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NONLINEAR OSCILLATOR EQUATION $\ddot{x} + x^{1/(2n+1)} = 0$.

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**ON THE PERIODS OF THE PERIODIC SOLUTIONS OF THE
NONLINEAR OSCILLATOR EQUATION $\ddot{x} + x^{1/(2n+1)} = 0$.**

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In a series of papers Mickens and his co-authors [1]-[3], and Awrejcewicz and Andrianov [4] considered nonlinear oscillator equations of the form

$$\ddot{x} + f(x) = 0 . \quad (1)$$

In particular the case $f(x) = x^{1/(2n+1)}$ with n a positive integer has been studied in [2], [3] and [4]. Using a generalized harmonic balance method (see [1]) approximations of the periodic solutions have been constructed in [2] and [3] in the form

$$x(t) \simeq \frac{A \cos(\omega t)}{1 + B \cos(2\omega t)} , \quad (2)$$

where A , B , and ω are to be determined as functions of the special initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$. The ultimate procedure used in [3] to calculate A , B , and ω is based on the numerical integration of the differential equation subject to $x(0) = x_0$ and $\dot{x}(0) = 0$. The "angular frequency" $\omega_n(x_0)$ of the periodic solution of (1) with $f(x) = x^{1/(2n+1)}$ was approximated in [2] by

$$\omega_n(x_0) \simeq \left[\frac{2^{2n}}{\binom{2n+1}{n} x_0^{2n}} \right]^{1/(4n+2)} . \quad (3)$$

From (3) the period $T_n(x_0)$ of the periodic solution of (1) with $f(x) = x^{1/(2n+1)}$ can be approximated by $\frac{2\pi}{\omega_n(x_0)}$. In [4] the so-called small δ -method has been applied to approximate $T_n(x_0)$.

In this paper an exact, analytical expression for $T_n(x_0)$ will be given, which easily can be approximated numerically (up to any desired accuracy). First of all it should be observed that (1) with $f(x) = x^{1/(2n+1)}$ has as an integrating factor \dot{x} . Using this integrating factor the following first integral is obtained

$$\frac{1}{2}(\dot{x})^2 + \frac{(2n+1)}{(2n+2)}x^{\frac{2n+2}{2n+1}} = c, \quad (4)$$

where c is a non-negative constant of integration. It follows from (4) that all orbits in the phase-plane (that is, in the (x, \dot{x}) -plane) are closed, and are symmetric with respect to the x -axis, and are symmetric with respect to the \dot{x} -axis. So, all solutions of (1) with $f(x) = x^{1/(2n+1)}$ are periodic. Without loss of generality it can be assumed that a periodic solution starts at $t = 0$ in $(x(0), \dot{x}(0)) = (x_0, 0)$ with $x_0 > 0$, and so c in (4) is equal to $\frac{(2n+1)}{(2n+2)}x_0^{\frac{2n+2}{2n+1}}$. Let $T_n(x_0)$ be the period of this periodic solution. Since the orbits in the phase-plane are symmetric with respect to the x -axis and the \dot{x} -axis it follows that $(x(\frac{T_n(x_0)}{2}), \dot{x}(\frac{T_n(x_0)}{2})) = (-x_0, 0)$. From (4) it then follows that

$$\frac{dx(t)}{dt} = \pm \sqrt{\frac{2(2n+1)}{2n+2}} \sqrt{x_0^{\frac{2n+2}{2n+1}} - x(t)^{\frac{2n+2}{2n+1}}},$$

or equivalently

$$\frac{1}{\sqrt{\frac{2n+1}{n+1}} \sqrt{x_0^{\frac{2n+2}{2n+1}} - x^{\frac{2n+2}{2n+1}}}} \frac{dx}{dt} = \pm 1. \quad (5)$$

Then, integrating (5) with respect to t from $t = 0$ to $\frac{T_n(x_0)}{2}$ yields

$$\frac{T_n(x_0)}{2} = \frac{1}{\sqrt{\frac{2n+1}{n+1}}} \int_{-x_0}^{x_0} \frac{1}{\sqrt{x_0^{\frac{2n+2}{2n+1}} - x^{\frac{2n+2}{2n+1}}}} dx,$$

and after introducing the new dimensionless variable $s = \frac{x}{x_0}$ instead of x the period $T_n(x_0)$ of the vibration becomes

$$T_n(x_0) = \frac{4x_0^{\frac{n}{2n+1}}}{\sqrt{\frac{2n+1}{n+1}}} \int_0^1 \frac{1}{\sqrt{1 - s^{\frac{2n+2}{2n+1}}}} ds. \quad (6)$$

To avoid computational difficulties when the integral in (6) is integrated numerically it should be observed that after some integrations by parts this

integral can be rewritten in

$$\int_0^1 \frac{1}{\sqrt{1 - s^{\frac{2n+2}{2n+1}}}} ds = \frac{(3n+2)}{n+1} \int_0^1 \sqrt{1 - s^{\frac{2n+2}{2n+1}}} ds . \quad (7)$$

From (6) and (7) it then follows that the period $T_n(x_0)$ of a periodic solution of $\ddot{x} + x^{1/(2n+1)} = 0$ (with $x(0) = x_0 > 0$, $\dot{x}(0) = 0$, and n a positive integer) is given by

$$T_n(x_0) = \frac{4(3n+2)x_0^{\frac{n}{2n+1}}}{\sqrt{(2n+1)(n+1)}} \int_0^1 \sqrt{1 - s^{\frac{2n+2}{2n+1}}} ds . \quad (8)$$

For $n = 0$ (the harmonic oscillator case) it follows from (8) that $T_0(x_0)$ is equal to the well-known value 2π . For large values of n (and for finite, n -independent, and fixed values of x_0) it also follows that $x_0^{\frac{n}{2n+1}} \rightarrow x_0^{\frac{1}{2}}$ and $\sqrt{1 - s^{\frac{2n+2}{2n+1}}} \rightarrow \sqrt{1 - s}$, and so, it follows from (8) that $T_n(x_0) \rightarrow 4\sqrt{2} x_0^{\frac{1}{2}}$ for $n \rightarrow \infty$. For other values of n the integral in (8) has to be calculated numerically. Using a standard numerical integration routine (as for instance available in the formula manipulation package Maple) the integral in (8) can easily be approximated numerically (up to any desired accuracy). For some values of n approximations of the period $T_n(x_0)$ and approximations of the "angular frequency" $\omega_n(x_0) = \frac{2\pi}{T_n(x_0)}$ are given in Table 1 up to five decimals. Also in Table 1 the approximations of $\omega_n(x_0)$ (as obtained in [2] and [3] by using a harmonic balance/numerical method, and given by (3)) are listed. As can be seen from this table the fractional errors of the approximations as obtained in [2] and [3] for $n \geq 1$ range from approximately 2 to approximately 11 percent. The approximations of $T_n(x_0)$ as obtained in [4] by using the small δ -method can also readily be compared with the accurate (up to 5 decimals) results as given in Table 1.

Finally it should be remarked that the analysis (to obtain periods of periodic solutions) as presented in this paper is not only restricted to a nonlinear oscillator equation (1) with $f(x) = x^{1/(2n+1)}$, but can be extended to more general, nonlinear oscillator equations.

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n	$T_n(x_0)$	$\omega_n(x_0)$	Approximation of $\omega_n(x_0)$ as given in [2] (see also (3))
0	2π	1.00000	1.00000
1	$x_0^{1/3}$ 5.86966	$x_0^{-1/3}$ 1.07045	$x_0^{-1/3}$ 1.04912
2	$x_0^{2/5}$ 5.78495	$x_0^{-2/5}$ 1.08613	$x_0^{-2/5}$ 1.04812
3	$x_0^{3/7}$ 5.74847	$x_0^{-3/7}$ 1.09302	$x_0^{-3/7}$ 1.04405
4	$x_0^{4/9}$ 5.72816	$x_0^{-4/9}$ 1.09689	$x_0^{-4/9}$ 1.04017
5	$x_0^{5/11}$ 5.71522	$x_0^{-5/11}$ 1.09938	$x_0^{-5/11}$ 1.03684
6	$x_0^{6/13}$ 5.70626	$x_0^{-6/13}$ 1.10110	$x_0^{-6/13}$ 1.03403
7	$x_0^{7/15}$ 5.69968	$x_0^{-7/15}$ 1.10238	$x_0^{-7/15}$ 1.03164
8	$x_0^{8/17}$ 5.69465	$x_0^{-8/17}$ 1.10335	$x_0^{-8/17}$ 1.02960
9	$x_0^{9/19}$ 5.69067	$x_0^{-9/19}$ 1.10412	$x_0^{-9/19}$ 1.02783
10	$x_0^{10/21}$ 5.68745	$x_0^{-10/21}$ 1.10474	$x_0^{-10/21}$ 1.02628
50	$x_0^{50/101}$ 5.66322	$x_0^{-50/101}$ 1.10947	$x_0^{-50/101}$ 1.00919
500	$x_0^{500/1001}$ 5.65750	$x_0^{-500/1001}$ 1.11059	$x_0^{-500/1001}$ 1.00149
∞	$x_0^{1/2}$ $4\sqrt{2}$	$x_0^{-1/2}$ $\frac{\pi}{2\sqrt{2}}$	$x_0^{-1/2}$ 1.00000

Table 1: The period $T_n(x_0)$ and the "angular frequency" $\omega_n(x_0)$ accurate up to five decimal places, and the approximations of $\omega_n(x_0)$ as given in [2].