) + \phi t_1$. Then it can easily be deduced from (12) that

$$\begin{aligned}
\dot{r}_1 &= \frac{4\alpha(4\omega_1 - \omega_2)}{9\omega_1}\sqrt{\frac{C - \omega_1 r_1^2}{\omega_2}}\sin(\Phi), \\
\dot{\Phi} &= \phi + \frac{4\alpha}{9}(4\omega_1 - \omega_2)\Big[\frac{r_2}{\omega_1 r_1} - \frac{r_1}{\omega_2 r_2}\Big]\cos(\Phi) + \tilde{P_1}^2\pi^4\Big[\frac{1}{32\omega_1}\Big(r_1^2 + 2\sum_{l=1}^{\infty} l^2 r_l^2\Big) - \\
&\quad \frac{1}{4\omega_2}\Big(2r_2^2 + \sum_{l=1}^{\infty} l^2 r_l^2\Big)\Big].
\end{aligned} \tag{13}$$

By introducing the following re-scalings $r_1(t_1) = \sqrt{\frac{c}{\omega_1}} R_1(s_2), \Phi(t_1) = \Psi(s_2)$ with $s_1 = \frac{4\alpha}{9\sqrt{\omega_1\omega_2}}$ $(4\omega_1 - \omega_2)t_1$, and $\frac{ds_2}{ds_1} = \frac{1}{R_1\sqrt{1-R_1^2}}$, and by using the first integrals $r_k(t_1) = r_k(0)$ for $k = 3, 4, 5, \ldots$, and $\omega_1 r_1^2 + \omega_2 r_2^2 = C$ it follows that (13) can be simplified to

$$\frac{dR_1}{ds_2} = R_1(1 - R_1^2)\sin(\Psi), \qquad \frac{d\Psi}{ds_2} = (1 - 2R_1^2)\cos(\Psi) + (k_1 R_1^2 + k_2)R_1\sqrt{1 - R_1^2}, \tag{14}$$

where $k_i = \frac{9\tilde{P_1}^2\pi^4\sqrt{\omega_1\omega_2}}{4\alpha(4\omega_1-\omega_2)}\bar{k}_i$ for $i = 1, 2$ and $\bar{k}_1 = \left(\frac{3}{32\omega_1} - \frac{1}{4\omega_2}\right)\frac{C}{\omega_1} - \left(\frac{1}{4\omega_1} - \frac{3}{2\omega_2}\right)\frac{C}{\omega_2}$, and $\bar{k}_2 = \left(\frac{1}{4\omega_1} - \frac{3}{2\omega_2}\right)\frac{C}{\omega_2} + \frac{\phi}{\tilde{P_1}^2\pi^2} + \left(\frac{1}{16\omega_1} - \frac{1}{4\omega_2}\right)\sum_{l=3}^{\infty} l^2 r_l(0)^2$. Since α and ϕ are both arbitrary it follows that k_1 and k_2 are arbitrary. However, the analysis can be restricted to the case $k_1 \geq 0$ and $-\infty < k_2 < \infty$, since for $k_1 < 0$ a simple rescaling ($\Psi := \Psi + \pi$, and $s_2 := -s_2$) leads again to the system (14) with $k_1 \geq 0$ and$-\infty < k_2 < \infty$. It turns out that a first integral for (14) can also be obtained. To obtain this first integral it should be observed from (14) that

$$\begin{aligned}
\frac{d\Psi}{dR_1} &= \frac{(1 - 2R_1^2)\cos(\Psi) + (k_1 R_1^2 + k_2)R_1\sqrt{1 - R_1^2}}{R_1(1 - R_1^2)\sin(\Psi)} \\
\Leftrightarrow \quad \frac{\sin(\Psi)d\Psi}{dR_1} &= \frac{(1 - 2R_1^2)\cos(\Psi) + (k_1 R_1^2 + k_2)R_1\sqrt{1 - R_1^2}}{R_1(1 - R_1^2)} \\
\Leftrightarrow -\frac{d(\cos(\Psi))}{dR_1} &= \frac{1 - 2R_1^2}{R_1(1 - R_1^2)}\cos(\Psi) + \frac{(k_1 R_1^2 + k_2)R_1\sqrt{1 - R_1^2}}{R_1(1 - R_1^2)} \\
\Leftrightarrow \quad \frac{d(\cos(\Psi))}{dR_1} &+ \frac{1 - 2R_1^2}{R_1(1 - R_1^2)}\cos(\Psi) = \frac{(k_1 R_1^2 + k_2)R_1\sqrt{1 - R_1^2}}{R_1(1 - R_1^2)}
\end{aligned} \tag{15}$$

which is a first order ODE in $\cos(\Psi)$. The solutions of this ODE(15) can readily be constructed, yielding

$$\cos(\Psi) = \frac{k_1}{3R_1\sqrt{1 - R_1^2}}\Big[R_1(1 - R_1^2)^{3/2} + \frac{2}{5}(1 - R_1^2)^{5/2}\Big] + \frac{k_2(1 - R_1^2)}{3R_1} + \frac{\widetilde{C}}{R_1\sqrt{1 - R_1^2}}, \tag{16}$$

where $\widetilde{C}$ is a constant of integration. In the following subsections an analysis of system (14) in the $(R_1, \Psi)-$phase plane will be given for different values of k_1 and k_2 with $k_1 \geq 0$ and $-\infty < k_2 < \infty$.

4.2.1 Equilibrium points of system (14)

The obvious equilibrium points of system (14) are $(R_1, \Psi) = (0, \pm\frac{n\pi}{2})$, and $(1, \pm\frac{n\pi}{2})$, with $n = 1, 3, 5, \ldots$. The less obvious equilibrium points (R_1, Ψ) are given by $\Psi = m\pi$ with $m \in \mathbb{Z}$, where R_1 with $0 < R_1 < 1$ follows from $(1 - 2R_1^2)\cos(m\pi) + (k_1 R_1^2 + k_2)R_1\sqrt{1 - R_1^2} = 0$. To determine the number of equilibrium points for a fixed value of m two cases have to be studied: (i) m is even, and (ii) m is odd. These two cases will now be studied.

(i) **The case $\Psi = m\pi$ with m even**
The R_1-values in this case follow from

$$1 - 2R_1^2 + (k_1R_1^2 + k_2)R_1\sqrt{1 - R_1^2} = 0 \quad \Leftrightarrow \quad \frac{1 - 2R_1^2}{R_1\sqrt{1 - R_1^2}} + k_1R_1^2 + k_2 = 0$$

$$\Leftrightarrow \quad \frac{1 - R_1^2 - R_1^2}{R_1\sqrt{1 - R_1^2}} + k_1R_1^2 + k_2 = 0 \quad \Leftrightarrow \quad \frac{\sqrt{1 - R_1^2}}{R_1} - \frac{R_1}{\sqrt{1 - R_1^2}} + k_1R_1^2 + k_2 = 0$$

$$\Leftrightarrow \quad \frac{\sqrt{z - z^2}}{z} - \frac{z}{\sqrt{z - z^2}} + k_1 z + k_2 = 0, \tag{17}$$

where $z = R_1^2$. To determine z from (17) is the same as determining the intersection point(s) of the following two curves: $y = k_1 z + k_2$, and $y = -(\frac{\sqrt{z-z^2}}{z} - \frac{z}{\sqrt{z-z^2}})$. For special values of k_1 and k_2, these two curves are given in Figure 2. By varying k_1 and k_2 it is possible to

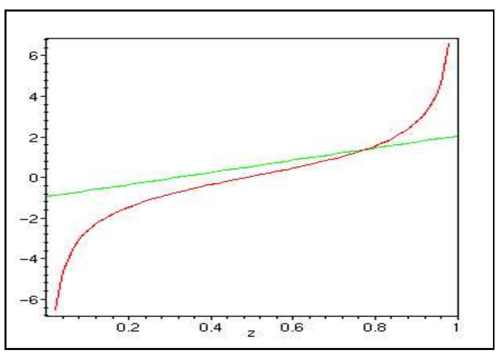


Figure 2: The curves $y = -(\frac{\sqrt{z-z^2}}{z} - \frac{z}{\sqrt{z-z^2}})$ and $y = k_1 z + k_2$ with $k_1 = 3$ and $k_2 = -1$.

obtain one, two, or three intersection points (i.e. equilibrium points). Observe also that as k_2 is getting larger, the intersection point tends to $z = 1$.

In the case that the straight line is tangent to the other curve, there will be two critical points. Assume that the straight line $y = k_1 z + k_2$ is tangent to $f(z) = -(\frac{\sqrt{z-z^2}}{z} - \frac{z}{\sqrt{z-z^2}})$ at the point $z = z_0$. It then follows that

$$k_1 \quad = \quad f'(z_0) = \frac{-1}{2z_0(z_0 - 1)\sqrt{-z_0(z_0 - 1)}},$$

$$k_2 = f(z_0) - z_0 f'(z_0) = \frac{4z_0^2 - 6z_0 + 3}{2(z_0-1)\sqrt{-z_0(z_0-1)}} = (4z_0^2 - 6z_0 + 3)(-z_0)k_1. \quad (18)$$

From the first equation in (18) z_0 can be determined, yielding

$$z_{0_{1,2}} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \sqrt[3]{16/k_1^2}}, \quad (19)$$

and then from the second equation in (18) it follows that

$$k_{2_1} = (-4z_{0_1}^3 + 6z_{0_1}^2 - 3z_{0_1})k_1, \quad k_{2_2} = (-4z_{0_2}^3 + 6z_{0_2}^2 - 3z_{0_2})k_1. \quad (20)$$

From (19) and from $0 < z_0 < 1$ it follows that $1 - \sqrt[3]{\frac{16}{k_1^3}} \geq 0$. Since $k_1 \geq 0$ it then follows that $k_1 \geq 4$. In Figure 3 the curves in the (k_1, k_2)-plane (as defined by (19) and(20)) are given on

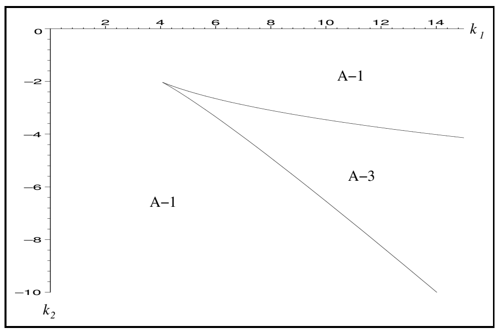


Figure 3: Bifurcation curve in the (k_1, k_2)-plane for the number of equilibrium points (R_1, Ψ) of system (14) with $\Psi = m\pi, m$ even and fixed.

which exactly two equilibrium points (R_1, Ψ) of system (14) can be found for $\Psi = m\pi$ with m even and fixed. Also in Figure 3 the region A-1 (A-3) is given in which exactly one (three) equilibrium point(s) of system (14) can be found for $\Psi = m\pi$ with m even and fixed.

(ii) **The case $\Psi = m\pi$ with m odd**
The R_1-values in this case follow from

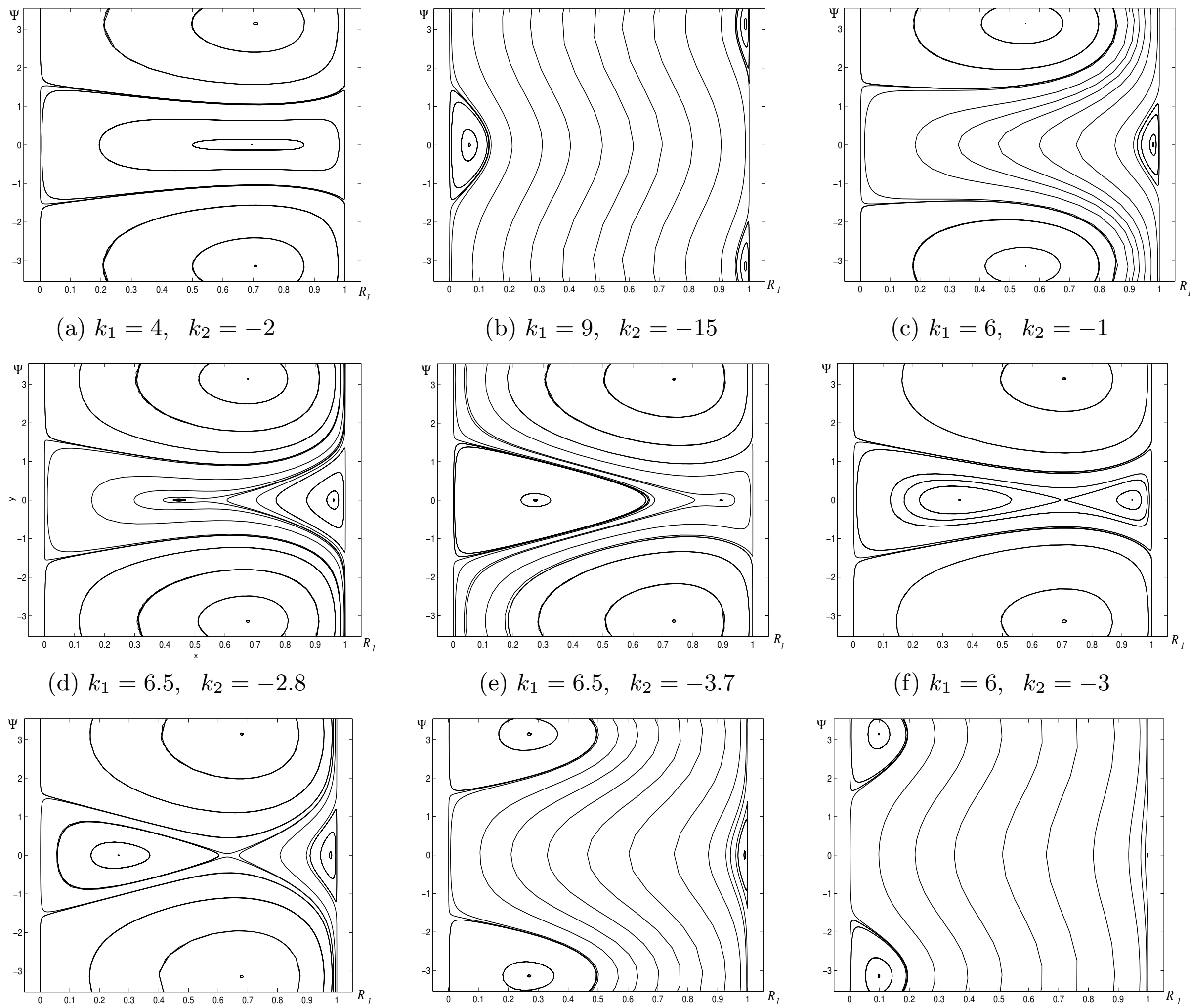
$$-(1 - 2R_1^2) + (k_1 R_1^2 + k_2)R_1\sqrt{1 - R_1^2} = 0, \quad (21)$$

which is equivalent to finding the intersection point(s) of the curves $y = -(k_1 z + k_2)$ and $y = -(\frac{\sqrt{z-z^2}}{z} - \frac{z}{\sqrt{z-z^2}})$, where $z = R_1^2$ (see also the previous case *(i)*). In this case always one equilibrium point will be found for $\Psi = m\pi$ with m odd and fixed since the straight line has a negative gradient.

4.2.2 The (R_1, Ψ)-phase plane of system (14)

In the previous subsection all equilibrium points of system (14) have been determined. In this subsection the orbits in the (R_1, Ψ)-phase plane for system (14) will be given for different values of k_1 and k_2. In Figure 4 these orbits are presented. It can be seen from Figure 4 that for large values of the detuning parameter ϕ (that is, for large values of $|k_2|$) $R_1(s_2)$, and so

$r_1(t_1)$ become constant. So, for large values of the detuning parameter ϕ the solutions of the "resonant" case (i.e. system (14)) tend to the solutions of the "non-resonant" case (i.e.system (9)). Figure 4 and the first integrals for system (14) also show that all solutions are bounded for this special value of $\Omega = \omega_2 - \omega_1 + \tilde{\epsilon}\phi$, which is of difference type. These results are in accordance with those obtained for the linearized problem (see [7]).



(a) $k_1 = 4, \quad k_2 = -2$ (b) $k_1 = 9, \quad k_2 = -15$ (c) $k_1 = 6, \quad k_2 = -1$

(d) $k_1 = 6.5, \quad k_2 = -2.8$ (e) $k_1 = 6.5, \quad k_2 = -3.7$ (f) $k_1 = 6, \quad k_2 = -3$

(g) $k_1 = 9, \quad k_2 = -4$ (h) $k_1 = 4, \quad k_2 = 3$ (i) $k_1 = 6, \quad k_2 = 10$

Figure 4: Orbits in the (R_1, Ψ) phase plane for system (14) for different values of k_1 and k_2 with $-\pi \leq \Psi \leq \pi$ (vertical axis) and $0 \leq R_1 \leq 1$ (horizontal axis).

4.3 The case $\Omega = \omega_2 + \omega_1 + \tilde{\epsilon}\phi$

At the end of section 3 it has been shown that resonances will occur when $\omega_n \pm \Omega = \pm\omega_k$, or when $\omega_l = \omega_k$. In this section the case $\Omega = \omega_2 + \omega_1 + \tilde{\epsilon}\phi$, will be studied, where ϕ is again a detuning parameter. By using this special value of Ω additional mode interactions will only occur between mode 1 and 2 as has been shown in [7]. Substituting $\Omega = \omega_2 + \omega_1 + \tilde{\epsilon}\phi$ into(8), taking apart those terms that cause resonances, and setting these terms equal to zero to avoid secular terms, the following set of equations for $A_{k0}(t_1)$ and $B_{k0}(t_1)$ will be obtained:

$$
\begin{aligned}
\dot{A}_{10} &= \frac{4\alpha(\omega_2+4\omega_1)}{9\omega_1}[B_{20}\cos(\phi t_1) - A_{20}\sin(\phi t_1)] - \frac{\tilde{P_1}^2\pi^4}{32\omega_1}B_{10}\Big[A_{10}^2 + B_{10}^2 + \\
&\quad 2\sum_{l=1}^{\infty} l^2(A_{l0}^2 + B_{l0}^2)\Big], \\
\dot{B}_{10} &= \frac{4\alpha(\omega_2+4\omega_1)}{9\omega_1}[A_{20}\cos(\phi t_1) + B_{20}\sin(\phi t_1)] + \frac{\tilde{P_1}^2\pi^4}{32\omega_1}A_{10}\Big[A_{10}^2 + B_{10}^2 + \\
&\quad 2\sum_{l=1}^{\infty} l^2(A_{l0}^2 + B_{l0}^2)\Big], \\
\dot{A}_{20} &= \frac{4\alpha(\omega_2+4\omega_1)}{9\omega_2}[B_{10}\cos(\phi t_1) - A_{10}\sin(\phi t_1)] - \frac{\tilde{P_1}^2\pi^4}{4\omega_2}B_{20}\Big[2(A_{20}^2 + B_{20}^2) + \\
&\quad \sum_{l=1}^{\infty} l^2(A_{l0}^2 + B_{l0}^2)\Big], \\
\dot{B}_{20} &= \frac{4\alpha(\omega_2+4\omega_1)}{9\omega_2}[A_{10}\cos(\phi t_1) + B_{10}\sin(\phi t_1)] + \frac{\tilde{P_1}^2\pi^4}{4\omega_2}A_{20}\Big[2(A_{20}^2 + B_{20}^2) + \\
&\quad \sum_{l=1}^{\infty} l^2(A_{l0}^2 + B_{l0}^2)\Big], \qquad (22)
\end{aligned}
$$

and (9) for $k = 3, 4, 5, \ldots$. By introducing polar coordinates in (9) for $k = 3, 4, 5, \ldots$ and (11), that is, $A_{k0}(t_1) = r_k(t_1)\sin(\phi_k(t_1))$ and $B_{k0}(t_1) = r_k(t_1)\cos(\phi_k(t_1))$ it follows that:

$$
\begin{aligned}
\dot{r}_1 &= \frac{4\alpha(\omega_2+4\omega_1)}{9\omega_1}r_2\sin(\phi_2+\phi_1+\phi t_1), \qquad \dot{r}_2 = \frac{4\alpha(\omega_2+4\omega_1)}{9\omega_2}r_1\sin(\phi_2+\phi_1+\phi t_1), \\
\dot{\phi}_1 &= \frac{4\alpha(\omega_2+4\omega_1)r_2}{9\omega_1 r_1}\cos(\phi_2+\phi_1+\phi t_1) - \frac{\tilde{P_1}^2\pi^4}{32\omega_1}\Big[r_1^2 + \sum_{l=1}^{\infty} l^2 r_l^2\Big], \\
\dot{\phi}_2 &= \frac{4\alpha(\omega_2+4\omega_1)r_1}{9\omega_2 r_2}\cos(\phi_2+\phi_1+\phi t_1) - \frac{\tilde{P_1}^2\pi^4}{4\omega_2}\Big[2r_2^2 + \sum_{l=1}^{\infty} l^2 r_l^2\Big], \qquad (23)
\end{aligned}
$$

where $r_l^2 = A_{l0}^2 + B_{l0}^2$, and $\dot{r}_k = 0$ for $k = 3, 4, 5, \ldots$. This implies that $r_k(t_1) = \tilde{K}$, where $\tilde{K}$ is a constant. From the first two equations in (23) a first integral can again be derived, yielding $\omega_1 r_1^2 - \omega_2 r_2^2 = K$, where K is a constant of integration. As in the previous section it will turn out that a phase plane analysis can be performed. To give this analysis three cases have to be distinguished: (i) $K > 0, (ii)$ $K = 0$, and (iii) $K < 0$.

4.3.1 The case $K > 0$

By using the first integrals and introducing $\Psi = \phi_2 + \phi_1 + \phi t_1$ a reduced system as in section 4.2 can be obtained from (23), that is;

$$
\dot{r}_1 = \frac{4\alpha}{9\omega_1}(4\omega_1+\omega_2)\sqrt{\frac{\omega_1 r_1^2 - K}{\omega_2}}\sin(\Psi),
$$

$$\dot{\Psi} = \phi + \frac{4\alpha}{9}(4\omega_1+\omega_2)\Big[\frac{2\omega_1 r_1^2 - K}{\omega_1\omega_2 r_1\sqrt{\frac{\omega_1 r_1^2-K}{\omega_2}}}\Big]\cos(\Psi) - \tilde{P_1}^2\pi^4\Big[\Big(\frac{3}{32\omega_1}+\frac{1}{4\omega_2}\Big)r_1^2 + \Big(\frac{1}{4\omega_1}+\frac{3}{2\omega_2}\Big)\frac{\omega_1 r_1^2 - K}{\omega_2} + \Big(\frac{1}{32\omega_1}+\frac{1}{4\omega_2}\Big)\sum_{l=3}^{\infty} l^2 r_l(0)^2\Big]. \tag{24}$$

A further simplification in (24) can be made by introducing the re-scalings $r_1(t_1) = \sqrt{\frac{K}{\omega_1}}R_1(s_2)$, $s_1 = \frac{4\alpha}{9\sqrt{\omega_1\omega_2}}(4\omega_1+\omega_2)t_1$, and $\frac{ds_2}{ds_1} = \frac{1}{R_1\sqrt{R_1^2-1}}$ which results in:

$$\frac{dR_1}{ds_2} = R_1(R_1^2-1)\sin(\Psi), \qquad \frac{d\Psi}{ds_2} = (2R_1^2-1)\cos(\Psi) - (k_1R_1^2+k_2)R_1\sqrt{R_1^2-1}, \tag{25}$$

where $k_i = \frac{9\tilde{P_1}^2\pi^4\sqrt{\omega_1\omega_2}}{4\alpha(\omega_2+4\omega_1)}\bar{k}_i$, for $i = 1,2, \bar{k}_1 = \Big(\frac{3}{32\omega_1}+\frac{1}{4\omega_2}\Big)\frac{K}{\omega_1} + \Big(\frac{1}{4\omega_1}+\frac{3}{2\omega_2}\Big)\frac{K}{\omega_2}$ and $\bar{k}_2 = \Big(\frac{1}{32\omega_1}+\frac{1}{4\omega_2}\Big)\sum_{l=3}^{\infty} l^2 r_l(0)^2 - \Big(\frac{1}{4\omega_1}+\frac{3}{2\omega_2}\Big)\frac{K}{\omega_2} - \frac{\phi}{\tilde{P_1}^2\pi^2}$. For the same reasons as given in section 4.2 the analysis can be restricted to the case $k_1 \geq 0$ and $-\infty < k_2 < \infty$. It should be observed that $K > 0$ implies that $R_1 > 1$. Using a similar method as described at the end of section 4.2, a first integral of (25) also can be derived, giving

$$\cos(\Psi) = \frac{1}{R_1\sqrt{R_1^2-1}}\Big[\frac{k_1}{7}R_1^7 + \frac{1}{5}(k_2-k_1)R_1^5 - \frac{k_2}{3}R_1^3 + \widehat{C}\Big], \tag{26}$$

where $\widehat{C}$ is a constant of integration. The equilibrium points of system (25) have to satisfy $\frac{dR_2}{ds_2} = 0$ and $\frac{d\Psi}{ds_2} = 0$. Since $R_1 > 1$ in this case it follows for the equilibrium points that $\Psi = m\pi$ with $m \in \mathbb{Z}$ and R_1 has to satisfy

$$\pm(2R_1^2-1) - (k_1R_1^2+k_2)R_1\sqrt{R_1^2-1} = 0, \tag{27}$$

where the '+' sign is associated with $\Psi = m\pi$ and m even, and the '−' sign is associated with $\Psi = m\pi$ and m odd. Introducing $z = R_1^2$ (27) becomes

$$\pm\Big(\frac{z}{\sqrt{z^2-z}} + \frac{\sqrt{z^2-z}}{z}\Big) - (k_1z+k_2) = 0. \tag{28}$$

The solution(s) of (28) will be the intersection point(s) of the curves given by $g_1(z) = \pm\Big(\frac{z}{\sqrt{z^2-z}} + \frac{\sqrt{z^2-z}}{z}\Big)$ and $g_2(z) = k_1z+k_2$. In case $\Psi = m\pi$ and m even always one intersection point will be found while in case $\Psi = m\pi$ and m odd zero, one or two intersection points can be found depending on the values of k_1 and k_2 (see also Figure 5). For $\Psi = m\pi$ with m odd exactly one intersection point will occur when the straight line is tangent to the other curve. Assume that the straight line $g_2(z) = k_1z+k_2$ is tangent to $g_1(z) = -\Big(\frac{\sqrt{z^2-z}}{z} + \frac{z}{\sqrt{z^2-z}}\Big)$ at the point $z = z_0$. It then follows that

$$\begin{aligned} k_1 &= g_1'(z_0) = \frac{1}{2}(z_0(z_0-1))^{-3/2}, \\ k_2 &= g_1(z_0) - z_0g_1'(z_0) = -(4z_0^3 - 6z_0^2 + 3z_0)k_1, \end{aligned} \tag{29}$$

where $z_0 > 1$. From the first equation in (29) it follows that $z_0 = \frac{1}{2} + \frac{1}{2}\sqrt{1+\sqrt[3]{\frac{16}{k_1^2}}}$, and then from the second equation in (29) it follows how the curve in the (k_1,k_2)-plane is defined on

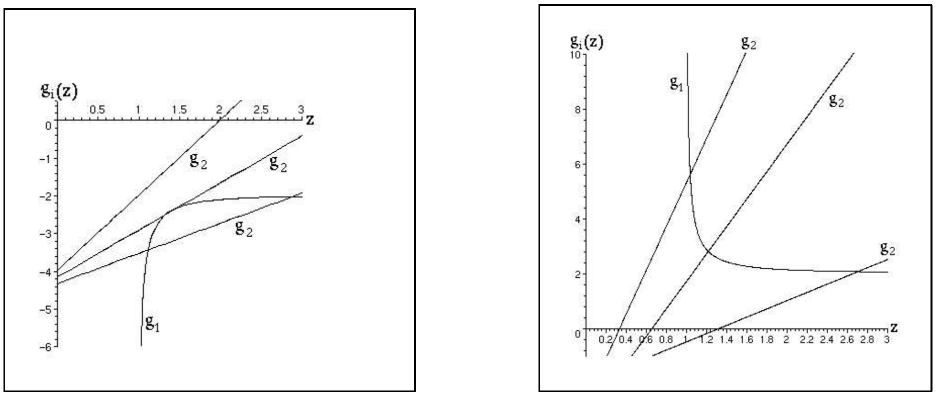


Figure 5: The functions $g_1(z) = \pm\left(\frac{\sqrt{z^2-z}}{z} + \frac{z}{\sqrt{z^2-z}}\right)$ and some functions $g_2(z) = k_1 z + k_2$. In the left graph g_1 is given with the '$-$' sign, and in the right graph g_1 is given with the '+' sign.

which exactly one equilibrium point (R_1, Ψ) of system (25) can be found for $\Psi = m\pi$ with m odd and fixed. In Figure 6 this curve has been plotted. Also in Figure 6 the region A-0 and A-2 are given in which zero and exactly two equilibrium points, respectively, of system (25) can be found for $\Psi = m\pi$ with m odd and fixed. In Figure 7 some phase portraits of system

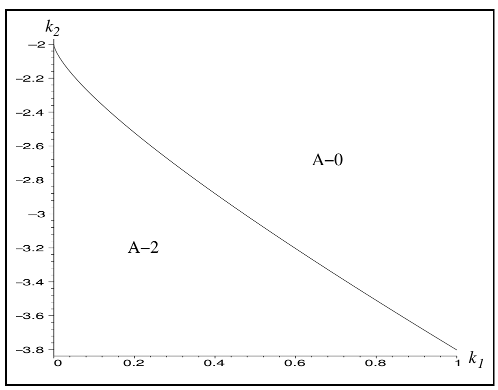


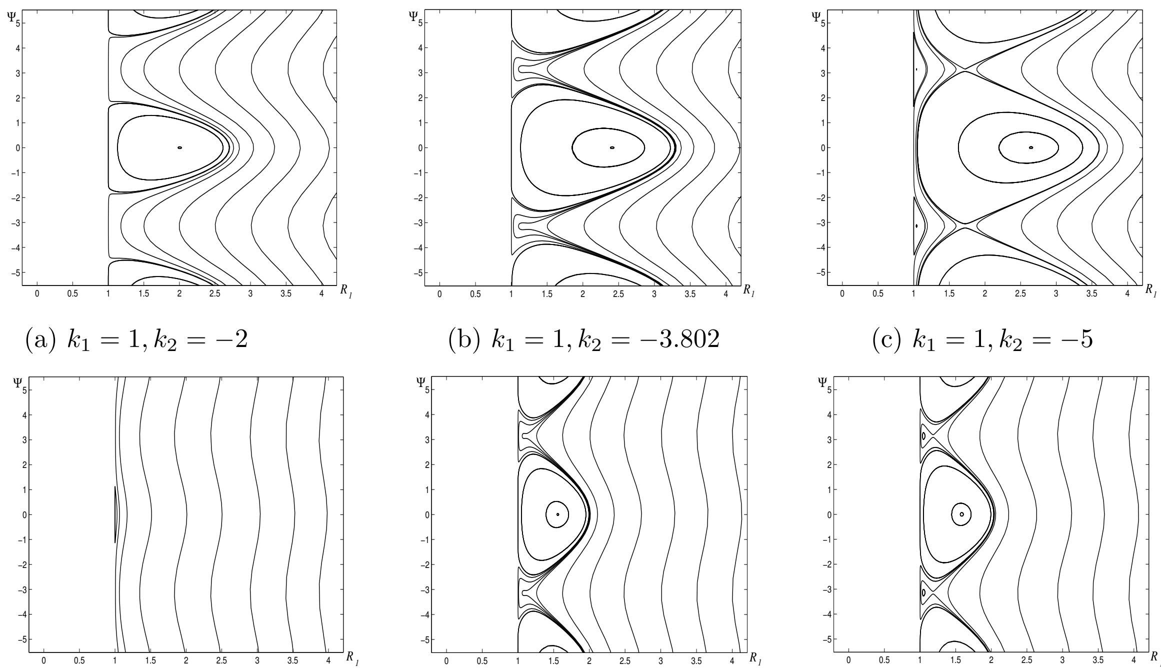
Figure 6: Bifurcation curve in the (k_1, k_2)-plane for the number of equilibrium points of system (25) with $\Psi = m\pi, m$ odd and fixed.

(25) have been given for different values of k_1 and k_2. It can also be seen in Figure 7 that all solutions for R_1 are bounded, and that for large $|k_2|$-values (that is, for large values of the detuning parameter) the behaviour of the solutions of system (25) resembles the solutions of the "non-resonant" system (9).

4.3.2 The case $K = 0$

By using the first integral $\omega_1 r_1^2 = \omega_2 r_2^2$ and by introducing $\Psi = \phi_2 + \phi_1 + \phi t_1$ a reduced system (as in section 4.2) can be obtained form (23), that is,

$$\dot{r}_1 = \frac{4\alpha}{9\sqrt{\omega_1\omega_2}}(4\omega_1 + \omega_2) r_1 \sin(\Psi),$$



(a) $k_1 = 1, k_2 = -2$ (b) $k_1 = 1, k_2 = -3.802$ (c) $k_1 = 1, k_2 = -5$

(d) $k_1 = 4, k_2 = 4$ (e) $k_1 = 4, k_2 = -7.657$ (f) $k_1 = 4, k_2 = -8$

Figure 7: Phase portraits of system (25) for different values of k_1 and k_2 (case $K > 0$).

$$\begin{aligned} \dot{\Psi} \;&=\; \frac{8\alpha}{9\sqrt{\omega_1\omega_2}}(4\omega_1+\omega_2)\cos(\Psi) - \tilde{P_1}^2\pi^4\Big[\Big\{\frac{3}{32\omega_1}+\frac{1}{4\omega_2}+\Big(\frac{1}{4\omega_1}+\frac{3}{2\omega_2}\Big)\frac{\omega_1}{\omega_2}\Big\}r_1^2 + \\ &\Big(\frac{1}{32\omega_1}+\frac{1}{4\omega_2}\Big)\sum_{l=3}^{\infty} l^2 r_l(0)^2 - \frac{\phi}{\tilde{P_1}^2\pi^4}\Big]. \end{aligned} \tag{30}$$

A further simplification in (30) can be made by introducing the re-scaling $s_1 = \frac{4\alpha}{9\sqrt{\omega_1\omega_2}}(4\omega_1+\omega_2)t_1$ which results in

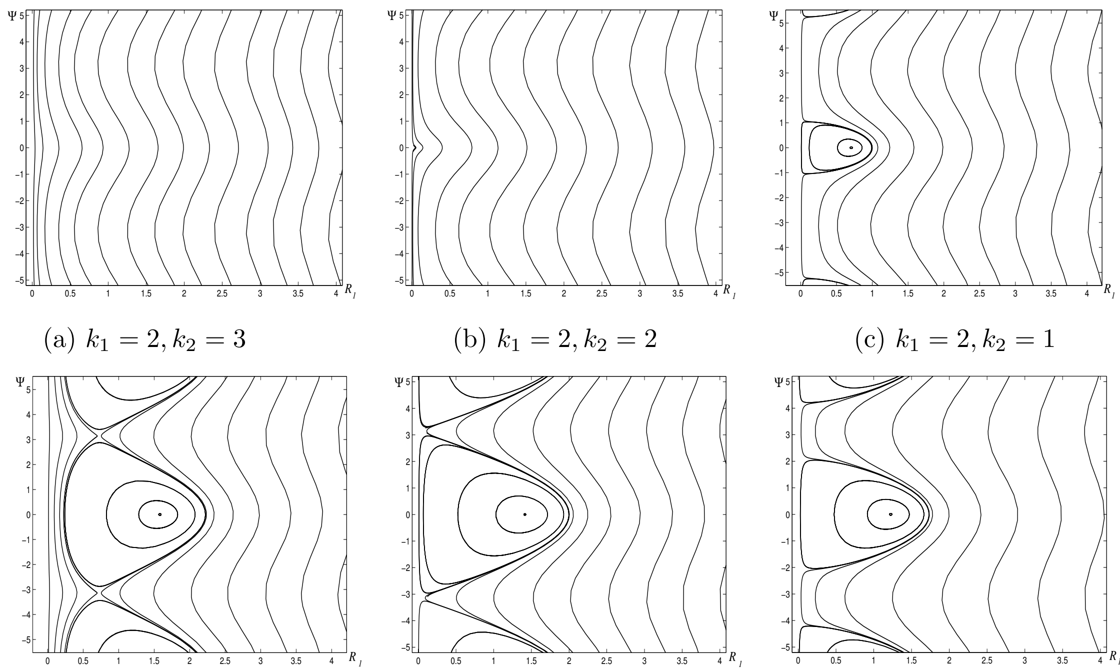
$$\frac{dr_1}{ds_1} = r_1\sin(\Psi), \qquad \frac{d\Psi}{ds_1} = 2\cos(\Psi) - (k_1 r_1^2 + k_2), \tag{31}$$

where $k_i = \frac{9\tilde{P_1}^2\pi^4\sqrt{\omega_1\omega_2}}{4\alpha(4\omega_1+\omega_2)}\bar{k}_i$, for $i = 1, 2$, and $\bar{k}_1 = \frac{3}{32\omega_1} + \frac{1}{4\omega_2} + \Big(\frac{1}{4\omega_1} + \frac{3}{2\omega_2}\Big)\frac{\omega_1}{\omega_2}, \bar{k}_2 = \Big(\frac{1}{32\omega_1} + \frac{1}{4\omega_2}\Big)\sum_{l=3}^{\infty} l^2 r_l(0)^2 - \frac{\phi}{\tilde{P_1}^2\pi^4}$. For the same reasons as given in section 4.2 the analysis can be restricted to the case $k_1 \geq 0$ and $-\infty < k_2 < \infty$. A first integral for system (31) can be computed as follows:

$$\begin{aligned} &\frac{d\Psi}{dr_1} = \frac{2\cos(\Psi)}{r_1\sin(\Psi)} - \frac{k_1r_1^2+k_2}{r_1\sin(\Psi)} \quad\Leftrightarrow\quad \sin(\Psi)\frac{d\Psi}{dr_1} = \frac{2\cos(\Psi)}{r_1} - \frac{k_1r_1^2+k_2}{r_1}, \\ &\Leftrightarrow \frac{d\cos(\Psi)}{dr_1} + \frac{2\cos(\Psi)}{r_1} = \frac{k_1r_1^2+k_2}{r_1}, \end{aligned} \tag{32}$$

which has as solution:

$$\cos(\Psi) = \frac{1}{r_1^2}\Big[\frac{k_1}{4}r_1^4 + \frac{k_2}{2}r_1^2 + C^*\Big], \tag{33}$$



(a) $k_1 = 2, k_2 = 3$ (b) $k_1 = 2, k_2 = 2$ (c) $k_1 = 2, k_2 = 1$

(d) $k_1 = 2, k_2 = -3$ (e) $k_1 = 2, k_2 = -2$ (f) $k_1 = 2, k_2 = -1$

Figure 8: Phase portraits of system (31) for different values of k_1 and k_2 (case $K = 0$).

where C^* is a constant of integration.

The equilibrium points of system (31) are given by $r_1 \sin(\Psi) = 0$ and $2\cos(\Psi) - (k_1 r_1^2 + k_2) = 0$. Elementarily it can be shown that the equilibrium points (r_1, Ψ) of system (31) are:

for $k_2 \leq -2$: $(r_1, \Psi) = \left(\sqrt{\frac{-2-k_2}{k_1}}, m\pi\right)$ with m odd, and
$(r_1, \Psi) = \left(\sqrt{\frac{2-k_2}{k_1}}, m\pi\right)$ with m even.

for $-2 \leq k_2 \leq 2$: $(r_1, \Psi) = (0, \Psi)$ with Ψ given by $\cos(\Psi) = \frac{k_2}{2}$, and
$(r_1, \Psi) = \left(\sqrt{\frac{-2-k_2}{k_1}}, m\pi\right)$ with m even.

for $k_2 > 2$: no equilibrium points.

In Figure 8 some phase portraits of system (31) have been given for different values of k_1 and k_2. It can also be seen in Figure 8 (and from (33)) that all solutions for r_1 are bounded, and that for large $|k_2|$-values (that is, for large values of the detuning parameter) the behaviour of the solution of system (31) resembles the behaviour of the solutions of the "non-resonant" system (9).

4.3.3 The case $K < 0$

From the first two equations in (23) a first integral $\omega_1 r_1^2 - \omega_2 r_2^2 = K$ can be derived. Substituting $K = -F$, with $F > 0$ into this first integral $\omega_2 r_2^2 = \omega_1 r_1^2 + F$ is obtained. By using this first integral and the other first integrals $r_k(t_1) = r_k(0)$ for $k \geq 2$, and by introducing

$\Phi = \phi_2 + \phi_1 + \phi t_1$ the following reduced system will be obtained:

$$\begin{aligned}
\dot{r}_1 &= \frac{4\alpha}{9\omega_1}(4\omega_1+\omega_2)\sqrt{\frac{\omega_1 r_1^2+F}{\omega_2}}\sin(\Phi),\\
\dot{\Phi} &= \phi + \frac{4\alpha}{9}(4\omega_1+\omega_2)\Big[\frac{2\omega_1 r_1^2+F}{\omega_1\omega_2 r_1\sqrt{\frac{\omega_1 r_1^2+F}{\omega_2}}}\Big]\cos(\Phi) - \tilde{P}_1^2\pi^4\Big[\Big(\frac{3}{32\omega_1}+\frac{1}{4\omega_2}\Big)r_1^2 + \\
&\qquad \Big(\frac{1}{4\omega_1}+\frac{3}{2\omega_2}\Big)\frac{\omega_1 r_1^2+F}{\omega_2} + \Big(\frac{1}{32\omega_1}+\frac{1}{4\omega_2}\Big)\sum_{l=3}^{\infty} l^2 r_l(0)^2\Big].
\end{aligned} \tag{34}$$

By introducing the following re-scalings $r_1(t_1) = \sqrt{\frac{F}{\omega_1}}R_1(s_2), \Phi(t_1) = \Psi(s_2)$ with $s_1 = \frac{4\alpha}{9\sqrt{\omega_1\omega_2}}(4\omega_1+\omega_2)t_1$, and $\frac{ds_2}{ds_1} = \frac{1}{R_1\sqrt{R_1^2+1}}$ system (34) becomes:

$$\begin{aligned}
\frac{dR_1}{ds_2} &= R_1(R_1^2+1)\sin(\Psi),\\
\frac{d\Psi}{ds_2} &= (2R_1^2+1)\cos(\Psi) - (k_1R_1^2+k_2)R_1\sqrt{R_1^2+1},
\end{aligned} \tag{35}$$

where $k_i = \frac{9\tilde{P_1}^2\pi^4\sqrt{\omega_1\omega_2}}{4\alpha(4\omega_1+\omega_2)}\bar{k}_i$ for $i = 1, 2$, and $\bar{k}_1 = \Big[\Big(\frac{3}{32\omega_1}+\frac{1}{4\omega_2}\Big) + \Big(\frac{1}{4\omega_1}+\frac{3}{2\omega_2}\Big)\frac{\omega_1}{\omega_2}\Big]\frac{F}{\omega_1}$ and $\bar{k}_2 = \Big(\frac{1}{4\omega_1}+\frac{3}{2\omega_2}\Big)\frac{F}{\omega_2} + \Big(\frac{1}{32\omega_1}+\frac{1}{4\omega_2}\Big)\sum_{l=3}^{\infty} l^2 r_l(0)^2 - \frac{\phi}{\tilde{P_1}^2\pi^4}$. For the same reasons as given in section 4.2 the analysis can be restricted to the case $k_1 \geq 0$ and $-\infty < k_2 < \infty$. Using a similar method as described at the end of section 4.2 a first integral of (35) can be derived, yielding

$$\cos(\Psi) = \frac{1}{R_1\sqrt{R_1^2+1}}\Big[\frac{k_1}{4}R_1^4 + \frac{k_2}{2}R_1^2 + C^{**}\Big], \tag{36}$$

where C^{**} is a constant of integration.

The equilibrium points of system (35) have to satisfy $R_1(R_1^2+1)\sin(\Psi) = 0$ and $(2R_1^2+1)\cos(\Psi) - (k_1R_1^2+k_2)R_1\sqrt{R_1^2+1} = 0$. From the first equation it follows that $R_1 = 0$ or $\Psi = m\pi$ with $m \in \mathbb{Z}$. For $R_1 = 0$ it follows from the second equation that $\cos(\Psi) = 0 \Rightarrow \Psi = \frac{(2n+1)}{2}\pi$ with $n \in \mathbb{Z}$. For $\Psi = m\pi$ it follows from the second equation that

$$(-1)^m(2R_1^2+1) - (k_1R_1^2+k_2)R_1\sqrt{R_1^2+1} = 0. \tag{37}$$

Following the analysis as given in subsection 4.3.1 it can be shown elementarily that

- (i) for m even and fixed there will be always exactly one equilibrium point,
- (ii) for m odd and fixed it is possible to have zero, one, or two equilibrium point(s) depending on the values of k_1 and k_2. In Figure 9 the bifurcation curve in the (k_1, k_2)-plane is given for which one equilibrium point occurs. Also in Figure 9 the regions A-0 and A-2 are given in which zero or two equilibrium points occur respectively.

In Figure 10 some phase portraits of system (35) are given for different values of k_1 and k_2. From these phase portraits and from (36) it can be deduced that R_1 remains bounded,

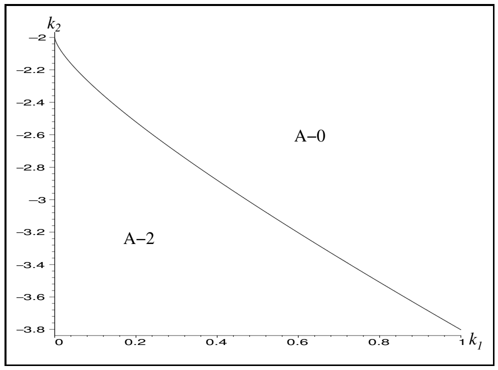
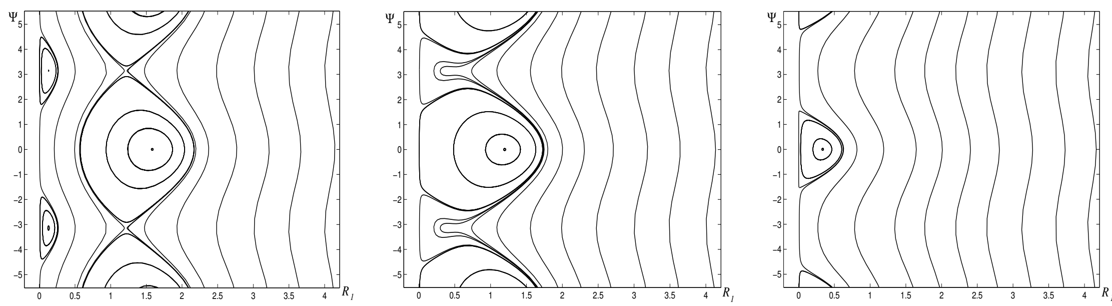


Figure 9: Bifurcation curve in the (k_1, k_2)-plane for the number of critical points of system (35) with $\Psi = m\pi, m$ odd and fixed.



(a) $k_1 = 4, k_2 = -8$ (b) $k_1 = 4, k_2 = -3.657$ (c) $k_1 = 4, k_2 = 3$

Figure 10: Phase portraits of system (35) for different values of k_1 and k_2 (case $K < 0$).

and so, all solutions of the problem with $\Omega = \omega_2 + \omega_1 + \epsilon\phi$ will remain bounded. These results are different from the ones found in the linearized case (see [7]). For the problem under consideration it can be concluded that the nonlinear terms "stabilize" the conveyor belt system.

4.4 The case $\Omega = \omega_3 + \omega_2 + \tilde{\epsilon}\phi$

The linearized problem with $\Omega = \omega_3 + \omega_2 + \tilde{\epsilon}\phi$ has been studied in [7]. It has been shown in [7] that for most parameter values only the second and the third mode will interact through an internal resonance and that for special values of the beam parameters there will be additional interactions. In this section it will be assumed that the beam parameters are such that only an interaction between the second and the third mode occurs due to velocity fluctuations with frequency $\Omega = \omega_3 + \omega_2 + \tilde{\epsilon}\phi$, where ϕ is a detuning parameter. In [7] it has been shown that for the linearized problem instabilities (that is, unbounded solutions) occur. For the nonlinear system (see (8)) with $\Omega = \omega_3 + \omega_2 + \tilde{\epsilon}\phi$ it can again be shown that in order to remove secular terms that A_{k0} and B_{k0} have to satisfy:

$$\dot{A}_{20} = \frac{12\alpha}{25\omega_2}(9\omega_2 + 4\omega_3)[B_{30}\cos(\phi t_1) - A_{30}\sin(\phi t_1)] - \frac{\tilde{P}_1^{\,2}\pi^4}{4\omega_2}B_{20}\Big[2(A_{20}^2 + B_{20}^2) +$$

$$\sum_{l=1}^{\infty} l^2(A_{l0}^2 + B_{l0}^2)\Big],$$

$$\dot{B}_{20} = \frac{12\alpha}{25\omega_2}(9\omega_2 + 4\omega_3)[A_{30}\cos(\phi t_1) + B_{30}\sin(\phi t_1)] + \frac{\tilde{P_1}^2\pi^4}{4\omega_2}A_{20}\Big[2(A_{20}^2 + B_{20}^2) + \sum_{l=1}^{\infty} l^2(A_{l0}^2 + B_{l0}^2)\Big],$$

$$\dot{A}_{30} = \frac{12\alpha}{25\omega_3}(9\omega_2 + 4\omega_3)[B_{20}\cos(\phi t_1) - A_{20}\sin(\phi t_1)] - \frac{9\tilde{P_1}^2\pi^4}{32\omega_3}B_{30}\Big[9(A_{30}^2 + B_{30}^2) + 2\sum_{l=1}^{\infty} l^2(A_{l0}^2 + B_{l0}^2)\Big],$$

$$\dot{B}_{30} = \frac{12\alpha}{25\omega_3}(9\omega_2 + 4\omega_3)[A_{20}\cos(\phi t_1) - B_{20}\sin(\phi t_1)] + \frac{9\tilde{P_1}^2\pi^4}{32\omega_3}A_{30}\Big[9(A_{30}^2 + B_{30}^2) + 2\sum_{l=1}^{\infty} l^2(A_{l0}^2 + B_{l0}^2)\Big]. \tag{38}$$

and $\dot{A}_{k0} = 0$ and $\dot{B}_{k0} = 0$ for $k = 1, 4, 5, 6, \ldots$. By introducing polar coordinates, that is, $A_{k0}(t_1) = r_k(t_1)\sin(\phi_k(t_1))$ and $B_{k0}(t_1) = r_k(t_1)\cos(\phi_k(t_1))$ it follows that system (38) becomes

$$\begin{aligned}
\dot{r}_2 &= \frac{12\alpha}{25\omega_2}(9\omega_2 + 4\omega_3)r_3\sin(\phi_2 + \phi_3 + \phi t_1),\\
\dot{r}_3 &= \frac{12\alpha}{25\omega_3}(9\omega_2 + 4\omega_3)r_2\sin(\phi_2 + \phi_3 + \phi t_1),\\
\dot{\phi_2} &= \frac{12\alpha}{25\omega_2}(9\omega_2 + 4\omega_3)\frac{r_3}{r_2}\cos(\phi_2 + \phi_3 + \phi t_1) - \frac{P_1^2\pi^4}{4\omega_2}\Big(2r_2^2 + \sum_{l=1}^{\infty} l^2 r_l^2\Big),\\
\dot{\phi_3} &= \frac{12\alpha}{25\omega_3}(9\omega_2 + 4\omega_3)\frac{r_2}{r_3}\cos(\phi_2 + \phi_3 + \phi t_1) - \frac{9P_1^2\pi^4}{32\omega_3}\Big(9r_3^2 + 2\sum_{l=1}^{\infty} l^2 r_l^2\Big),
\end{aligned} \tag{39}$$

and $\dot{r}_{k0} = 0$ for $k = 1, 4, 5, 6, \ldots$. It follows from the first two equations in (39) that $\omega_2 r_2 \dot{r}_2 - \omega_3 r_3 \dot{r}_3 = 0$ which leads to the first integral $\omega_2 r_2^2 - \omega_3 r_3^2 = \widetilde{K}$, where $\widetilde{K}$ is a constant of integration.

Now it should be observed that system (39) and system (23) are of the same from. So, the analysis as presented in section 4.3 can be repeated leading to the same conclusions (see the end of section 4.3).

5 Conclusions and remarks

In this paper a weakly nonlinear model describing the transversal vibrations of a conveyor belt with a low and time-varying velocity has been studied. The equations of motion have been derived using Hamilton's principle leading to a system of partial differential equations describing the longitudinal and the transversal displacements of the conveyor belt. Using

Kirchhoff's assumption the system of partial differential equations has been reduced to a single fourth order, weakly nonlinear beam equation, which describes the transversal vibrations of the belt system. In the analysis it has been assumed that the belt moves with a time-varying velocity $V(t) = \tilde{\epsilon}(V_0 + \alpha \sin(\Omega t))$, where $\tilde{\epsilon}, V_0$, and α are constants with $|\alpha| < V_0$ and $0 < \tilde{\epsilon} \ll 1$. The value of $\tilde{\epsilon}$ can be considered to be a measure of the smallness of the belt speed compared to the wave speed. Further it has been assumed that the vertical and the longitudinal displacement are of order $\tilde{\epsilon}$ and of order $\tilde{\epsilon}^2$ respectively, and that $P_0^2 = \frac{EI}{T_0 L^2}$ and $P_1^2 = \frac{EA}{T_0}$ are of order 1 and of order $\frac{1}{\tilde{\epsilon}}$ respectively. Complicated dynamical behaviour of the belt system occurs when the frequency Ω of the belt speed fluctuations is the sum or difference of any two natural frequencies of the belt system with velocity equal to zero. In [7] it has been shown for a linear model that the behaviour of the system will be unstable for frequencies Ω of sum type. In this paper it has been shown for a weakly nonlinear model that the behaviour of the system will always be stable for $\Omega = \omega_2 - \omega_1 + \tilde{\epsilon}\phi$, or $\Omega = \omega_2 + \omega_1 + \tilde{\epsilon}\phi$, or $\Omega = \omega_3 + \omega_2 + \tilde{\epsilon}\phi$, where ϕ is a detuning parameter. It is expected that for other values of Ω the same techniques (as presented in this paper) can be applied to determine the stability properties of the belt system. Finally it should be remarked that other order assumptions on the longitudinal and the vertical displacement, and on P_0^2 and P_1^2 lead to other model equations. These model problems will be the subject for future research.

References

[1] F. Pellicano and F. Vestroni 2000 *Journal of Vibration and Acoustics* **122**, 21-30. Nonlinear Dynamics and Bifurcations of an Axially Moving Beam.

[2] J.A. Wickert 1992 *International Journal of Nonlinear Mechanics* **27**(3), 503-517. Nonlinear Vibrations of a Traveling Tensioned Beam.

[3] J.A. Wickert and C.D. Mote, Jr. 1988 *Journal of the Acoustic Society of America* **84**, 963-969. Linear transverse vibration of an axially moving string-particle system.

[4] G. Lodewijks 1996 *Dynamics of Belt Systems*. PhD Thesis, Delft Universiteitsdrukkerij TU Delft, Delft.

[5] S. Abrate 1992 *Mech. Mach. Theory* **27**(6), 645-659. Vibrations of belts and belt drives.

[6] G. Suweken and W.T. van Horssen 2002 *Journal of Sound and Vibration*. On The Transversal Vibrations of A Conveyor Belt with A Low and Time-Varying Velocity, Part I: The String-Like Case (to be published).

[7] G. Suweken and W.T. van Horssen 2002 *Journal of Sound and Vibration*. On The Transversal Vibrations of A Conveyor Belt with A Low and Time-Varying Velocity, Part II: The Beam-Like Case (submitted).

[8] G. Suweken and W.T. van Horssen 2000 *Mathematical Journal of The Indonesian Mathematical Society* **6**(5), 427-433. On The Transversal Vibrations of A Conveyor Belt.

[9] H.R. Öz and H. Boyaci 2000 *Journal of Sound and Vibration* **236**(2), 259-276. Transverse Vibrations of Tensioned Pipes Conveying Fluid with Time-Dependent Velocity.

[10] H.R. Öz and M. Pakdemirli 1999 *Journal of Sound and Vibration* **227**(2), 239-257. Vibrations of an Axially Moving Beam with Time-Dependent Velocity.

[11] R.G. Parker and Y. Lin 2001 *Journal of Applied Mechanics* **68**, 49-57. Parametric Instability of Axially Moving Media Subjected to Multifrequency Tension and Speed Fluctuations.

[12] H.R. Öz, M. Pakdemirli, and H. Boyaci 2001 *International Journal of Non-linear Mechanics* **36**, 107-115. Non-linear vibrations and stability of an axially moving beam with time-dependent velocity.

[13] R.A. Sack 1954 *British Applied Physics* **5**, 224-226. Transverse oscillations in travelling strings.

[14] F.R. Archibald, and A.G. Emslie 1958 *Applied Mechanics* **25**, 347-348. The vibrations of a string having a uniform motion along its length.

[15] S. Mahalingham 1957 *British Journal of Applied Physics* **8**, 145-148. Transverse vibrations of power transmission chains.

[16] A.L. Thurman and C.D. Mote 1969 *Journal of Applied Mechanics.* 83-91. Free, Periodic, Nonlinear Oscillation of an Axially Moving Strip.

[17] G.J. Boertjens and W.T. van Horssen 2000 *SIAM Journal on Applied Mathematics* **60**, 602-632. An asymptotic theory for a beam equation with a quadratic perturbation.

[18] W.T. van Horssen 1992 *Nonlinear Analysis* **19**,501-530. Asymptotics for a class of semilinear hyperbolic equations with an application to a problem with a quadratic nonlinearity.

[19] A.H. Nayfeh and D.T. Mook 1979 *Nonlinear Oscillations.* New York: Wiley & Sons.

[20] J. Kevorkian and J.D. Cole 1996 *Multiple Scale and Singular Perturbation Methods.* New York: Springer-Verlag.

[21] M. Pakdemirli and A.G. Ulsoy 1999 *Journal of Sound and Vibration* **203**(5), 815-832. Stability analysis of an axially accelerating string.

[22] M. Pakdemirli, A.G. Ulsoy, and A. Ceranoglu 1994 *Journal of Sound and Vibration* **169**(2), 179-196. Transverse vibration of an axially accelerating string.

[23] G.J. Boertjens and W.T. van Horssen 2000 *Journal of Sound and Vibration* **235**(2), 201-217. On interactions of Oscillation modes for a weakly nonlinear undamped elastic beam with an external force.

[24] G.J. Boertjens and W.T. van Horssen 1998 *Nonlinear Dynamics* **17**, 23-40. On mode interactions for a weakly nonlinear beam equation.

[25] W.T. van Horssen and A.H.P. van der Burgh 1988 *SIAM Journal on Applied Mathematics* **48**, 719-736. On initial-boundary value problems for weakly semi-linear telegraph equations. Asymptotic theory and application.