On A Characteristic Layer Problem For A Weakly Damped String

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Abstract. In this paper initial-boundary value problems for a string (a wave) equation are considered. One end of the string is assumed to be fixed and the other end of the string is attached to a dashpot system, where the damping generated by the dashpot system is assumed to be small, and is assumed to be proportional to the vertical and the angular velocity of the string in the endpoint. This problem can be regarded as a rather simple model describing oscillations of flexible structures such as for instance overhead power transmission lines. A semigroup approach will be used to prove the wellposedness of the singularly perturbed problem. It will be shown how a multiple scales perturbation method can be used effectively to construct asymptotic approximations of the solution on long timescales. Based on these asymptotic results the effectiveness of the dashpot system is discussed.

Keywords: Characteristic layer, boundary damping, asymptotics, perturbation method.

1. Introduction

There are examples of flexible structures such as suspension bridges and overhead transmission lines that are subjected to oscillations due to different causes. Simple models which describe these oscillations can be expressed in initial-boundary value problems for wave equations like in [1, 4–6, 10–13, 16, 17] or for beam equations like in [2, 3, 9, 14, 15, 24]. To suppress the oscillations various types of boundary damping can be used (such as described in [4, 6, 14–18]). For instance in [4] a spring-mass-dashpot system at one end of a string has been applied. In [4] it has been assumed that the damping generated by the dashpot is only proportional to the vertical velocity of the string in the endpoint, and it has been shown in [4] that the energy of the string decays to zero, but does not decay uniformly to zero. When the damping generated by the dashpot is also proportional to the angular velocity in the endpoint it has been shown in [6] that for certain values of the dashpot-parameters the energy decays uniformly to zero. How the oscillations of the string tend to zero remains an open problem in [6].

In most cases simple, classical boundary conditions are applied (such as in [1–3, 10, 12, 13, 24]) to construct approximations of the oscillations. For more complicated, non-classical boundary conditions (see for instance [6, 11, 14–18]) it is usually not possible to construct explicit approximations of the oscillations. In this paper, we will study such an initial- boundary value problem with a non-classical boundary condition and we will construct explicit asymptotic approximations of the solution, which are valid on a long time-scale. We will consider a string which is fixed at $x = 0$ and attached to a dashpot system at $x = \pi$.

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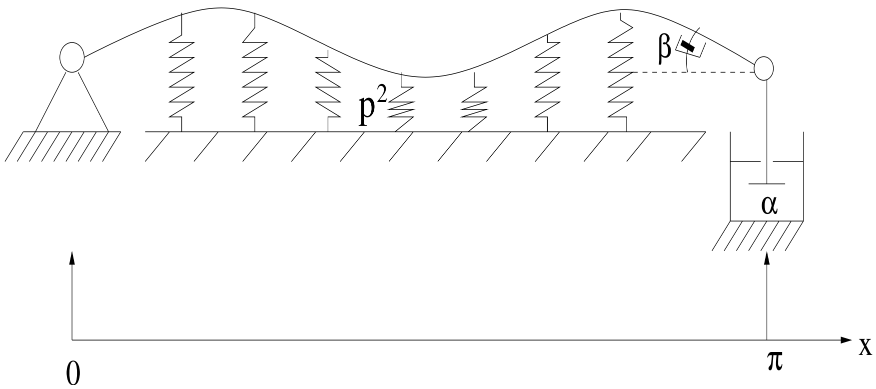
To derive a model for flexible structures such as suspension bridges or overhead transmission lines we refer to [2, 3, 12, 13]. It is assumed that l (the length of the string), ρ (the mass-density of the string per unit length), T (the tension in the string), and $\tilde{\alpha}$, $\tilde{\beta}$ (the damping coefficients of the dashpot which are assumed to be small), and k (for instance the stiffness of the stays of the bridge) are all positive constants. Furthermore, we only consider the vertical displacement $\tilde{u}(\tilde{x}, \tilde{t})$ of the string, where $\tilde{x}$ is the place along the string, and $\tilde{t}$ is time.



*F*igure 1. A simple model for a flexible structure.

After applying a simple rescaling in time, space, and in displacement ($t = \sqrt{\frac{T}{\rho}}\frac{\pi}{l}\tilde{t}, x = \frac{\pi}{l}\tilde{x}$, $\tilde{u}(\tilde{x}, \tilde{t}) = u(x, t)$), putting $\tilde{\beta} = \epsilon\left(\sqrt{T\rho}\frac{l}{\pi}\right)\beta$, $\tilde{\alpha} = \epsilon\sqrt{T\rho}\alpha$, and $p^2 = (l/\pi)^2\frac{k}{T}$) we obtain as a simple model for the oscillation of the string the following initial-boundary value problem for $u = u(x, t)$:

$$\begin{aligned}
u_{tt} - u_{xx} + p^2 u &= 0,\ 0 < x < \pi,\ t > 0, &(1)\\
u(0, t) &= 0,\ t \geq 0, &(2)\\
u_x(\pi, t) &= -\epsilon\left(\beta u_{xt}(\pi, t) + \alpha u_t(\pi, t)\right),\ t \geq 0, &(3)\\
u(x, 0) &= \phi(x),\ 0 < x < \pi, &(4)\\
u_t(x, 0) &= \psi(x),\ 0 < x < \pi, &(5)
\end{aligned}$$

where ϵ is a small, positive parameter with $0 < \epsilon \ll 1$, where α and β are positive constants, $p^2 \geq 0$, and where ϕ and ψ are the initial displacement and the initial velocity of the string respectively. The functions ϕ and ψ are assumed to be sufficiently smooth and to be of order one.

By using a semigroup approach (as described in [8, 22]) the well-posedness of problem (1) - (5) will be proved for all $t > 0$. The presence of the term u_{xt} in the boundary condition at $x = \pi$ will give rise to a singularly perturbed problem. In fact, it will turn out that a characteristic layer near $x = \pi$ will play an important role in the construction of an approximation of the exact solution. To construct formal approximations of the solution of the singularly perturbed problem the method of multiple scales or that of matched asymptotic expansions can be used. However, when the method of matched asymptotic expansions (as for instance described in [7, 19–21, 23, 24]) is used at least two (inner and outer) expansions(which must be matched and assumed to have an overlap domain) have to be generated. Usually it is extremely difficult to give a justification of the asymptotic results when the method of matching is applied to problems for partial differential equations. When the method of multiple scales is used the problem with matching can more or less be avoided. Moreover, the asymptotic validity of the results can be obtained more easy as will be shown in this paper. So for that reason we will use the method of multiple scales to construct asymptotic approximations of the solution of the initial - boundary value problem (1) - (5).

This paper is organized as follows. In section 2 we will show that the solution of the initial-boundary value problem (1) - (5) is bounded. In section 3 we will prove the well-posedness of the problem for all $t \geq 0$. In section 4 a formal approximation of the solution of the problem will be constructed explicitly, and in section 5 the asymptotic validity of the approximations will be proved on a long time-scale. Finally, in section 6 some remarks will be made and some conclusions will be drawn.

2. On the boundedness of solutions

In this section we will show that the solution $u(x,t)$ is bounded if an expression which is related to the initial energy is bounded. By multiplying the PDE (1) with u_t and then by integrating the so-obtained equation with respect to x from 0 to π and by taking into account the boundary conditions (2) and (3) we obtain

$$\frac{\partial}{\partial t}\left[\int_0^\pi \left(u_t^2(x,t)+u_x^2(x,t)+p^2u^2(x,t)\right)dx+\frac{\beta}{\alpha}u_x^2(\pi,t)\right]=-\frac{2}{\varepsilon\alpha}u_x^2(\pi,t), \tag{6}$$

where α and β are positive constants. By integrating (6) with respect to t from 0 to t we then obtain

$$\begin{aligned}\rho(t) &\equiv \int_0^\pi \left(u_t^2(x,t)+u_x^2(x,t)+p^2u^2(x,t)\right)dx+\frac{\beta}{\alpha}u_x^2(\pi,t)\\ &= \rho(0)-\frac{2}{\varepsilon\alpha}\int_0^t u_x^2(\pi,s)ds \leq \rho(0).\end{aligned} \tag{7}$$

Note that $\rho(t) \geq 0$. By using the Cauchy-Schwarz inequality it now follows that

$$|u(x,t)| = \left|\int_0^x u_s(s,t)ds\right| \leq \sqrt{\pi\int_0^\pi u_s^2(s,t)ds} \leq \sqrt{\pi\rho(t)} \leq \sqrt{\pi\rho(0)}. \tag{8}$$

And so, $u(x,t)$ is bounded if the initial energy of the string and $u_x(\pi,0)$ are bounded.

3. The well-posedness of the problem.

In this section we will show that the initial - boundary value problem (1) - (5) with $0 < \varepsilon \ll 1$ is well-posed for all $t > 0$. To show the well-posedness we will use a semigroup approach (as for instance described in [8, 22]). For that purpose we define the following auxiliary functions

$$a(t) = u(\bullet, t), \quad b(t) = u_t(\bullet, t), \quad \eta(t) = \varepsilon\beta u_x(\pi, t). \tag{9}$$

We note that these auxiliary functions are functionals which map $[0, \pi]$ into $\Re$. For simplicity, we denote a, b, η for $a(t), b(t), \eta(t)$ respectively. By differentiating these functions with respect to t we obtain

$$\begin{pmatrix} a_t \\ b_t \\ \eta_t \end{pmatrix} = \begin{pmatrix} b \\ a_{xx} - p^2 a \\ -a_x(\pi) - \varepsilon\alpha b(\pi) \end{pmatrix}. \tag{10}$$

Next, we define the following function spaces:

$$\mathcal{V} := \{a \in H^1[0, \pi],\, a(0) = 0\}, \quad \mathcal{H} := \{y(t) = (a, b, \eta) \in \mathcal{V} \times L^2[0, \pi] \times \Re\}. \tag{11}$$

We equip the space $\mathcal{H}$ with the inner product $\langle \cdot\, , \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \Re$ which is defined by

$$\langle y\, , \tilde{y} \rangle := \varepsilon^2\beta^2 \int_0^\pi (a_x\tilde{a}_x + b\tilde{b} + p^2 a\tilde{a})dx + \frac{\beta}{\alpha}\eta\tilde{\eta}, \tag{12}$$

where $y = (a, b, \eta)$ and $\tilde{y} = (\tilde{a}, \tilde{b}, \tilde{\eta})$ are in $\mathcal{H}$. Observe that this inner product is based upon a function which is related to the energy of the string (see also (7)). The space $\mathcal{H}$ together with the inner product $\langle \cdot\, , \cdot \rangle$ is a Hilbert space. Next we define the unbounded operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ by

$$Ay(t) := \begin{pmatrix} b \\ a_{xx} - p^2 a \\ -a_x(\pi) - \varepsilon\alpha b(\pi) \end{pmatrix}, \; y \in D(A), \tag{13}$$

where $D(A) := \{y(t) = (a, b, \eta) \in (H^2[0, \pi] \cap \mathcal{V}) \times \mathcal{V} \times \Re;\, \eta = \varepsilon\beta a_x(\pi)\}$. Using (13) it then follows that (10) can be rewritten in the abstract Cauchy problem

$$\frac{dy}{dt} = \dot{y} \;=\; Ay, \tag{14}$$

$$y(0) \;=\; \Phi = \begin{pmatrix} \phi \\ \varphi \\ \eta(0) \end{pmatrix}. \tag{15}$$

THEOREM 1. *The operator $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$ defined by (13) generates a C_o semigroup of contractions $T(t)$ on the space $\mathcal{H}$.*

Proof. According to the Lumer-Philips theorem (see [8], p. 26) it is sufficient to show that the operator A is a $m-$*dissipative* operator and that the domain $D(A)$ is dense in $\mathcal{H}$. After a straightforward computation we obtain

$$\langle Ay, y \rangle = -\frac{\varepsilon\beta^2}{\alpha}a_x^2. \tag{16}$$

It should also be observed that the equation

$$(I - A)\, y = 0, \tag{17}$$

is solvable in $\mathcal{H}$. From (16) - (17) it then follows that A is $m-dissipative$ operator. To prove that $D(A)$ is dense in $\mathcal{H}$ it is sufficient to show that $D(A)^{\perp} = \{0\}$(see [22], p.186). From the definition of $\mathcal{H}$ equipped with the inner product (12) it easily follows that $D(A)$ is dense in $\mathcal{H}$. $\square$

If A is a linear operator on $\mathcal{H}$ generating the C_o semigroup $T(t)$ and if the function y_o is in $D(A)$ then we can show that T$(t)y_o$ is in $D(A)$. Moreover, we have the following lemma.

LEMMA 1. *Let A be the infinitesimal generator of the C_o semigroup $T(t)$. Then for any $f \in D(A)$ we have $T(t)f \in D(A)$ and the function $[0,\infty) \ni t \mapsto T(t)f \in \mathcal{H}$ is differentiable. In fact,* $\frac{d}{dt}T(t)f = AT(t)f = T(t)Af$.

Proof. See [22] p. 398.

For $y_o \in D(A)$ we define

$$(a, b, \eta) = y(t) := T(t)y_o. \tag{18}$$

Applying the lemma we find that

$$a \in C^2\left(\Re^+; L^2[0,\pi]\right) \cap C\left(\Re^+; V \cap H^2(0,\pi)\right). \tag{19}$$

So $y(t) = T(t)y_o$ is a strong solution of (14) - (15) for all $y_o \in D(A)$. But for $y_o \in D(A^2)$, applying the lemma twice we have

$$\begin{pmatrix} a_{tt} \\ b_{tt} \\ \eta_{tt} \end{pmatrix} = A \begin{pmatrix} b \\ a_{xx} - p^2 a \\ -a_x(\pi) - \varepsilon\alpha b(\pi) \end{pmatrix} = \begin{pmatrix} a_{xx} - p^2 a \\ b_{xx} \\ -b_x(\pi) - \varepsilon\alpha\left(a_{xx}(\pi) - p^2 a(\pi)\right) \end{pmatrix}. \tag{20}$$

From (20) and the definition of $D(A)$ we obtain

$$t \longmapsto a(t) \in C^1\left(\Re^+, H^2 \cap V\right), \text{and} \quad t \longmapsto a_{xx} - p^2 a \in V. \tag{21}$$

On the other hand, we also have

$$t \longmapsto a(t) \in C^2\left(\Re^+, V\right). \tag{22}$$

From (21) - (22), it follows that

$$a \in C^2\left(\Re^+; V\right) \cap C^1\left(\Re^+; V \cap H^2(0,\pi)\right) \cap C\left(\Re^+; H^3(0,\pi) \cap V\right). \tag{23}$$

So from (18) and (23), for all $y_o \in D(A^2)$, $y(t) = T(t)y_o$, we have the equivalence between problem (1) - (5) and problem (14) - (15). So, the following theorem has now been established.

THEOREM 2. *Let $\Phi \in D(A^2)$, then the initial - boundary value problem (1) - (5) and the initial value problem (14) - (15) are equivalent.*

Next, we will show that the solution of the initial - boundary value problem (1) - (5) depends continuously on the initial values. Let $\hat{y}(t)$ satisfy (14) with the initial values $\hat{y}(0) = \hat{\Phi}$, where $\hat{\Phi} = \begin{pmatrix} \hat{\phi} \\ \hat{\psi} \\ \hat{\eta}(0) \end{pmatrix}$, $(\hat{\phi}, \hat{\psi}) \in (C^2[0,\pi] \cap \mathcal{V}) \times \mathcal{V}$. Now we approximate the difference between $y(t)$ and $\hat{y}(t)$ as follows: $\|y(t) - \hat{y}(t)\|_{\mathcal{H}} \leq \left\|T(t)(\Phi - \hat{\Phi})\right\|_{\mathcal{H}} \leq \left\|\Phi - \hat{\Phi}\right\|_{\mathcal{H}}$ for all $t \geq 0$. This means that small differences between the initial values cause small differences between the solutions $y(t)$ and $\hat{y}(t)$ for all $t \geq 0$. We observe that if we take $\phi(x) \in H^3(0,\pi)$, $\phi(0) = \phi''(0) = 0$ and $\psi(x) \in H^2(0,\pi) \cap \mathcal{V}$ then we have Φ in the domain A^2. So, we can now formulate the following theorem on the well-posedness of the initial-boundary value problem (1) - (5).

THEOREM 3. *Suppose* $\phi(x) \in H^3(0,\pi)$, $\phi(0) = \phi''(0) = 0$ *and* $\psi(x) \in H^2(0,\pi) \cap \mathcal{V}$, $\psi(0)=0$, *then problem* (1) - (5) *has a unique and twice continuously differentiable solution for* $x \in [0,\pi]$ *and* $t \geq 0$. *Moreover, this solution depends continuously on the initial values.*

It should be observed from (3) that it is necessary to assume additionally that $\phi'(\pi)$ is of order ϵ. This assumption will influence the calculations in the next section.

4. The construction of a formal approximation

In this section a formal approximation of the solution of the initial-boundary value problem (1)-(5) will be constructed for arbitrary α and β. For $\alpha = 0$ an exact solution can be constructed as will be shown in section 4.1. This exact solution for $\alpha = 0$ gives a good indication what scalings are necessary to construct approximations of the solution of the initial-boundary value problem (1)-(5) for arbitrary α and β. Using a multiple scales perturbation method these approximations of the solution will be constructed in section 4.2 for arbitrary α and β. The asymptotic validity of these approximations on long time-scales will be proved in section 5. In section 4.3 we will present shortly an alternative way to approximate the solution by applying the method of separation of variables to the problem. This approach of course can only be applied to linear problems, whereas the method as presented in section 4.2 can also be applied to weakly nonlinear problems.

4.1. The case $\alpha = 0$.

Using the method of separation of variables the exact solution of the initial-boundary value problem (1)-(5) with $\alpha = 0$ will be constructed in this section. Firstly a nontrivial solution in the form $X(x)T(t)$ of the boundary value problem (1)-(3) will be constructed. Substituting $X(x)T(t)$ into (1)-(3) it follows that

$$\frac{X''}{X} = \frac{T''}{T} + p^2 = -\lambda, \tag{24}$$

$$X(0) = 0, \tag{25}$$

$$X'(\pi)(T(t) + \epsilon\beta T'(t)) = 0 \Leftrightarrow \; X'(\pi) = 0 \quad \text{or} \tag{26}$$

$$T(t) + \epsilon\beta T'(t) = 0, \tag{27}$$

where λ is a separation parameter. The problem (24)-(26) is in fact a classical boundary value problem with a Dirichlet condition at $x = 0$ and a Neumann condition at $x = \pi$. This problem can readily be solved, and a nontrivial solution of the initial-boundary problem (1)-(3) is

$$\left(A_n \cos(\sqrt{(n-1/2)^2 + p^2}t) + B_n \sin(\sqrt{(n-1/2)^2 + p^2}t)\right) \sin((n-1/2)x) \tag{28}$$

with $n = 1, 2, 3, \cdots$, and A_n and B_n arbitrary constants. The problem (24), (25), and (27) can be solved in the following way. From (27) it follows that $T(t) = T(0) \exp\left(\frac{-t}{\epsilon\beta}\right)$ and

$$T'(t) = -\frac{1}{\epsilon\beta}T(t) \Rightarrow T''(t) = -\frac{1}{\epsilon\beta}T'(t) = \frac{1}{\epsilon^2\beta^2}T(t). \tag{29}$$

Substituting (29) into (24) yields

$$\frac{X''}{X} = \frac{1}{\epsilon^2\beta^2} + p^2 \Rightarrow X(x) = A\sinh(\sqrt{1 + (\epsilon\beta p)^2}\frac{x}{\epsilon\beta}) + B\cosh(\sqrt{1 + (\epsilon\beta p)^2}\frac{x}{\epsilon\beta}) \tag{30}$$

with A and B arbitrary constants. From (25) it follows that $B = 0$. So, a nontrivial solution of the boundary value problem (1)-(3) is also

$$A\sinh(\sqrt{1 + (\epsilon\beta p)^2}\frac{x}{\epsilon\beta}), \tag{31}$$

where A is an arbitrary constant. The general solution of the boundary value problem (1)-(3) then follows from (28) and (31), yielding

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty}(A_n \cos(\sqrt{(n-1/2)^2 + p^2}t) + B_n \sin(\sqrt{(n-1/2)^2 + p^2}t)) \sin((n-1/2)x) \\ &\quad + A\sinh(\sqrt{1 + (\epsilon\beta p)^2}\frac{x}{\epsilon\beta}) \exp\left(\frac{-t}{\epsilon\beta}\right), \end{aligned} \tag{32}$$

where A_n, B_n, and A are constants which are determined by the initial conditions (4) and (5). The constant A follows from (4) and (32), that is, from

$$u_x(\pi, 0) = \phi'(\pi) \quad \Rightarrow \quad A = \frac{\epsilon\beta\phi'(\pi)}{\sqrt{1 + (\epsilon\beta p)^2}\cosh(\sqrt{1 + (\epsilon\beta p)^2}\pi/(\epsilon\beta))}. \tag{33}$$

The constants A_n and B_n then easily follow from (4), (5), (32), and (33), yielding

$$A_n = \frac{2}{\pi}\left(\int_0^{\pi}\phi(x)\sin\left((n-\frac{1}{2})x\right)dx - \frac{(-)^n(\epsilon\beta)^2\phi'(\pi)}{1 + (\epsilon\beta p)^2 + (n-\frac{1}{2})^2(\epsilon\beta)^2}\right), \tag{34}$$

$$B_n = \frac{2}{\pi\sqrt{(n-\frac{1}{2})^2 + p^2}}\left(\int_0^{\pi}\psi(x)\sin\left((n-\frac{1}{2})x\right)dx + \frac{(-)^n(\epsilon\beta)\phi'(\pi)}{1 + (\epsilon\beta p)^2 + (n-\frac{1}{2})^2(\epsilon\beta)^2}\right). \tag{35}$$

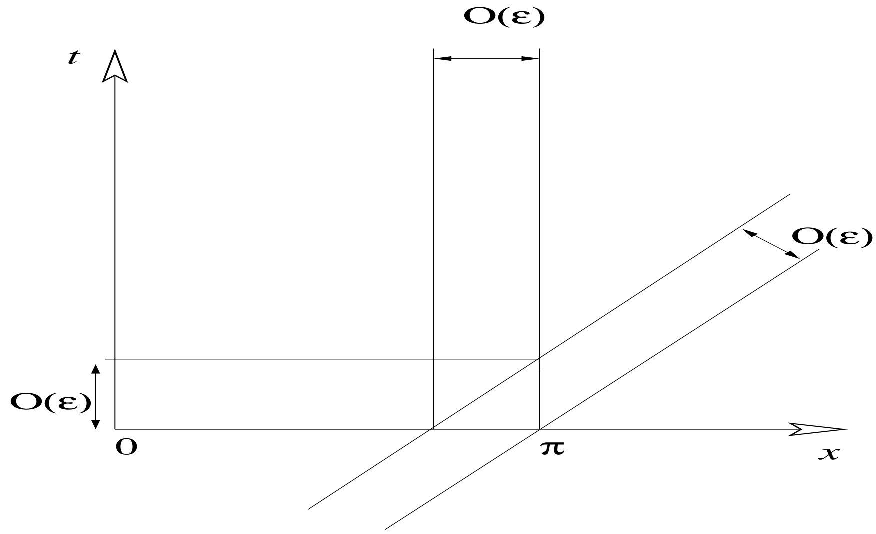


Figure 2. A characteristic layer $x - t - \pi = O(\epsilon)$ in the neighborhood of $x = \pi$ and $t = 0$.

From (32) and (33) it can be seen that the angular velocity damper only plays a significant role in a very small neighborhood of the boundary at $x = \pi$. The damper induces a small characteristic layer $x - t - \pi = O(\epsilon)$(see also figure 2) in which the angle of the string tends to zero in a very short time. Outside this layer the vibrations are undamped when $\alpha = 0$.

4.2. THE CASE $\alpha > 0$ AND $\beta > 0$.

Using the multiple scales perturbation method an approximation of the solution of the initial-boundary value problem (1)-(5) with $\alpha > 0$ and $\beta > 0$ will be constructed in this section. To describe the characteristic layer near $x = \pi$ correctly it follows from section 4.1 that the following scalings are needed: $\bar{x} = \frac{x}{\epsilon}$, x, $\bar{t} = \frac{t}{\epsilon}$, and t. The single term $-\epsilon\alpha u_t$ in boundary condition (3) does not lead to a singularly perturbed problem, that is, for this term the following scalings $x, t, \tau = \epsilon t, \mu = \epsilon^2 t, \cdots$ are usually needed to approximate the solution correctly for large values of t (see also [1–5, 12, 13]). Based on these observations it is reasonable to put $u(x,t;\varepsilon) = v(\bar{x}, x, \bar{t}, t, \tau, \mu; \varepsilon)$, and to assume that the function v can be approximated by a formal expansion in ϵ, that is, by $u_o + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$, where each u_i is a function of $\bar{x}, x, \bar{t}, t, \tau$, and μ, and where each u_i and its derivatives are of order one. By substituting the formal expansion into the initial - boundary value problem (1) - (5), and by taking terms of equal power in ϵ together the following problems are obtained

$$\begin{aligned} u_{i_{\bar{t}\bar{t}}} - u_{i_{\bar{x}\bar{x}}} &= 2\left(u_{(i-1)_{\bar{x}x}} - u_{(i-1)_{\bar{t}t}}\right) - u_{(i-2)_{tt}} + u_{(i-2)_{xx}} - p^2 u_{(i-2)} - 2u_{(i-2)_{\bar{t}\tau}} \quad (36)\\ &\quad -2\left(u_{(i-3)_{\bar{t}\mu}} + u_{(i-3)_{t\tau}}\right) - \left(2u_{(i-4)_{t\mu}} + u_{(i-4)_{\tau\tau}}\right) - 2u_{(i-5)_{\tau\mu}} - u_{(i-6)_{\mu\mu}}, \\ &\quad 0 < x < \pi,\ t > 0, \\ u_i(\bar{x}, x, \bar{t}, t, \tau, \mu) &= 0,\ x = 0,\ \bar{x} = 0,\ t \geq 0, \quad (37) \end{aligned}$$

$$\begin{aligned} u_{i_{\bar{x}}} + \beta u_{i_{\bar{x}\bar{t}}} &= -u_{(i-1)_x} - \beta u_{(i-1)_{\bar{x}t}} - \beta u_{(i-1)_{x\bar{t}}} - \alpha u_{(i-1)_{\bar{t}}} - \beta u_{(i-2)_{\bar{x}\tau}} - \beta u_{(i-2)_{xt}} \quad (38)\\ &\quad -\alpha u_{(i-2)_t} - \beta u_{(i-3)_{\bar{x}\mu}} - \beta u_{(i-3)_{x\tau}} - \alpha u_{(i-3)_\tau} - \beta u_{(i-4)_{x\mu}} - \alpha u_{(i-4)_\mu},\\ &\quad x = \pi,\ t \geq 0,\\ u_i(\bar{x}, x, 0, 0, 0, 0) &= \phi_i(x),\ 0 < x < \pi, \quad (39)\\ u_{i_{\bar{t}}}(\bar{x}, x, 0, 0, 0, 0) &= \psi_{i-1}(x) - u_{(i-1)_t}(\bar{x}, x, 0, 0, 0, 0) - u_{(i-2)_\tau}(\bar{x}, x, 0, 0, 0, 0)\\ &\quad -u_{(i-3)_\mu}(\bar{x}, x, 0, 0, 0, 0),\ 0 < x < \pi, \quad (40) \end{aligned}$$

for $i = 0, 1, 2, 3, \cdots$, and where it is assumed that $\phi(x) = \phi_o(x) + \epsilon\phi_1(x) + \cdots$, and $\psi(x) = \psi_o(x) + \epsilon\psi_1(x) + \cdots$. For a negative index the function u_i is defined to be identically equal to zero. For instance, $u_{-1} \equiv 0, u_{-2_t} \equiv 0, u_{-2_{\bar{x}\tau}} \equiv 0$, and so on. From (3) for $t \downarrow 0$ it follows that $\phi'(\pi) = -\epsilon\beta\psi'(\pi) - \epsilon\alpha\psi(\pi)$, and so $\phi'(\pi)$ is of order ϵ. To describe the characteristic layer sufficiently accurate it follows from (32) and (33) that an approximation has to be constructed at least up to order ϵ^2. For that reason a secular free approximation $\bar{u} = u_0 + \epsilon u_1 + \epsilon^2 u_2$ will be constructed in this section. From (36)-(40) it follows that $u_0(\bar{x}, x, \bar{t}, t, \tau, \mu)$ has to satisfy

$$\begin{aligned} u_{0_{\bar{t}\bar{t}}} - u_{0_{\bar{x}\bar{x}}} &= 0, \quad 0 < x < \pi,\ t > 0, \quad (41)\\ u_0(\bar{x}, x, \bar{t}, t, \tau) &= 0,\ x = 0,\ \bar{x} = 0,\ t \geq 0, \quad (42)\\ u_{0_{\bar{x}}} + \beta u_{0_{\bar{x}\bar{t}}} &= 0, \quad x = \pi,\ t \geq 0, \quad (43)\\ u_0(\bar{x}, x, 0, 0, 0, 0) &= \phi_o(x),\ 0 < x < \pi, \quad (44)\\ u_{0_{\bar{t}}}(\bar{x}, x, 0, 0, 0, 0) &= 0,\ 0 < x < \pi. \quad (45) \end{aligned}$$

To solve (41)-(45) the method of separation of variables will be used. Firstly nontrivial solutions of the boundary-value problem (41)-(43) will be constructed by separating space and time variables in the form $X(\bar{x}, x)T(\bar{t}, t, \tau, \mu)$. By substituting $X(\bar{x}, x)T(\bar{t}, t, \tau, \mu)$ into (41)-(43) the following eigenvalue problem is obtained

$$\begin{aligned} \frac{T_{\bar{t}\bar{t}}(\bar{t}, t, \tau\mu)}{T(t, t, \tau, \mu)} = \frac{X_{\bar{x}\bar{x}}(\bar{x}, x)}{X(\bar{x}, x)} &= -\sigma^2, \quad 0 < x < \pi,\ t > 0, \quad (46)\\ X(0, 0) &= 0, \quad (47)\\ X_{\bar{x}}(\tfrac{\pi}{\epsilon}, \pi)\,(T + \beta T_{\bar{t}}) = 0 &\Leftrightarrow\ X_{\bar{x}}(\tfrac{\pi}{\epsilon}, \pi) = 0 \quad \text{or} \quad (48)\\ &\qquad T + \beta T_{\bar{t}} = 0, \quad t \geq 0, \quad (49) \end{aligned}$$

where $\sigma \in \mathbb{C}$ is a separation parameter. It should be observed that this parameter is ϵ-independent since all ϵ-dependence is already included in the variables $\bar{x}, x, \bar{t}, t, \tau$ and μ, and in the formal expansion $u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots$ for u. Nontrivial solutions of the boundary-value problem (46)-(48) for $X(\bar{x}, x)$ are given by

$$a_0(x)\bar{x} + b_0(x) \quad \text{and} \quad \tilde{A}_{0\sigma}(x)\cos(\sigma\bar{x}) + \tilde{B}_{0\sigma}(x)\sin(\sigma\bar{x}), \quad (50)$$

for $\sigma = 0$, and for $\sigma \neq 0$ respectively. From (47) and (48) it follows that

$$b_0(0) = \tilde{A}_{0\sigma}(0) = 0, \quad \text{and} \quad a_0(\pi) = -\tilde{A}_{0\sigma}(\pi)\sin(\sigma\tfrac{\pi}{\epsilon}) + \tilde{B}_{0\sigma}(\pi)\cos(\sigma\tfrac{\pi}{\epsilon}) = 0 \quad (51)$$

respectively. Since σ is ϵ-independent it then follows that $\tilde{A}_{0\sigma}(\pi) = \tilde{B}_{0\sigma}(\pi) = 0$. And so, nontrivial solutions of the boundary value problem (46)-(48) for $X(\bar{x}, x)$ are given by (50) with

$$b_0(0) = \tilde{A}_{0\sigma}(0) = 0, \quad \text{and} \quad a_0(\pi) = \tilde{A}_{0\sigma}(\pi) = \tilde{B}_{0\sigma}(\pi) = 0. \tag{52}$$

Nontrivial solutions of the problem (46) for $T(\bar{t}, t, \tau, \mu)$ are given by

$$c_0(t,\tau,\mu)\bar{t} + d_0(t,\tau,\mu) \quad \text{and} \quad \tilde{C}_{0\sigma}(t,\tau,\mu)\cos(\sigma\bar{t}) + \tilde{D}_{0\sigma}(t,\tau,\mu)\sin(\sigma\bar{t}), \tag{53}$$

for $\sigma = 0$, and for $\sigma \neq 0$ respectively. Another nontrivial solution follows from (49), that is, $T(\bar{t}, t, \tau, \mu) = \tilde{T}(t, \tau, \mu)\exp\left(-\frac{\bar{t}}{\beta}\right)$. From (46) it then follows that $X_{\bar{x}\bar{x}} = (1/\beta^2)\, X$, and so

$$X(\bar{x}, x) = \tilde{A}_0(x)\exp\left(\frac{\bar{x}}{\beta}\right) + \tilde{B}_0(x)\exp\left(\frac{-\bar{x}}{\beta}\right). \tag{54}$$

From (47) it follows that

$$\tilde{A}_0(0) + \tilde{B}_0(0) = 0. \tag{55}$$

Obviously u_0 can not contain secular terms. For that reason we have to take $a_0(x) \equiv 0$ in (50), and $c_0(t,\tau,\mu) \equiv 0$ in (53). By adding up all nontrivial, non-secular solutions $X(\bar{x}, x)T(\bar{t}, t, \tau, \mu)$ as given by (50)-(54) it follows that the general solution of the boundary value problem (41)-(43) for u_0 is given by

$$\begin{aligned} u_0(\bar{x}, x, \bar{t}, t, \tau, \mu) = & \sum_{\substack{\sigma \\ \sigma\neq 0 \\ \sigma^2 \neq \frac{-1}{\beta^2}}} ((A_{0\sigma}\cos(\sigma\bar{x})\cos(\sigma\bar{t}) + B_{0\sigma}\cos(\sigma\bar{x})\sin(\sigma\bar{t}) + C_{0\sigma}\sin(\sigma\bar{x})\cos(\sigma\bar{t}) \\ & + D_{0\sigma}\sin(\sigma\bar{x})\sin(\sigma\bar{t}))) + \left(y_0\exp\left(\frac{\bar{x}}{\beta}\right) + z_0\exp\left(\frac{-\bar{x}}{\beta}\right)\right)\exp\left(-\frac{\bar{t}}{\beta}\right) \\ & + v_0(x, t, \tau, \mu), \end{aligned} \tag{56}$$

where $A_{0\sigma}, B_{0\sigma}, C_{0\sigma}, D_{0\sigma}, y_0$, and z_0 are functions of x, t, τ, and μ, satisfying the following conditions

$$A_{0\sigma}(0,t,\tau,\mu) = B_{0\sigma}(0,t,\tau,\mu) = y_0(0,t,\tau,\mu) + z_0(0,t,\tau,\mu) = v_0(0,t,\tau,\mu) = 0, \tag{57}$$
$$A_{0\sigma}(\pi,t,\tau,\mu) = B_{0\sigma}(\pi,t,\tau,\mu) = C_{0\sigma}(\pi,t,\tau,\mu) = D_{0\sigma}(\pi,t,\tau,\mu) = 0, \tag{58}$$

and where u_0 satisfies (44) and (45). The still arbitrary functions $A_{0\sigma}, B_{0\sigma}, C_{0\sigma}, D_{0\sigma}, y_0, z_0$, and v_0 in (56) will be used to avoid secular terms in u_1, u_2, and so on. It now follows from (36)-(40) and (56) that the function $u_1(\bar{x}, x, \bar{t}, t, \tau, \mu)$ has to satisfy

$$\begin{aligned} u_{1_{\bar{t}\bar{t}}} - u_{1_{\bar{x}\bar{x}}} = 2 & \sum_{\substack{\sigma \\ \sigma\neq 0 \\ \sigma^2 \neq \frac{-1}{\beta^2}}} \sigma\left[A_0\sin(\sigma(\bar{x}+\bar{t})) + B_0\sin(\sigma(\bar{x}-\bar{t})) + C_0\cos(\sigma(\bar{x}+\bar{t}))\right. \\ & \left. + D_0\cos(\sigma(\bar{x}-\bar{t}))\right] + \frac{2}{\beta}\left((y_{0_x} + y_{0_t})\exp\left(\frac{\bar{x}-\bar{t}}{\beta}\right) + (-z_{0_x} + z_{0_t})\exp\left(\frac{-(\bar{x}+\bar{t})}{\beta}\right)\right), \\ & 0 < x < \pi,\ t > 0, \end{aligned} \tag{59}$$

$$u_1(\bar{x}, x, \bar{t}, t, \tau, \mu) = 0, \quad x = 0,\ \bar{x} = 0,\ t \geq 0, \tag{60}$$

$$u_{1_{\bar{x}}} + \beta u_{1_{\bar{x}\bar{t}}} = -\sum_{\substack{\sigma \\ \sigma\neq 0 \\ \sigma^2\neq\frac{-1}{\beta^2}}} \left(\left([A_{0\sigma_x} + \beta\sigma B_{0\sigma_x}]\cos(\sigma\tfrac{\pi}{\epsilon}) + [C_{0\sigma_x} + \beta\sigma D_{0\sigma_x}]\sin(\sigma\tfrac{\pi}{\epsilon})\right)\cos(\sigma\bar{t})\right.$$

$$+ \left.\left([B_{0\sigma_x} - \beta\sigma A_{0\sigma_x}]\cos(\sigma\tfrac{\pi}{\epsilon}) + [D_{0\sigma_x} - \beta\sigma C_{0\sigma_x}]\sin(\sigma\tfrac{\pi}{\epsilon})\right)\sin(\sigma\bar{t})\right) +$$

$$\left(\left(\frac{\alpha}{\beta}y_0 - y_{0_t}\right)\exp\left(\frac{\pi}{\epsilon\beta}\right) + \left(\frac{\alpha}{\beta}z_0 + z_{0_t}\right)\exp\left(\frac{-\pi}{\epsilon\beta}\right)\right)\exp\left(\frac{-\bar{t}}{\beta}\right) - v_{0_x},\ x = \pi,\ t \geq 0, \quad (61)$$

$$u_1(\bar{x}, x, 0, 0, 0, 0) = \phi_1(x), \quad 0 < x < \pi, \quad (62)$$

$$u_{1_{\bar{t}}}(\bar{x}, x, 0, 0, 0, 0) = \psi_0(x) - u_{0_t}, \quad 0 < x < \pi, \quad (63)$$

where $A_0/2 = -A_{0\sigma_x} - D_{0\sigma_t} + D_{0\sigma_x} + A_{0\sigma_t}, B_0/2 = -A_{0\sigma_x} - D_{0\sigma_t} - D_{0\sigma_x} - A_{0\sigma_t}, C_0/2 = C_{0\sigma_x} - B_{0\sigma_t} - B_{0\sigma_x} + C_{0\sigma_t}, D_0/2 = C_{0\sigma_x} - B_{0\sigma_t} + B_{0\sigma_x} - C_{0\sigma_t}$.

It shoud be observed that equation (61) can be integrated once with respect to $\bar{t}$. In this way an inhomogeneous Neumann condition for u_1 at $x = \pi$ is obtained, that is, $u_{1_{\bar{x}}}$ is prescribed at $x = \pi$. This expression for $u_{1_{\bar{x}}}$ at $x = \pi$ will contain terms of the form $\bar{t}\exp\left(\frac{-\bar{t}}{\beta}\right)$. These terms are of a secular nature and will violate the asymptotic ordering principle. For that reason the coefficient of $\exp\left(\frac{-\bar{t}}{\beta}\right)$ in (61) has to be taken equal to zero, that is,

$$\left(\frac{\alpha}{\beta}y_0 - y_{0_t}\right)\exp\left(\frac{\pi}{\epsilon\beta}\right) + \left(\frac{\alpha}{\beta}z_0 + z_{0_t}\right)\exp\left(\frac{-\pi}{\epsilon\beta}\right) = 0, \quad x = \pi,\ t \geq 0. \quad (64)$$

It is also easy to see that the term $-v_{0_x}(\pi, t, \tau, \mu)$ in (61) will lead to a "particular" solution $-\bar{x}v_{0_x}(\pi, t, \tau, \mu)$ of the boundary value problem (59)-(61). Also this solution has a secular behaviour. For that reason $-v_{0_x}(\pi, t, \tau, \mu)$ has to be taken equal to zero, that is,

$$v_{0_x}(\pi, t, \tau, \mu) = 0, \quad t \geq 0. \quad (65)$$

Furthermore, it should be observed that all terms in the right handside of the PDE (59) are solution of the associated, homogeneous PDE. Obviously all these terms will lead to secular terms in the solution u_1. To avoid these secular terms the coefficients $A_0, B_0, C_0, D_0, y_{0_x}+y_{0_t}$, and $-z_{0_x} + z_{0_t}$ have to be taken equal to zero, or equivalently

$$A_{0\sigma_x} + D_{0\sigma_t} = 0, \quad A_{0\sigma_t} + D_{0\sigma_x} = 0, \quad 0 < x < \pi,\ t > 0, \quad (66)$$

$$B_{0\sigma_x} - C_{0\sigma_t} = 0, \quad B_{0\sigma_t} - C_{0\sigma_x} = 0, \quad 0 < x < \pi,\ t > 0, \quad (67)$$

$$y_{0_x} + y_{0_t} = 0, \quad z_{0_x} - z_{0_t} = 0, \quad 0 < x < \pi,\ t > 0. \quad (68)$$

By using (58) it follows from (66) and (67) for $x = \pi$ that

$$A_{0\sigma_x}(\pi, t, \tau, \mu) = B_{0\sigma_x}(\pi, t, \tau, \mu) = C_{0\sigma_x}(\pi, t, \tau, \mu) = D_{0\sigma_x}(\pi, t, \tau, \mu) = 0, \quad t \geq 0. \quad (69)$$

From (66) and (67) it also follows that

$$A_{0\sigma_{xx}} - A_{0\sigma_{tt}} = 0, \quad B_{0\sigma_{tt}} - B_{0\sigma_{xx}} = 0, \quad 0 < x < \pi, \quad t > 0. \quad (70)$$

It is clear from (57), (58), (69) and (70) that only trivial solution will be obtained for $A_{0\sigma}$ and $B_{0\sigma}$ (that is, $A_{0\sigma}$ and $B_{0\sigma}$ are identically zero). It then also follows from (58), (66), and (67) that $C_{0\sigma}$ and $D_{0\sigma}$ are identically zero. From (56) it should be observed that $y_0 \exp(-\bar{t}/\beta)$ is at most of order one for all $t \geq 0$ since the solution is bounded. It then follows from (57) that $z_0 \exp(-\bar{t}/\beta)$ is also at most of order one for all $t \geq 0$. This implies that the second term in the left handside of (64) is asymptotically small for all $t \geq 0$. For that reason, the second term in the left handside of (64) can be neglected. Then it follows from (57), (64), and (68) that

$$y_0(x,t,\tau,\mu) = y_{01}(\tau,\mu) \exp\left(\frac{\alpha}{\beta}(t-x)\right), \tag{71}$$

$$z_0(x,t,\tau,\mu) = -y_{01}(\tau,\mu) \exp\left(\frac{\alpha}{\beta}(t+x)\right), \tag{72}$$

where $y_{01}(\tau,\mu)$ is still an arbitrary function which can be used to avoid secular in the higher order approximations. So far it can be concluded that

$$u_0(\bar{x},x,\bar{t},t,\tau,\mu) = \left(y_0 \exp\left(\frac{\bar{x}}{\beta}\right) + z_0 \exp\left(\frac{-\bar{x}}{\beta}\right)\right) \exp\left(-\frac{\bar{t}}{\beta}\right) + v_0(x,t,\tau,\mu), \tag{73}$$

$$\begin{aligned} u_1 = \sum_{\substack{\sigma \\ \sigma\neq 0 \\ \sigma^2 \neq \frac{-1}{\beta^2}}} &((A_{1\sigma}\cos(\sigma\bar{x})\cos(\sigma\bar{t}) + B_{1\sigma}\cos(\sigma\bar{x})\sin(\sigma\bar{t}) + C_{1\sigma}\sin(\sigma\bar{x})\cos(\sigma\bar{t}) \\ &+D_{1\sigma}\sin(\sigma\bar{x})\sin(\sigma\bar{t}))) + \left(y_1 \exp\left(\frac{\bar{x}}{\beta}\right) + z_1 \exp\left(\frac{-\bar{x}}{\beta}\right)\right) \exp\left(-\frac{\bar{t}}{\beta}\right) + v_1(x,t,\tau,\mu), \end{aligned} \tag{74}$$

where $A_{1\sigma}, B_{1\sigma}, C_{1\sigma}, D_{1\sigma}, y_1, z_1$ and v_1 are arbitrary functions in $x, t, \tau,$ and μ, satisfying (62), and (63) for $\bar{t} = t = \tau = \mu = 0$, and the following conditions

$$A_{1\sigma}(0,t,\tau,\mu) = B_{1\sigma}(0,t,\tau,\mu) = y_1(0,t,\tau,\mu) + z_1(0,t,\tau,\mu) = v_1(0,t,\tau,\mu) = 0, \tag{75}$$

$$A_{1\sigma}(\pi,t,\tau,\mu) = B_{1\sigma}(\pi,t,\tau,\mu) = C_{1\sigma}(\pi,t,\tau,\mu) = D_{1\sigma}(\pi,t,\tau,\mu) = 0, \tag{76}$$

and where y_0 and z_0 are given by (71) and (72) respectively. The still undetermined functions y_{01}, and v_0 in u_0, $A_{1\sigma}, B_{1\sigma}, C_{1\sigma}, D_{1\sigma}, y_1, z_1$, and v_1 in u_1 will be used to avoid secular terms in u_2, u_3, and so on. After solving the problem for u_1 it follows from (36)-(40) that the function $u_2(\bar{x},x,\bar{t},t,\tau,\mu)$ has to satisfy

$$\begin{aligned} u_{2_{\bar{t}\bar{t}}} - u_{2_{\bar{x}\bar{x}}} = 2 &\sum_{\substack{\sigma \\ \sigma\neq 0 \\ \sigma^2 \neq \frac{-1}{\beta^2}}} \sigma\left[A_1 \sin(\sigma(\bar{x}+\bar{t})) + B_1 \sin(\sigma(\bar{x}-\bar{t})) + C_1 \cos(\sigma(\bar{x}+\bar{t}))\right. \\ &\left.+D_1\cos(\sigma(\bar{x}-\bar{t}))\right] + \frac{2}{\beta}\left(\left(y_{1_x} + y_{1_t} + y_{0_\tau} - \frac{\beta p^2}{2} y_0\right) \exp\left(\frac{(\bar{x}-\bar{t})}{\beta}\right)\right. \\ &\left.+\left(-z_{1_x} + z_{1_t} + z_{0_\tau} - \frac{\beta p^2}{2} z_0\right) \exp\left(\frac{-(\bar{x}+\bar{t})}{\beta}\right)\right) - \left(v_{0_{tt}} - v_{0_{xx}} + p^2 v_0\right), \\ &0 < x < \pi,\ t > 0, \end{aligned} \tag{77}$$

$$u_2(0,0,\bar{t},t,\tau)=0, \quad t\geq 0, \tag{78}$$

$$u_{2_{\bar{x}}}+\beta u_{2_{\bar{x}\bar{t}}}=-\sum_{\substack{\sigma\\ \sigma\neq 0\\ \sigma^2\neq\frac{-1}{\beta^2}}}\left(\left([A_{1\sigma_x}+\beta\sigma B_{1\sigma_x}]\cos(\sigma\frac{\pi}{\epsilon})+[C_{1\sigma_x}+\beta\sigma D_{1\sigma_x}]\sin(\sigma\frac{\pi}{\epsilon})\right)\cos(\sigma\bar{t})\right.$$

$$\left.+\left([B_{1\sigma_x}-\beta\sigma A_{1\sigma_x}]\cos(\sigma\frac{\pi}{\epsilon})+[D_{1\sigma_x}-\beta\sigma C_{1\sigma_x}]\sin(\sigma\frac{\pi}{\epsilon})\right)\sin(\sigma\bar{t})\right)$$

$$+\left(\left(\frac{\alpha}{\beta}y_1-y_{1_t}-y_{0_\tau}\right)\exp\left(\frac{\pi}{\epsilon\beta}\right)+\left(\frac{\alpha}{\beta}z_1+z_{1_t}-2\alpha z_{0_t}+z_{0_\tau}\right)\exp\left(\frac{-\pi}{\epsilon\beta}\right)\right)\exp\left(\frac{-\bar{t}}{\beta}\right)$$

$$-\left(v_{1_x}+\alpha v_{0_t}\right), \quad x=\pi, \quad t\geq 0, \tag{79}$$

$$u_2(\bar{x},x,0,0,0,0)=\phi_2(x), \quad 0<x<\pi,\ t=0, \tag{80}$$

$$u_{2_{\bar{t}}}(\bar{x},x,0,0,0,0)=\psi_1(x)-u_{1_t}-u_{o\tau}, \quad 0<x<\pi,\ t=0, \tag{81}$$

where $A_1/2=-A_{1\sigma_x}-D_{1\sigma_t}+D_{1\sigma_x}+A_{1\sigma_t}, B_1/2=-A_{1\sigma_x}-D_{1\sigma_t}-D_{1\sigma_x}-A_{1\sigma_t}, C_1/2=C_{1\sigma_x}-B_{1\sigma_t}-B_{1\sigma_x}+C_{1\sigma_t}, D_1/2=C_{1\sigma_x}-B_{1\sigma_t}+B_{1\sigma_x}-C_{1\sigma_t}$. As before it should be observed that (79) can be integrated once with respect to $\bar{t}$. In this way an inhomogeneous neumann condition for u_2 at $x=\pi$ is obtained, that is, $u_{2_{\bar{x}}}$ is prescribed at $x=\pi$. This expression for $u_{2_{\bar{x}}}$ at $x=\pi$ will contain terms of the form $\bar{t}\exp(-\bar{t}/\beta)$. These terms are of secular nature and will violate the asymptotic ordering principle. For that reason the coefficient of $\exp(-\bar{t}/\beta)$ in (79) has to be taken equal to zero, that is,

$$\left(\frac{\alpha}{\beta}y_1-y_{1_t}-y_{0_\tau}\right)\exp\left(\frac{\pi}{\epsilon\beta}\right)+\left(\frac{\alpha}{\beta}z_1+z_{1_t}-2\alpha z_{0_t}+z_{0_\tau}\right)\exp\left(\frac{-\pi}{\epsilon\beta}\right)=0,\ x=\pi,\ t\geq 0. \tag{82}$$

It can also easily be seen that the term $v_{1_x}+\alpha v_{0_t}$ at $x=\pi$ in (79) will lead to a "particular" solution $-\bar{x}\left(v_{1_x}(\pi,t,\tau,\mu)+\alpha v_{0_t}(\pi,t,\tau,\mu)\right)$ of the the boundary value problem (77)-(79). To avoid this secular behaviour it follows that

$$v_{1_x}(\pi,t,\tau,\mu)+\alpha v_{0_t}(\pi,t,\tau,\mu)=0, \quad t\geq 0. \tag{83}$$

Furthermore, it should be observed that all terms in the right handside of the PDE (77) are solutions of the associated, homogeneous PDE. Obviously all these terms will lead to secular terms in the solution u_2. To avoid these secular terms the coefficients $A_1, B_1, C_1, D_1, y_{1_x}+y_{1_t}+y_{0_\tau}-\frac{\beta p^2}{2}y_0, -z_{1_x}+z_{1_t}+z_{0_\tau}-\frac{\beta p^2}{2}z_0$, and $v_{0_{tt}}-v_{0_{xx}}+p^2v_0$ have to be taken equal to zero, or equivalently

$$A_{1\sigma_x}+D_{1\sigma_t}=0, \quad A_{1\sigma_t}+D_{1\sigma_x}=0, \quad 0<x<\pi,\ t>0, \tag{84}$$

$$B_{1\sigma_x}-C_{1\sigma_t}=0, \quad B_{1\sigma_t}-C_{1\sigma_x}=0, \quad 0<x<\pi,\ t>0, \tag{85}$$

$$y_{1_x}+y_{1_t}+y_{0_\tau}-\frac{\beta p^2}{2}y_0=0, \quad z_{1_x}-z_{1_t}-z_{0_\tau}+\frac{\beta p^2}{2}z_0=0, \quad 0<x<\pi,\ t>0, \tag{86}$$

$$v_{0_{tt}}-v_{0_{xx}}+p^2v_0=0, \quad 0<x<\pi,\ t>0. \tag{87}$$

Using a similar argument as has been used in (64) it follows from (82) that $\frac{\alpha}{\beta}y_1-y_{1_t}-y_{0_\tau}=0$ for $x=\pi$ and $t\geq 0$. By integrating this expression with respect to t it follows that $y_1(\pi,t,\tau,\mu)$ can be found. However, $y_1(\pi,t,\tau,\mu)$ will contain secular terms of the form $t\exp(\alpha t/\beta)$. To avoid these secular terms it follows that $y_{0_\tau}(\pi,t,\tau,\mu)$ has to be equal to zero. It then follows from (71) and (72) that

$$y_0(x,t,\tau,\mu) = y_{012}(\mu)\exp\left(\frac{\alpha}{\beta}(t-x)\right), \tag{88}$$

and

$$z_0(x,t,\tau,\mu) = -y_{012}(\mu)\exp\left(\frac{\alpha}{\beta}(t+x)\right) \tag{89}$$

respectively, and from (75) and (86) that y_1 and z_1 are given by

$$y_1(x,t,\tau,\mu) = \left(y_{11}(\tau,\mu)\exp\left(\frac{\alpha\pi}{\beta}\right) + \frac{\beta p^2}{2}y_{012}(\mu)(x-\pi)\right)\exp\left(\frac{\alpha}{\beta}(t-x)\right), \tag{90}$$

and

$$z_1(x,t,\tau,\mu) = \left(-y_{11}(\tau,\mu)\exp\left(\frac{\alpha\pi}{\beta}\right) + \frac{\beta p^2}{2}y_{012}(\mu)(x+\pi)\right)\exp\left(\frac{\alpha}{\beta}(t+x)\right) \tag{91}$$

respectively. To determine the functions $A_{1\sigma}, B_{1\sigma}, C_{1\sigma}$, and $D_{1\sigma}$ a completely similar analysis as given for $A_{0\sigma}, B_{0\sigma}, C_{0\sigma}$, and $D_{0\sigma}$ can be given (see also (66)-(70)), yielding

$$A_{1\sigma}(x,t,\tau,\mu) = B_{1\sigma}(x,t,\tau,\mu) = C_{1\sigma}(x,t,\tau,\mu) = B_{1\sigma}(x,t,\tau,\mu) \equiv 0. \tag{92}$$

It follows from (57), (65), and (87) that v_0 is given by

$$v_0(x,t,\tau,\mu) = \sum_{n=1}^{\infty}\left(A_{0n}(\tau,\mu)\cos(\sqrt{\lambda_n}t) + B_{0n}(\tau,\mu)\sin(\sqrt{\lambda_n}t)\right)\sin((n-1/2)x), \tag{93}$$

where $\lambda_n = p^2 + (n-1/2)^2$, and where the functions A_{0n} and B_{0n} are still arbitrary functions which can be used to avoid secular terms in v_1 and v_2. After avoiding secular terms in u_2 the initial-boundary value problem (77)-(81) can now be solved and the general solution can be obtained. So far it can be concluded that

$$u_0(\bar{x},x,\bar{t},t,\tau,\mu) = \left(y_0\exp\left(\frac{\bar{x}}{\beta}\right) + z_0\exp\left(\frac{-\bar{x}}{\beta}\right)\right)\exp\left(-\frac{\bar{t}}{\beta}\right) + v_0(x,t,\tau,\mu), \tag{94}$$

$$u_1(\bar{x},x,\bar{t},t,\tau,\mu) = \left(y_1\exp\left(\frac{\bar{x}}{\beta}\right) + z_1\exp\left(\frac{-\bar{x}}{\beta}\right)\right)\exp\left(-\frac{\bar{t}}{\beta}\right) + v_1(x,t,\tau,\mu), \tag{95}$$

$$\begin{aligned} u_2(\bar{x},x,\bar{t},t,\tau,\mu) = & \sum_{\substack{\sigma \\ \sigma\neq 0 \\ \sigma^2\neq\frac{-1}{\beta^2}}} ((A_{2\sigma}\cos(\sigma\bar{x})\cos(\sigma\bar{t}) + B_{2\sigma}\cos(\sigma\bar{x})\sin(\sigma\bar{t}) \\ & +C_{2\sigma}\sin(\sigma\bar{x})\cos(\sigma\bar{t}) + D_{2\sigma}\sin(\sigma\bar{x})\sin(\sigma\bar{t}))) \\ & + \left(y_2\exp\left(\frac{\bar{x}}{\beta}\right) + z_2\exp\left(\frac{-\bar{x}}{\beta}\right)\right)\exp\left(-\frac{\bar{t}}{\beta}\right) + v_2(x,t,\tau,\mu), \end{aligned} \tag{96}$$

where $y_0, z_0,$ and $v_0(x,t,\tau,\mu)$ are given by (88), (89), and (93) respectively, and where y_1, z_1 are given by (90), (91) respectively, where v_1 satisfies (75) and (83), and where $A_{2\sigma}, B_{2\sigma}, C_{2\sigma}, D_{2\sigma}, y_2, z_2$ and v_2 are arbitrary functions depending on x, t, τ, and μ satisfying

$$A_{2\sigma}(0,t,\tau,\mu) = B_{2\sigma}(0,t,\tau,\mu) = y_2(0,t,\tau,\mu) + z_2(0,t,\tau,\mu) = v_2(0,t,\tau,\mu) = 0,\ t \geq 0, \quad (97)$$

$$A_{2\sigma}(\pi,t,\tau,\mu) = B_{2\sigma}(\pi,t,\tau,\mu) = C_{2\sigma}(\pi,t,\tau,\mu) = D_{2\sigma}(\pi,t,\tau,\mu) = 0, \quad t \geq 0. \quad (98)$$

The still arbitrary functions in $u_0, u_1,$ and u_2 will be used to avoid secular terms in u_3, u_4 and so on. After removing secular terms in u_2 it follows from (36)- (40) that the function $u_3(\bar{x}, x, \bar{t}, t, \tau, \mu)$ has to satisfy

$$\begin{aligned}
&u_{3_{\bar{t}\bar{t}}} - u_{3_{\bar{x}\bar{x}}} = 2 \sum_{\substack{\sigma \\ \sigma\neq 0 \\ \sigma^2 \neq \frac{-1}{\beta^2}}} \sigma\left[A_2 \sin(\sigma(\bar{x}+\bar{t})) + B_2 \sin(\sigma(\bar{x}-\bar{t})) + C_2\cos(\sigma(\bar{x}+\bar{t}))\right. && (99)\\
&\left.+D_2\cos(\sigma(\bar{x}-\bar{t}))\right] + \frac{2}{\beta}\left(\left(y_{2_x} + y_{2_t} + y_{1_\tau} - \frac{\beta p^2}{2} y_1 - \frac{\beta^2 p^2}{2} y_{0_t} + y_{0_\mu}\right)\exp\left(\frac{\bar{x}-\bar{t}}{\beta}\right)\right. \\
&\left.+\left(-z_{2_x} + z_{2_t} + z_{1_\tau} - \frac{\beta p^2}{2} z_1 - \frac{\beta^2 p^2}{2} z_{0_t} + z_{0_\mu}\right)\exp\left(\frac{-(\bar{x}+\bar{t})}{\beta}\right)\right) \\
&-\left(v_{1_{tt}} - v_{1_{xx}} + p^2 v_1 + 2v_{0_{t\tau}}\right), \quad 0 < x < \pi, \quad t > 0, \\
&u_3(\bar{x}, x, \bar{t}, t, \tau) = 0, \quad x = 0,\ \bar{x} = 0,\ t \geq 0, && (100)\\
&u_{3_{\bar{x}}} + \beta u_{3_{\bar{x}\bar{t}}} = -\sum_{\substack{\sigma \\ \sigma\neq 0 \\ \sigma^2 \neq \frac{-1}{\beta^2}}} \left(\left([A_{2\sigma_x} + \beta\sigma B_{2\sigma_x}]\cos(\sigma\frac{\pi}{\epsilon}) + [C_{2\sigma_x} + \beta\sigma D_{2\sigma_x}]\sin(\sigma\frac{\pi}{\epsilon})\right)\cos(\sigma\bar{t})\right. \\
&\left.+\left([B_{2\sigma_x} - \beta\sigma A_{2\sigma_x}]\cos(\sigma\frac{\pi}{\epsilon}) + [D_{2\sigma_x} - \beta\sigma C_{2\sigma_x}]\sin(\sigma\frac{\pi}{\epsilon})\right)\sin(\sigma\bar{t})\right) && (101)\\
&+\left(\left(\frac{\alpha}{\beta}y_2 - y_{2_t} - y_{1_\tau} - \beta y_{1_{xt}} - \alpha y_{1_t} - y_{0_\mu}\right)\exp\left(\frac{\pi}{\epsilon\beta}\right) + \left(\frac{\alpha}{\beta}z_2 + z_{2_t} + z_{1_\tau} - \beta z_{1_{xt}}\right.\right. \\
&\left.\left.-\alpha z_{1_t} + z_{0_\mu}\right)\exp\left(\frac{\pi}{\epsilon\beta}\right)\right)\exp\left(\frac{-\bar{t}}{\beta}\right) - \left(v_{2_x} + \beta v_{1_{xt}} + \alpha v_{1_t} + \beta v_{0_{x\tau}} + \alpha v_{0_\tau}\right), x = \pi,\ t \geq 0, \\
&u_3(\bar{x}, x, 0, 0, 0, 0) = \phi_3(x), \quad 0 < x < \pi,\ t = 0, && (102)\\
&u_{3_{\bar{t}}}(\bar{x}, x, 0, 0, 0, 0) = \psi_2(x) - u_{2_t} - u_{1\tau} - u_{0_\mu}, \quad 0 < x < \pi,\ t = 0, && (103)
\end{aligned}$$

where $A_2/2 = -A_{2\sigma_x} - D_{2\sigma_t} + D_{2\sigma_x} + A_{2\sigma_t}, B_2/2 = -A_{2\sigma_x} - D_{2\sigma_t} - D_{2\sigma_x} - A_{2\sigma_t}, C_2/2 = C_{2\sigma_x} - B_{2\sigma_t} - B_{2\sigma_x} + C_{2\sigma_t}, D_2/2 = C_{2\sigma_x} - B_{2\sigma_t} + B_{2\sigma_x} - C_{2\sigma_t}$. To avoid secular terms in u_3 an almost completely similar analysis as used to avoid secular terms in u_1 and u_2 can be given. From the boundary condition (101) it then will follow that

$$\begin{aligned}
&\left(\frac{\alpha}{\beta}y_2 - y_{2_t} - y_{1_\tau} - \beta y_{1_{xt}} - \alpha y_{1_t} - y_{0_\mu}\right)\exp\left(\frac{\pi}{\epsilon\beta}\right) + \left(\frac{\alpha}{\beta}z_2 + z_{2_t} + z_{1_\tau} - \beta z_{1_{xt}}\right. \\
&\left.-\alpha z_{1_t} + z_{0_\mu}\right)\exp\left(\frac{-\pi}{\epsilon\beta}\right) = 0, \quad x = \pi, \quad t \geq 0, && (104)\\
&v_{2_x} + \beta v_{1_{xt}} + \alpha v_{1_t} + \beta v_{0_{x\tau}} + \alpha v_{0_\tau} = 0, \quad x = \pi, \quad t \geq 0. && (105)
\end{aligned}$$

From (104) it follows that $\frac{\alpha}{\beta}y_2 - y_{2_t} - y_{1_\tau} - \beta y_{1_{xt}} - \alpha y_{1_t} - y_{0_\mu} = 0$, for $x = \pi$ and for $t \geq 0$ since $\exp\left(\frac{-\pi}{\epsilon\beta}\right)$ is asymptotically small. This expression for y_2 at $x = \pi$ and for $t \geq 0$ can be integrated with respect to t, and it turns out that secular terms of the form $t\exp(\alpha t/\beta)$ will occur in y_2. To avoid these secular terms $-y_{1_\tau} - \beta y_{1_{xt}} - \alpha y_{1_t} - y_{0_\mu}$ has to be taken equal to zero at $x = \pi$ and for $t \geq 0$. The equation $-y_{1_\tau} - \beta y_{1_{xt}} - \alpha y_{1_t} - y_{0_\mu} = 0$ at $x = \pi, t \geq 0$ can be integrated with respect to τ, and will give rise to secular terms in y_1 of the form $\tau\exp(\alpha t/\beta)$. To avoid these secular terms in y_1 and by noticing that $-y_{1_\tau} - \beta y_{1_{xt}} - \alpha y_{1_t} - y_{0_\mu} = -\left(y_{11_\tau}(\tau,\mu) + y_{012_\mu}(\mu) + (\alpha\beta p^2/2)y_{012}(\mu)\right)\exp\left(\frac{\alpha}{\beta}(t-\pi)\right)$ it follows that $y_{012_\mu}(\mu) + (\alpha\beta p^2/2)y_{012}(\mu) = 0$. The last expression determines the function y_0 with respect to μ and the function y_1 with respect to τ. After avoiding secular terms in y_1 the functions y_0 and z_0 with respect to μ can now be determined. It turns out that y_0 and z_0 are given by

$$y_0(x,t,\tau,\mu) = y_{012}(0)\exp\left(\frac{\alpha}{\beta}\left(t - x - \frac{(\beta p)^2}{2}\mu\right)\right), \tag{106}$$

and

$$z_0(x,t,\tau,\mu) = -y_{012}(0)\exp\left(\frac{\alpha}{\beta}\left(t + x - \frac{(\beta p)^2}{2}\mu\right)\right), \tag{107}$$

respectively. From (106), (107) and using (45) it follows that

$$y_{012}(0) = 0. \tag{108}$$

From (90), (91) and (108) y_1 and z_1 now become

$$y_1(x,t,\tau,\mu) = y_{112}(\mu)\exp\left(\frac{\alpha}{\beta}(t-x)\right), \tag{109}$$

and

$$z_1(x,t,\tau,\mu) = -y_{112}(\mu)\exp\left(\frac{\alpha}{\beta}(t+x)\right), \tag{110}$$

respectively. To avoid secular terms in u_3 due to the terms in the right handside of the PDE (99) it follows elementarily from (99) that $A_{2\sigma}, B_{2\sigma}, C_{2\sigma}, D_{2\sigma}, y_2, z_2$, and v_1 have to satisfy

$$A_{2\sigma}(x,t,\tau,\mu) = B_{2\sigma}(x,t,\tau,\mu) = C_{2\sigma}(x,t,\tau,\mu) = D_{2\sigma}(x,t,\tau,\mu) \equiv 0, \tag{111}$$

$$y_2(x,t,\tau,\mu) = \left(y_{21}(\tau,\mu)\exp\left(\frac{\alpha\pi}{\beta}\right) + \frac{\beta p^2}{2}y_{112}(\mu)(x-\pi)\right)\exp\left(\frac{\alpha}{\beta}\right)(t-x), \tag{112}$$

$$z_2(x,t,\tau,\mu) = \left(-y_{21}(\tau,\mu)\exp\left(\frac{\alpha\pi}{\beta}\right) + \frac{\beta p^2}{2}y_{112}(\mu)(x+\pi)\right)\exp\left(\frac{\alpha}{\beta}\right)(t+x), \tag{113}$$

$$v_{1_{tt}} - v_{1_{xx}} + p^2 v_1 + 2v_{0_{t\tau}} = 0, \quad 0 < x < \pi,\ t > 0. \tag{114}$$

The function v_1 can now be determined with respect to x and t by solving (114) with respect to boundary conditions (75) and (83). To obtain v_1 the boundary condition for v_1 at $x = \pi$ is made homogeneous by introducing the following transformation

$$\bar{v}(x,t,\tau,\mu) = v_1(x,t,\tau,\mu) + \alpha x v_{0_t}(\pi,t,\tau,\mu), \quad 0 < x < \pi, \quad t > 0. \tag{115}$$

The boundary value problem for v_1 now becomes a boundary value problem for $\bar{v}$ with a homogeneous Dirichlet condition at $x = 0$ and a homogeneous Neumann condition at $x = \pi$. The inhomogeneous part of PDE for $\bar{v}$ now contains terms which all give rise to secular terms. To remove these secular terms it can be shown elementarily that $A_{0n}(\tau, \mu)$ and $B_{0n}(\tau, \mu)$ have to satisfy

$$A_{0n}(\tau, \mu) = A_{0n}(0, \mu) \exp\left((-\frac{\alpha}{\pi})\tau\right), \tag{116}$$

and

$$B_{0n}(\tau, \mu) = B_{0n}(0, \mu) \exp\left((-\frac{\alpha}{\pi})\tau\right), \tag{117}$$

respectively. Elementary v_1 can now be constructed, yielding

$$\begin{aligned} v_1(x, t, \tau, \mu) = & \sum_{n=1}^{\infty} \Big(A_{1n}(\tau, \mu)\cos(\sqrt{\lambda_n}t) + B_{1n}(\tau, \mu)\sin(\sqrt{\lambda_n}t) + S(t, \tau, \mu)\Big)\sin((n-1/2)x) \\ & -\alpha x v_{0_t}(\pi, t, \tau, \mu), \end{aligned} \tag{118}$$

where

$$S(t, \tau, \mu) = \exp\left(-(\frac{\alpha}{\pi})\tau\right) \sum_{\substack{m=0 \\ n \neq n}}^{\infty} G_{mn}\left(A_{0m}(0, \mu)\sin(\sqrt{\lambda_m}t) - B_{0m}(0, \mu)\cos(\sqrt{\lambda_m}t)\right) \tag{119}$$

with $G_{mn} = 2\alpha(-1)^{m+n}\frac{\sqrt{\lambda_m}}{\pi(\lambda_n - \lambda_m)}\left(\frac{m+\frac{1}{2}}{n+\frac{1}{2}}\right)^2$, and where A_{1n}, B_{1n} are still arbitrary functions in τ and μ which can be used to remove secular terms in v_2. After avoiding secular terms in u_3 the initial-boundary value problem (99)-(103) can now be solved and the general solution can be obtained. So far it can be concluded that

$$\begin{aligned} u_0(\bar{x}, x, \bar{t}, t, \tau, \mu) &= \exp\left(-\frac{\alpha}{\pi}\tau\right)\sum_{n=1}^{\infty}\left(A_{0n}(0, \mu)\cos(\sqrt{\lambda_n}t) + B_{0n}(0, \mu)\sin(\sqrt{\lambda_n}t)\right) \quad (120) \\ & \quad \times \sin((n-1/2)x), \\ u_1(\bar{x}, x, \bar{t}, t, \tau, \mu) &= \left(y_1 \exp\left(\frac{\bar{x}}{\beta}\right) + z_1 \exp\left(\frac{-\bar{x}}{\beta}\right)\right)\exp\left(-\frac{\bar{t}}{\beta}\right) + v_1(x, t, \tau, \mu), \quad (121) \\ u_2(\bar{x}, x, \bar{t}, t, \tau, \mu) &= \left(y_2 \exp\left(\frac{\bar{x}}{\beta}\right) + z_2 \exp\left(\frac{-\bar{x}}{\beta}\right)\right)\exp\left(-\frac{\bar{t}}{\beta}\right) + v_2(x, t, \tau, \mu), \quad (122) \end{aligned}$$

and

$$\begin{aligned} u_3(\bar{x}, x, \bar{t}, t, \tau, \mu) = & \sum_{\substack{\sigma \\ \sigma \neq 0 \\ \sigma^2 \neq \frac{-1}{\beta^2}}} ((A_{3\sigma}\cos(\sigma\bar{x})\cos(\sigma\bar{t}) + B_{3\sigma}\cos(\sigma\bar{x})\sin(\sigma\bar{t}) + C_{3\sigma}\sin(\sigma\bar{x})\cos(\sigma\bar{t}) \\ & + D_{3\sigma}\sin(\sigma\bar{x})\sin(\sigma\bar{t}))) + \left(y_3 \exp\left(\frac{\bar{x}}{\beta}\right) + z_3 \exp\left(\frac{-\bar{x}}{\beta}\right)\right)\exp\left(-\frac{\bar{t}}{\beta}\right) + v_3(x, t, \tau, \mu), \end{aligned} \tag{123}$$

where y_1, z_1, and v_1 are given by (109), (110), and (118) respectively, where y_2 and z_2 are given by (112) and (113) respectively, where v_2 satisfies (97) and (105), and where

$A_{3\sigma}, B_{3\sigma}, C_{3\sigma}, D_{3\sigma}, y_3, z_3$ and v_3 are functions in x, t, τ, and μ satisfying

$$A_{3\sigma}(0,t,\tau,\mu) = B_{3\sigma}(0,t,\tau,\mu) = y_3(0,t,\tau,\mu) + z_3(0,t,\tau,\mu) = v_3(0,t,\tau,\mu) = 0,\ t \geq 0, \tag{124}$$

$$A_{3\sigma}(\pi,t,\tau,\mu) = B_{3\sigma}(\pi,t,\tau,\mu) = C_{3\sigma}(\pi,t,\tau,\mu) = D_{3\sigma}(\pi,t,\tau,\mu) = 0, \quad t \geq 0. \tag{125}$$

The still arbitrary functions in u_0, u_1, u_2, and in u_3 will be used to avoid secular terms in higher order approximations. After removing secular term in u_3 it then follows from (36)-(40) and from u_0, u_1, u_2, and u_3 that the function $u_4(\bar{x}, x, \bar{t}, t, \tau, \mu)$ has to satisfy

$$u_{4_{\bar{t}\bar{t}}} - u_{4_{\bar{x}\bar{x}}} = 2 \sum_{\substack{\sigma \\ \sigma \neq 0 \\ \sigma^2 \neq \frac{-1}{\beta^2}}} \sigma \left[A_3 \sin(\sigma(\bar{x}+\bar{t})) + B_3 \sin(\sigma(\bar{x}-\bar{t})) + C_3 \cos(\sigma(\bar{x}+\bar{t}))\right. \tag{126}$$

$$\left. + D_3 \cos(\sigma(\bar{x}-\bar{t}))\right] + \frac{2}{\beta}\left(\left(y_{3_x} + y_{3_t} + y_{2_\tau} - \frac{\beta p^2}{2} y_2 - \frac{\beta^2 p^2}{2} y_{1_t} + y_{1_\mu}\right) \exp\left(\frac{\bar{x}-\bar{t}}{\beta}\right)\right.$$

$$\left. + \left(-z_{3_x} + z_{3_t} + z_{2_\tau} - \frac{\beta p^2}{2} z_2 - \frac{\beta^2 p^2}{2} z_{1_t} + z_{1_\mu}\right) \exp\left(\frac{-(\bar{x}+\bar{t})}{\beta}\right)\right)$$

$$- \left(v_{2_{tt}} - v_{2_{xx}} + p^2 v_2 + 2 v_{1_{t\tau}} + 2 v_{0_{t\mu}} + v_{0_{\tau\tau}}\right), \quad 0 < x < \pi, \quad t > 0,$$

$$u_4(\bar{x}, x, \bar{t}, t, \tau) = 0, \quad x = 0,\ \bar{x} = 0,\ t \geq 0, \tag{127}$$

$$u_{4_{\bar{x}}} + \beta u_{4_{\bar{x}\bar{t}}} = - \sum_{\substack{\sigma \\ \sigma \neq 0 \\ \sigma^2 \neq \frac{-1}{\beta^2}}} \left(\left(\left[A_{3\sigma_x} + \beta\sigma B_{3\sigma_x}\right]\cos(\sigma\frac{\pi}{\epsilon}) + \left[C_{3\sigma_x} + \beta\sigma D_{3\sigma_x}\right]\sin(\sigma\frac{\pi}{\epsilon})\right)\cos(\sigma\bar{t})\right.$$

$$\left. + \left(\left[B_{3\sigma_x} - \beta\sigma A_{3\sigma_x}\right]\cos(\sigma\frac{\pi}{\epsilon}) + \left[D_{3\sigma_x} - \beta\sigma C_{3\sigma_x}\right]\sin(\sigma\frac{\pi}{\epsilon})\right)\sin(\sigma\bar{t})\right)$$

$$+ \left(\left(\frac{\alpha}{\beta} y_3 - y_{3_t} - y_{2_\tau} - \beta y_{2_{xt}} - \alpha y_{2_t} - y_{1_\mu}\right)\exp\left(\frac{\pi}{\epsilon\beta}\right) + \left(\frac{\alpha}{\beta} z_3 + z_{3_t} + z_{2_\tau} - \beta z_{2_{xt}}\right.\right.$$

$$\left.\left. - \alpha z_{2_t} + z_{1_\mu}\right)\exp\left(-\frac{\pi}{\epsilon\beta}\right)\right)\exp\left(\frac{-\bar{t}}{\beta}\right)$$

$$- \left(v_{3_x} + \beta v_{2_{xt}} + \alpha v_{2_t} + \beta v_{1_{x\tau}} + \alpha v_{1_\tau} + \beta v_{0_{x\mu}} + \alpha v_{0_\mu}\right), \quad x = \pi,\ t \geq 0, \tag{128}$$

$$u_4(\bar{x}, x, 0, 0, 0, 0) = \phi_4(x), \quad 0 < x < \pi,\ t = 0, \tag{129}$$

$$u_{4_{\bar{t}}}(\bar{x}, x, 0, 0, 0, 0) = \psi_3(x) - u_{3_t} - u_{2_\tau} - u_{1_\mu}, \quad 0 < x < \pi,\ t = 0, \tag{130}$$

where $A_3/2 = -A_{3\sigma_x} - D_{3\sigma_t} + D_{3\sigma_x} + A_{3\sigma_t}, B_3/2 = -A_{3\sigma_x} - D_{3\sigma_t} - D_{3\sigma_x} - A_{3\sigma_t}, C_3/2 = C_{3\sigma_x} - B_{3\sigma_t} - B_{3\sigma_x} + C_{3\sigma_t}, D_3/2 = C_{3\sigma_x} - B_{3\sigma_t} + B_{3\sigma_x} - C_{3\sigma_t}$. It should be noted that the boundary value problem (99) - (101) and the boundary value problem (126) - (128) are similar. To recognize and to avoid secular terms in u_4 a completely similar analysis as given for u_3 can now be given. By removing secular terms in u_4 a part of the arbitrary functions in u_0, u_1, u_2, and u_3 can be determined. By removing secular terms originating from the boundary condition (128) at $x = \pi$ the arbitrary functions in y_2, z_2 with respect to τ and the arbitrary functions in y_1, z_1 with respect to μ can be determined, and the arbitrary functions

y_3 can be prescribed at $x = \pi$ for $t \geq 0$, yielding

$$y_1(x,t,\tau,\mu) = y_{112}(0) \exp\left(\frac{\alpha}{\beta}\left(t - x - \frac{(\beta p)^2}{2}\mu\right)\right), \tag{131}$$

$$z_1(x,t,\tau,\mu) = -y_{112}(0) \exp\left(\frac{\alpha}{\beta}\left(t + x - \frac{(\beta p)^2}{2}\mu\right)\right), \tag{132}$$

and

$$y_2(x,t,\tau,\mu) = \left(y_{212}(\mu) + \frac{\beta p^2}{2}(x-\pi)y_{112}(0)\exp\left(-\frac{\alpha\beta p^2}{2}\mu\right)\right)\exp\left(\frac{\alpha}{\beta}(t-x)\right), \tag{133}$$

$$z_2(x,t,\tau,\mu) = \left(-y_{212}(\mu) + \frac{\beta p^2}{2}(x+\pi)y_{112}(0)\exp\left(-\frac{\alpha\beta p^2}{2}\mu\right)\right)\exp\left(\frac{\alpha}{\beta}(t+x)\right), \tag{134}$$

and

$$y_3(\pi,t,\tau,\mu) = y_{31}(\tau,\mu)\exp\left(\frac{\alpha}{\beta}t\right), \quad t \geq 0. \tag{135}$$

In section 3 it follows from the wellposedness of the problem that the tangent of initial displacement at $x = \pi$, that is, $\phi'(\pi)$ is of order ϵ. This implies that $u_{1_{\bar{x}}}$ has to be equal to zero at $\bar{x} = \pi/\epsilon, x = \pi$ and for $\bar{t} = t = \tau = \mu = 0$. It then follows from (121), (131), and (132) that

$$y_1(x,t,\tau,\mu) = z_1(x,t,\tau,\mu) \equiv 0, \quad t \geq 0. \tag{136}$$

Next, we continue to remove secular terms in u_4 originating from the right handside of the PDE (126). In this step the arbitrary functions $A_{3\sigma}, B_{3\sigma}, C_{3\sigma}, D_{3\sigma}$ with respect to x, t, τ, and μ, and the arbitrary functions y_3, z_3, v_2 with respect to x, t can be determined, yielding

$$A_{3\sigma}(x,t,\tau,\mu) = B_{3\sigma}(x,t,\tau,\mu) = C_{3\sigma}(x,t,\tau,\mu) = D_{2\sigma}(x,t,\tau,\mu) \equiv 0, \tag{137}$$

$$y_3(x,t,\tau,\mu) = \left(y_{31}(\tau,\mu)\exp\left(\frac{\alpha\pi}{\beta}\right) + \frac{\beta p^2}{2}y_{212}(\mu)(x-\pi)\right)\exp\left(\frac{\alpha}{\beta}\right)(t-x), \tag{138}$$

$$z_3(x,t,\tau,\mu) = \left(-y_{31}(\tau,\mu)\exp\left(\frac{\alpha\pi}{\beta}\right) + \frac{\beta p^2}{2}y_{212}(\mu)(x+\pi)\right)\exp\left(\frac{\alpha}{\beta}\right)(t+x), \tag{139}$$

$$v_{2_{tt}} - v_{2_{xx}} + p^2 v_2 + 2v_{1_{t\tau}} + 2v_{0_{t\mu}} + v_{0_{\tau\tau}} = 0, \quad 0 < x < \pi, \quad t > 0. \tag{140}$$

To obtain v_2 the boundary condition for v_2 at $x = \pi$ is made homogeneous by introducing the following transformation

$$\begin{aligned} \tilde{v}(x,t,\tau,\mu) = {} & v_2(x,t,\tau,\mu) + x\left(\beta v_{1_{xt}}(\pi,t,\tau,\mu) + \alpha v_{1_t}(\pi,t,\tau,\mu) + \alpha v_{0_\tau}(\pi,t,\tau,\mu)\right), \\ & 0 < x < \pi, \quad t > 0. \end{aligned} \tag{141}$$

The boundary value problem for v_2 now becomes a boundary value problem for $\tilde{v}$ with a homogeneous Dirichlet condition at $x = 0$ and a homogeneous Neumann condition at $x = \pi$. The inhomogeneous part of the PDE for $\tilde{v}$ now contains terms which give rise to secular terms.

To remove these secular terms it can be shown elementarily that $A_{0n}(\tau,\mu), B_{0n}(\tau,\mu), A_{1n}(\tau,\mu)$, and $B_{1n}(\tau,\mu)$ have to satisfy

$$A_{0n_\mu}(\tau,\mu) - H(n)B_{0n}(\tau,\mu) = 0, \quad B_{0n_\mu}(\tau,\mu) + H(n)A_{0n}(\tau,\mu) = 0, \quad \tau,\ \mu > 0, \tag{142}$$

$$A_{1n_\tau}(\tau,\mu) + \frac{\alpha}{\pi}A_{1n}(\tau,\mu) = 0, \quad B_{1n_\tau}(\tau,\mu) + \frac{\alpha}{\pi}B_{1n}(\tau,\mu) = 0, \quad \tau,\ \mu > 0, \tag{143}$$

where

$$H(n) = \frac{2\alpha\sqrt{\lambda_n}}{\pi}\left[\beta + \alpha\pi + \sum_{\substack{k=0\\k\neq n}}^{\infty}\frac{2\alpha}{\lambda_k - \lambda_n}\left(\frac{n+\frac{1}{2}}{k+\frac{1}{2}}\right)^2\right]. \tag{144}$$

It should be observed from (116), (117), (142), and (143) that if $A_{0n}(0,0) = B_{0n}(0,0) = A_{1n}(0,0) = B_{1n}(0,0) = 0$ then $A_{0n}(\tau,\mu) = B_{0n}(\tau,\mu) = A_{1n}(\tau,\mu) = B_{1n}(\tau,\mu) \equiv 0$ for $\forall\tau,\ \mu > 0$. This means that if we start with zero initial energy in the nth mode, there will be no energy present up to $O(\varepsilon^2)$ on a time scale of order ε^{-1}. In this case we say the coupling between the modes is of $O(\varepsilon^2)$. This allows truncation to those modes with non-zero initial energy. It follows from (142), (143), (116), and (117) that

$$A_{0n}(\tau,\mu) = \left(A_{0n}(0,0)\cos(H(n)\mu) + \frac{A_{0n_\mu}(0,0)}{H(n)}\sin(H(n)\mu)\right)\exp\left(-\frac{\alpha}{\pi}\tau\right), \tag{145}$$

$$B_{0n}(\tau,\mu) = \left(B_{0n}(0,0)\cos(H(n)\mu) + \frac{B_{0n_\mu}(0,0)}{H(n)}\sin(H(n)\mu)\right)\exp\left(-\frac{\alpha}{\pi}\tau\right), \tag{146}$$

$$A_{1n}(\tau,\mu) = A_{1n}(0,\mu)\exp\left(-\frac{\alpha}{\pi}\tau\right) \quad \text{and} \quad B_{1n}(\tau,\mu) = B_{1n}(0,\mu)\exp\left(-\frac{\alpha}{\pi}\tau\right). \tag{147}$$

After removing secular terms and after lengthy but still rather elementary calculations it follows that $v_2(x,t,\tau,\mu)$ is given by

$$\begin{aligned} v_2(x,t,\tau,\mu) = {} & \sum_{n=0}^{\infty}\omega_n(t,\tau,\mu)\sin\left(\left(n-\frac{1}{2}\right)x\right) - x\left(\beta v_{1_{xt}}(\pi,t,\tau,\mu) + \alpha v_{1_t}(\pi,t,\tau,\mu)\right. \\ & \left. + \alpha v_{0_\tau}(\pi,t,\tau,\mu)\right), \end{aligned} \tag{148}$$

where

$$\begin{aligned} \omega_n(t,\tau,\mu) = {} & E_n(\tau,\mu)\cos(\sqrt{\lambda_n}t) + F_n(\tau,\mu)\sin(\sqrt{\lambda_n}t) - \sum_{m\neq n}^{\infty}\frac{1}{\lambda_n - \lambda_m}\left(\gamma_{1_{mn}}\cos(\sqrt{\lambda_m}t)\right. \\ & \left. + \gamma_{2_{mn}}\sin(\sqrt{\lambda_m}t)\right) - \sum_{k=0}^{\infty}\sum_{\substack{m=0\\m\neq k,n}}^{\infty}\frac{1}{\lambda_n - \lambda_m}\left(\eta_{1_{kmn}}\cos(\sqrt{\lambda_m}t) + \eta_{2_{kmn}}\sin(\sqrt{\lambda_m}t)\right) \end{aligned}$$

with

$$\gamma_{1_{mn}} = \frac{2\alpha(-1)^{m+n}}{\pi(n+\frac{1}{2})^2}\left[A_{m_\tau}\frac{(m+\frac{1}{2})^2\lambda_n - \lambda_m^2}{\lambda_n - \lambda_m} + \sqrt{\lambda_m}\left(\sqrt{\lambda_m}(\beta+\alpha\pi)A_m + D_m\right)\right],$$

$$\begin{aligned}
\gamma_{2_{mn}} &= \frac{2\alpha(-1)^{m+n}}{\pi(n+\frac{1}{2})^2}\left[B_{m_\tau}\frac{(m+\frac{1}{2})^2\lambda_n-\lambda_m^2}{\lambda_n-\lambda_m}+\sqrt{\lambda_m}\left(\sqrt{\lambda_m}(\beta+\alpha\pi)B_m-C_m\right)\right],\\
\eta_{1_{kmn}} &= \left(\frac{2\alpha}{\pi}\right)^2(-1)^2\left(\frac{(m+\frac{1}{2})^2}{(n+\frac{1}{2})(k+\frac{1}{2})}\right)^2\frac{\lambda_m}{\lambda_k-\lambda_m}A_m,\\
\eta_{2_{kmn}} &= \left(\frac{2\alpha}{\pi}\right)^2(-1)^2\left(\frac{(m+\frac{1}{2})^2}{(n+\frac{1}{2})(k+\frac{1}{2})}\right)^2\frac{\lambda_m}{\lambda_k-\lambda_m}B_m.
\end{aligned}$$

As for u_0, u_1, u_2, and u_3 it follows that the general solution of the initial-boundary value problem (126) - (130) for u_4 is given by

$$\begin{aligned}
u_4(\bar{x},x,\bar{t},t,\tau,\mu) = & \sum_{\substack{\sigma\\ \sigma\neq 0\\ \sigma^2\neq\frac{-1}{\beta^2}}}((A_{4\sigma}\cos(\sigma\bar{x})\cos(\sigma\bar{t})+B_{4\sigma}\cos(\sigma\bar{x})\sin(\sigma\bar{t})+C_{4\sigma}\sin(\sigma\bar{x})\cos(\sigma\bar{t})\\
& +D_{4\sigma}\sin(\sigma\bar{x})\sin(\sigma\bar{t})))\\
& +\left(y_4\exp\left(\frac{\bar{x}}{\beta}\right)+z_4\exp\left(\frac{-\bar{x}}{\beta}\right)\right)\exp\left(-\frac{\bar{t}}{\beta}\right)+v_4(x,t,\tau,\mu),
\end{aligned} \tag{149}$$

where $A_{4\sigma}, B_{4\sigma}, C_{4\sigma}, D_{4\sigma}, y_4, z_4$ and v_4 are functions of x, t, τ, and μ satisfying

$$A_{4\sigma}(0,t,\tau,\mu)=B_{4\sigma}(0,t,\tau,\mu)=y_4(0,t,\tau,\mu)+z_4(0,t,\tau,\mu)=v_4(0,t,\tau,\mu)=0,\ t\geq 0, \tag{150}$$
$$A_{4\sigma}(\pi,t,\tau,\mu)=B_{4\sigma}(\pi,t,\tau,\mu)=C_{4\sigma}(\pi,t,\tau,\mu)=D_{4\sigma}(\pi,t,\tau,\mu)=0,\quad t\geq 0. \tag{151}$$

The still arbitrary functions in u_0, u_1, u_2, u_3 and u_4 can be used to avoid secular terms in higher order approximations.

Now the solution for u_0 of the problem (41) - (45) has been determined completely. Observing the solutions for u_1 and u_2 of the initial-boundary value problems (59) - (63) and the initial-boundary value problem (77) - (81) respectively it should be noted that the functions A_{1n}, B_{1n} are still arbitrary functions in μ and that A_{2n}, B_{2n} and are still arbitrary functions in τ and μ, and that the "singular" part of u_2 still contains an arbitrary function y_{212} in μ. These functions have to be used to avoid secular terms in the higher order approximations. It is, however, our purpose to construct a formal approximation of the solution that satisfies the differential equation up to order ϵ^2. For that reason, we take A_{1n}, B_{1n} to be equal to $A_{1n}(\tau,0)$, $B_n(\tau,0)$ respectively, and A_{2n}, B_{2n}, and y_{212} to be equal to their initial values $A_{2n}(0,0), B_{2n}(0,0), y_{212}(0)$ respectively. By differentiating u_2 with respect to $\bar{x}$, putting $x=\pi$ and $\bar{t}=t=\tau=\mu=0$, and by applying (3) the arbitrary constant $y_{212}(0)$ in the "singular" part of u_2 can be determined, yielding

$$y_{212}(0)=\frac{\beta(\phi_0'(\pi)+\alpha\psi_0(\pi))}{\cosh\left(\frac{1}{\beta}\left(\frac{\pi}{\epsilon}-\alpha\pi\right)\right)}. \tag{152}$$

To determine the arbitrary constants $A_{0n}(0,0), B_{0n}(0,0), A_{1n}(0,0), B_{1n}(0,0), A_{2n}(0,0)$ and $B_{2n}(0,0)$ we can use the initial conditions (44)-(45), (62)-(63), and (80)-(81) respectively.

So far we have constructed a formal approximation $\bar{u}$ of the solution of the initial-boundary value problem (1) - (5) up to order ϵ^2, that is, $\bar{u} = u_0 + \epsilon u_1 + \epsilon^2 u_2$, in which u_0, u_1, and u_2 are given by

$$u_0(\bar{x}, x, \bar{t}, t, \tau, \mu) = \exp\left(-\frac{\alpha}{\pi}\tau\right) \sum_{n=1}^{\infty} \Big(A_{0n}(0,0) \cos\left(\sqrt{\lambda_n} t + H(n)\mu\right) + B_{0n}(0,0) \sin\left(\sqrt{\lambda_n} t + H(n)\mu\right)\Big) \sin((n-1/2)\, x), \tag{153}$$

$$u_1(\bar{x}, x, \bar{t}, t, \tau, \mu) = \exp\left(-\frac{\alpha}{\pi}\tau\right) \sum_{n=1}^{\infty} \Big(A_{1n}(0,0) \cos\left(\sqrt{\lambda_n} t\right) + B_{1n}(0,0) \sin\left(\sqrt{\lambda_n} t\right) + S(t,\tau,\mu)\Big) \sin((n-1/2)\, x) - \alpha x v_{0_t}(\pi, t, \tau, \mu), \tag{154}$$

and

$$u_2(\bar{x}, x, \bar{t}, t, \tau, \mu) = y_{212}(0) \left(\sinh\left(\frac{1}{\beta}(\bar{x} - \alpha x) \right) \right) \exp\left(-\frac{1}{\beta}(\bar{t} - \alpha t) \right) + v_2(x, t, \tau, \mu), \tag{155}$$

where H_n is given by (144), $S(t,\tau,\mu)$ is given by (119), where $y_{212}(0)$ is given by (152), where v_2 is given by (148), where v_0 is given by (93), $A_{0n}(0,0) = \frac{2}{\pi}\int_0^\pi \phi_0(x) \sin\left(\left(n - \frac{1}{2}\right)x\right) dx$, $B_{0n}(0,0) = \frac{2}{\pi\left(n-\frac{1}{2}\right)}\int_0^\pi \psi_0(x) \sin\left(\left(n - \frac{1}{2}\right)x\right) dx$, $A_{1n}(0,0) = \frac{2}{\pi}\int_0^\pi \phi_1(x) \sin\left(\left(n - \frac{1}{2}\right)x\right) dx$, $B_{1n}(0,0) = \frac{2}{\pi\left(n-\frac{1}{2}\right)}\int_0^\pi \psi_1(x) \sin\left(\left(n - \frac{1}{2}\right)x\right) dx + \frac{\alpha}{\pi\left(n-\frac{1}{2}\right)} A_{0n}(0,0)$ respectively, and where $\lambda_n = p^2 + \left(n - \frac{1}{2}\right)^2$. By considering the "singular" part of the approximation we can conclude that the angular velocity damper (β) plays only an important role in a very small region in the (x,t)-plane near $x = \pi$. This damper makes the angle of the string at $x = \pi$ zero in a very short time. On the other hand, the velocity damper (α) suppresses the oscillations in the entire string. However, this damping takes place on a time-scale of order ϵ^{-1}.

4.3. Separation of variables.

Since the initial-boundary value problem (1)-(5) is linear the method of separation of variables can also be used to solve problem (1)-(5) approximately. In this section we will shortly outline how nontrivial solutions of the boundary value problem (1)-(3) can be obtained in the form $X(x)T(t)$. By Substituting $X(x)T(t)$ into (1)-(3) the following eigenvalue problem is obtained

$$\frac{X''(x)}{X(x)} = \rho^2 = \frac{T''(t)}{T(t)} + p^2, \quad \rho \in \mathbb{C}, \quad 0 < x < \pi, \quad t \geq 0, \tag{156}$$

$$X(0) = 0, \tag{157}$$

$$X'(\pi)T(t) = -\left(\epsilon\beta X'(\pi) + \epsilon\alpha X(\pi)\right) T'(t), \tag{158}$$

where $\rho \in \mathbb{C}$ is a separation parameter. From (156) and (157) it follows that $X(x) = A\sinh(\rho x)$ with A an arbitrary constant. By substituting $X(x) = A\sinh(\rho x)$ into (158) the following expression is obtained

$$\rho \cosh(\rho\pi) T(t) = -\left(\epsilon\beta\rho\cosh(\rho\pi) + \epsilon\alpha\sinh(\rho\pi)\right) T'(t). \tag{159}$$

It should be observed that for $\epsilon\beta\rho\cosh(\rho\pi) + \epsilon\alpha\sinh(\rho\pi) = 0$ only trivial solutions for the boundary-value problem (156)-(158) are obtained. By differentiating (159) with respect to t and by using (159) again it follows that

$$T''(t) = \frac{\rho^2\cosh^2(\rho\pi)}{\left(\epsilon\beta\rho\cosh(\rho\pi) + \epsilon\alpha\sinh(\rho\pi)\right)^2} T(t). \tag{160}$$

By substituting (160) into (156) it then follows that ρ has to satisfy

$$\rho^2 = \frac{\rho^2\cosh^2(\rho\pi)}{\left(\epsilon\beta\rho\cosh(\rho\pi) + \epsilon\alpha\sinh(\rho\pi)\right)^2} + p^2 \tag{161}$$

with $\rho \in \mathbb{C}$. Approximations of ρ can be obtained from (161) in the following way. First (161) is rewritten in

$$\left(\rho^2 - p^2\right)\left(\epsilon\beta\rho\cosh(\rho\pi) + \epsilon\alpha\sinh(\rho\pi)\right)^2 = \rho^2\cosh^2(\rho\pi), \tag{162}$$

and then the following two cases $\rho\cosh(\rho\pi) = 0 + O(\epsilon)$ and $\rho\cosh(\rho\pi) \neq 0 + O(\epsilon)$ are considered.

4.3.1. *The case* $\rho\cosh(\rho\pi) = 0 + O(\epsilon)$.

For this case we have to consider two subcases, that is, $\cosh(\rho\pi) = 0 + O(\epsilon)$ or $\rho = 0 + O(\epsilon)$. Dividing (162) by $\rho^2 - p^2$, putting $\rho = a + bi$ with $a, b \in \Re$, and then taking apart the real part and the imaginary part of the so-obtained equation, it follows that

$$X_0X_1 + Y_0Y_1 = \epsilon^2\left(X_5 - X_6 + X_4\right), \tag{163}$$

and

$$X_0Y_1 - X_1Y_0 = \epsilon^2\left(X_7 - X_8 + Y_4\right), \tag{164}$$

where

$$\begin{aligned}
X_0 &= \frac{\left(a^2-b^2\right)\left(a^2-b^2-p^2\right)+4a^2b^2}{\left(a^2-b^2-p^2\right)+4a^2b^2}, \quad Y_0 = \frac{2abp^2}{\left(a^2-b^2-p^2\right)+4a^2b^2},\\
X_1 &= \cos(2b\pi)\cosh(2a\pi) + 1, \quad Y_1 = \sin(2b\pi)\sinh(2a\pi),\\
X_4 &= \beta^2\left(a^2-b^2\right) + \alpha^2, \quad Y_4 = 2ab\beta^2,\\
X_5 &= \cos(2b\pi)\left(\left[\beta^2\left(a^2-b^2\right)+\alpha^2\right]\cosh(2a\pi) + 2a\alpha\beta\sinh(2a\pi)\right),\\
X_6 &= 2b\beta\sin(2b\pi)\left(a\beta\sinh(2a\pi) + \alpha\cosh(2a\pi)\right),\\
X_7 &= 2b\beta\cos(2b\pi)\left(a\beta\cosh(2a\pi) + \alpha\sinh(2a\pi)\right),\\
X_8 &= \sin(2b\pi)\left(\left[\beta^2\left(a^2-b^2\right)+\alpha^2\right]\sinh(2a\pi) + 2a\alpha\beta\cosh(2a\pi)\right).
\end{aligned}$$

It is assumed that the eigenvalue $\rho = a + bi$ can be expanded in a power series in ϵ, that is,

$$a = a_0 + \epsilon a_1 + \epsilon^2 a_2 + \dots, \tag{165}$$
$$b = b_0 + \epsilon b_1 + \epsilon^2 b_2 + \dots. \tag{166}$$

To approximate ρ (163) and (164) are then expanded in power series in ϵ. For the case $\cosh(\rho\pi) = 0 + O(\epsilon)$ it follows that $\rho = a + ib = i(n - \frac{1}{2}) + O(\epsilon)$. By substituting (165) and (166) (in this case $a_0 = 0$ and $b_0 = n - 1/2$ with $n \in \mathbb{Z}$) into (163)-(164) and equating the coefficients of ϵ^n for $n = 0, 1, 2, \dots$ we obtain

$$a_1 = \pm\frac{\alpha}{\pi}\sqrt{1 + \frac{p^2}{(n-\frac{1}{2})^2}}, \quad a_2 = 0, \dots, \tag{167}$$

and

$$b_1 = 0, \quad b_2 = \frac{\alpha}{\pi}\left(\beta(n - \frac{1}{2})\left(1 + \frac{p^2}{(n-\frac{1}{2})^2}\right) + \frac{\alpha}{\pi}\frac{p^2}{(n-\frac{1}{2})^3}\right), \dots. \tag{168}$$

So for the case $cosh(\rho\pi) = 0 + O(\epsilon)$ the approximations for ρ are given by

$$\begin{aligned}\rho = {} & \frac{\alpha}{\pi}\sqrt{1 + \frac{p^2}{(n-\frac{1}{2})^2}}\epsilon \\ & + \left(n - \frac{1}{2} + \frac{\alpha}{\pi}\left(\beta(n - \frac{1}{2})\left(1 + \frac{p^2}{(n-\frac{1}{2})^2}\right) + \frac{\alpha}{\pi}\frac{p^2}{(n-\frac{1}{2})^3}\right)\epsilon^2\right) i + O(\epsilon^3)\end{aligned} \tag{169}$$

or

$$\begin{aligned}\rho = {} & -\frac{\alpha}{\pi}\sqrt{1 + \frac{p^2}{(n-\frac{1}{2})^2}}\epsilon \\ & + \left(n - \frac{1}{2} + \frac{\alpha}{\pi}\left(\beta(n - \frac{1}{2})\left(1 + \frac{p^2}{(n-\frac{1}{2})^2}\right) + \frac{\alpha}{\pi}\frac{p^2}{(n-\frac{1}{2})^3}\right)\epsilon^2\right) i + O(\epsilon^3),\end{aligned} \tag{170}$$

with $n \in \mathbb{Z}$.

For the case $\rho = 0 + O(\epsilon)$ it follows that $a_0 = b_0 = 0$. Again to approximate the eigenvalue ρ (165) and (166) are substituted into (163)-(164). After expanding (165) and (166) into power series in ϵ and equating the coefficients of ϵ^n for $n = 0, 1, 2, \dots$ we obtain

$$a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ \dots, \tag{171}$$

and

$$b_1 = 0,\ b_2 = 0,\ b_3 = 0\ \dots. \tag{172}$$

For the case $\rho = 0 + O(\epsilon)$ only the trivial solution of the boundary value problem (156) - (158) will be obtained. So, for $\rho = 0 + O(\epsilon)$ no eigenvalues are found.

Now we can approximate the solution for the case $\rho\cosh(\rho\pi) = 0 + O(\epsilon)$. For instance, if we approximate the eigenvalue up to order ϵ, that is, $\rho = (n - 1/2)i + \frac{\alpha}{\pi}\sqrt{1 + \frac{p^2}{(n-\frac{1}{2})^2}}\epsilon$ or

$\rho = (n-1/2)i - \frac{\alpha}{\pi}\sqrt{1+\frac{p^2}{(n-\frac{1}{2})^2}}\epsilon$ with $n \in \mathbb{Z}$ it follows from (156) that $T(t)$ and $X(x)$ are approximated by

$$A_n \exp\left(\left(\frac{\alpha}{\pi}\epsilon + \sqrt{p^2+\left(n-\frac{1}{2}\right)^2} i\right) t\right) + B_n \exp\left(-\left(\frac{\alpha}{\pi}\epsilon + \sqrt{p^2+\left(n-\frac{1}{2}\right)^2} i\right) t\right), \quad (173)$$

and

$$\sinh\left(\left((n-1/2)i + \frac{\alpha}{\pi}\sqrt{1+\frac{p^2}{(n-\frac{1}{2})^2}\epsilon}\right) x\right), \quad (174)$$

or by

$$C_n \exp\left(\left(\frac{\alpha}{\pi}\epsilon - \sqrt{p^2+\left(n-\frac{1}{2}\right)^2} i\right) t\right) + D_n \exp\left(-\left(\frac{\alpha}{\pi}\epsilon - \sqrt{p^2+\left(n-\frac{1}{2}\right)^2} i\right) t\right), \quad (175)$$

and

$$\sinh\left(\left((n-1/2)i - \frac{\alpha}{\pi}\sqrt{1+\frac{p^2}{(n-\frac{1}{2})^2}\epsilon}\right) x\right) \quad (176)$$

respectively. By applying (158) it follows that $A_n = C_n = 0$. By taking $\bar{A_n} = B_n - D_n$ and $\bar{B_n} = -i(B_n + D_n)$ and using the superposition principle the general, approximate solution for the case $\rho\cosh(\rho\pi) = 0 + O(\epsilon)$ is given by

$$\begin{aligned}
&\exp\left(-\frac{\alpha}{\pi}\epsilon t\right)\sum_{n=1}^{\infty}\Bigg[\exp\left(\frac{\alpha}{\pi}\sqrt{\left(1+\frac{p^2}{\left(n-\frac{1}{2}\right)^2}\right)}\epsilon x\right)\Bigg[\bar{A_n}\left(\cos\left(\sqrt{\lambda_n}t\right)\cos\left(\left(n-\frac{1}{2}\right)x\right)+\right.\\
&\left.\sin\left(\sqrt{\lambda_n}t\right)\sin\left(\left(n-\frac{1}{2}\right)x\right)\right)+\bar{B_n}\left(\sin\left(\sqrt{\lambda_n}t\right)\cos\left(\left(n-\frac{1}{2}\right)x\right)-\right.\\
&\left.\cos\left(\sqrt{\lambda_n}t\right)\sin\left(\left(n-\frac{1}{2}\right)x\right)\right)\Bigg]-\\
&\exp\left(-\frac{\alpha}{\pi}\sqrt{\left(1+\frac{p^2}{\left(n-\frac{1}{2}\right)^2}\right)}\epsilon x\right)\Bigg[\bar{A_n}\left(\cos\left(\sqrt{\lambda_n}t\right)\cos\left(\left(n-\frac{1}{2}\right)x\right)-\right.\\
&\left.\sin\left(\sqrt{\lambda_n}t\right)\sin\left(\left(n-\frac{1}{2}\right)x\right)\right)+\bar{B_n}\left(\sin\left(\sqrt{\lambda_n}t\right)\cos\left(\left(n-\frac{1}{2}\right)x\right)+\right.\\
&\left.\cos\left(\sqrt{\lambda_n}t\right)\sin\left(\left(n-\frac{1}{2}\right)x\right)\right)\Bigg]\Bigg], \quad (177)
\end{aligned}$$

where $\lambda_n = p^2 + (n-1/2)^2$. The constants $\bar{A_n}$ and $\bar{B_n}$ can be determined by using the initial conditions (4) and (5).

4.3.2. *The case* $\rho\cosh(\rho\pi) \neq 0 + O(\epsilon)$.
Dividing (162) by $\rho\cosh(\rho\pi)$ it follows that

$$\left(1 - \frac{p^2}{\rho^2}\right)(\epsilon\beta\rho + \epsilon\alpha\tanh(\rho\pi))^2 = 1. \tag{178}$$

It should be observed in this case that the first order approximation of ρ is proportional to $\frac{1}{\epsilon\beta}$. For that reason the eigenvalue ρ is approximated by

$$\rho = \frac{\rho_{-1}}{\epsilon} + \rho_0 + \epsilon\rho_1 + \epsilon^2\rho_2 + \dots. \tag{179}$$

Now we will consider two cases for the real part of the eigenvalue ρ. In the first case the real part of the eigenvalue ρ is assumed to be positive, and in the second case the real part of the eigenvalue ρ is assumed to be negative. For the positive real part it is clear from (179) that

$$\lim_{\epsilon\downarrow 0}\tanh(\rho\pi) = 1. \tag{180}$$

So for $\epsilon \downarrow 0$ it then follows from (178) and (180) that

$$\left(1 - \frac{p^2}{\rho^2}\right)(\epsilon\beta\rho + \epsilon\alpha - 2\epsilon\alpha\frac{e^{-2\rho\pi}}{1 + e^{-2\rho\pi}})^2 = 1. \tag{181}$$

Rescaling ρ by $\frac{\tilde{\rho}}{\epsilon}$, where $\tilde{\rho} = a + ib$ is of order one it follows that

$$\eta\left((\beta a + \epsilon\alpha)^2 - (\beta b)^2\right) - 2\beta b(\beta a + \epsilon\alpha)\theta + i\left(\theta\left((\beta a + \epsilon\alpha)^2 - (\beta b)^2\right) + 2\beta b(\beta a + \epsilon\alpha)\eta\right) = 1, \tag{182}$$

where $\eta = 1 - \frac{p^2(a^2-b^2)}{(a^2+b^2)^2}$ and $\theta = \frac{2abp^2}{(a^2+b^2)^2}$, and where terms which are exponentially small (that is, terms of order $\epsilon e^{\left(\frac{-2\tilde{\rho}\pi}{\epsilon}\right)}$) have been neglected. It then follows from (182) that the real and the imaginary part of (182) have to satisfy

$$\eta\left((\beta a + \epsilon\alpha)^2 - (\beta b)^2\right) - 2\beta b(\beta a + \epsilon\alpha)\theta = 1, \tag{183}$$

and

$$\theta\left((\beta a + \epsilon\alpha)^2 - (\beta b)^2\right) + 2\beta b(\beta a + \epsilon\alpha)\eta = 0, \tag{184}$$

respectively. To approximate $\tilde{\rho}$ the power series representations (165) and (166) are used again. By substituting (165) and (166) into (183) and (184) the following results are obtained

$$a_0 = \frac{1}{\beta},\ a_1 = -\frac{\alpha}{\beta},\ a_2 = \frac{\beta p^2}{2},\ a_3 = \alpha\beta p^2 \tag{185}$$

and

$$b_0 = b_1 = b_2 = b_3 = \dots = 0. \tag{186}$$

From (185) and (186) the approximation of ρ up to order ϵ^3 is given by

$$\rho = \frac{1}{\epsilon\beta} - \frac{\alpha}{\beta} + \frac{\beta p^2}{2}\epsilon + \alpha\beta p^2\epsilon^2 + O(\epsilon^3). \tag{187}$$

When the eigenvalue ρ has a negative real part a compeletely similar analysis can be given, yielding

$$\rho = -\frac{1}{\epsilon\beta} + \frac{\alpha}{\beta} - \frac{\beta p^2}{2}\epsilon - \alpha\beta p^2\epsilon^2 + O(\epsilon^3). \tag{188}$$

Now it should be observed that the approximation of ρ as given by (187) does not approximate an eigenvalue of the boundary value problem (156)-(158) since the boundary condition (158) is not satisfied. As in section 4.3.1 a nontrivial solution $X(x)T(t)$ of the boundary value problem (156) - (158) can again be constructed (using (188)), yielding

$$E\sinh\left(\left(1 - \epsilon\alpha + \frac{\beta^2 p^2\epsilon^2}{2}\right)\frac{x}{\epsilon\beta}\right)\exp\left(-\left(\frac{1}{\epsilon\beta} - \frac{\alpha}{\beta}\right)t\right), \tag{189}$$

where E is a constant which follows by evaluating u_x at $x = \pi$ and $t = 0$ (see also section 4.2). An approximation of the solution of the initial-boundary value problem (1)-(5) follows from (177) and (189). The results can readily be compared with the results obtained in section 4.1 and in section 4.2.

5. The validity of formal approximations.

Since the initial-boundary value problem (1)-(5) contains a small parameter ϵ perturbation methods may be applied to construct approximations of the solution. In most perturbation methods a function is constructed that satisfies the differential equation, the initial conditions, and the boundary conditions up to some order depending on the small parameter ϵ. Such a function is usually called a formal approximation. To show that this formal approximation is an asymptotic approximation (as $\epsilon \to 0$) requires an additional analysis. The formal approximation $\bar{u}(x,t;\varepsilon) = u_o + \epsilon u_1 + \epsilon^2 u_2$ which has been constructed in section 4.2 satisfies the following initial-boundary value problem

$$\begin{aligned}
\bar{u}_{tt} - \bar{u}_{xx} + p^2\bar{u} &= \varepsilon^3 C_o(x,t;\varepsilon),\ 0 < x < \pi,\ t > 0, && (190)\\
\bar{u}(0,t;\varepsilon) &= 0,\ t \geq 0, && (191)\\
\bar{u}_x(\pi,t;\varepsilon) &= -\varepsilon\left(\beta\bar{u}_{xt}(\pi,t;\varepsilon) + \alpha\bar{u}_t(\pi,t;\varepsilon)\right) + \varepsilon^3 C_1(t;\varepsilon),\ t \geq 0, && (192)\\
\bar{u}(x,0;\varepsilon) &= \phi(x) - \varepsilon^3 C_2(x;\varepsilon),\ 0 < x < \pi, && (193)\\
\bar{u}_t(x,0;\varepsilon) &= \varphi(x) - \varepsilon^3 C_3(x;\varepsilon),\ 0 < x < \pi, && (194)
\end{aligned}$$

where C_o, C_1, C_2, and C_3 satisfy $C_o(0,t;\varepsilon) = C_2(0;\varepsilon) = C_3(0;\varepsilon) = 0$.

To prove the asymptotic validity of $\bar{u}(x,t;\varepsilon)$ we define some auxiliary functions,

$$\bar{a}(t) = \bar{u}(\cdot,t;\varepsilon), \quad \bar{b}(t) = \bar{u}_t(\cdot,t;\varepsilon), \quad \bar{\eta}(t) = \varepsilon\beta\bar{u}_x(\pi,t;\varepsilon). \tag{195}$$

We also denote $\bar{a}$, $\bar{b}$ and $\bar{\eta}$ for $\bar{a}(t)$, $\bar{b}(t)$ and $\bar{\eta}(t)$, respectively. By differentiating these functions with respect to t we obtain

$$\begin{pmatrix} \bar{a}_t \\ \bar{b}_t \\ \bar{\eta}_t \end{pmatrix} = \begin{pmatrix} \bar{b} \\ \bar{a}_{xx} - p^2\bar{a} \\ -\bar{a}(\pi) - \varepsilon\alpha\bar{b}(\pi) \end{pmatrix} + \begin{pmatrix} 0 \\ C_o \\ C_1 \end{pmatrix}. \tag{196}$$

We also define the same operator A as in section 3, i.e.

$$A\bar{y} = \begin{pmatrix} \bar{b} \\ \bar{a}_{xx} - p^2\bar{a} \\ -\bar{a}(\pi) - \varepsilon\alpha\bar{b}(\pi) \end{pmatrix}, \tag{197}$$

where $\bar{y} = \begin{pmatrix} \bar{a} \\ \bar{b} \\ \bar{\eta} \end{pmatrix}$. By using (153) - (154) the following abstract-Cauchy problem is obtained

$$\frac{d\bar{y}}{dt} = A\bar{y} + \varepsilon^3\Theta, \tag{198}$$

$$\bar{y}(0) = \bar{\Phi}, \tag{199}$$

where $\Theta = \begin{pmatrix} 0 \\ C_o \\ C_1 \end{pmatrix}$, and $\bar{\Phi} = \Phi - \varepsilon^3 K$ with $K = \begin{pmatrix} C_2 \\ C_3 \\ \varepsilon\beta C_2' \end{pmatrix}$, where $'$ in C_2' denotes differentiation of C_2 with respect to x. The solution of the initial value problem (198) - (199) is given by

$$\bar{y} = T(t)\bar{\Phi} + \varepsilon^3 \int_0^t T(t-s)\Theta(s)ds, \tag{200}$$

where $T(t)$ is defined as in section 3. For $0 \le x \le \pi$ and $0 \le t \le L\varepsilon^{-1}$, (where L is a positive and ϵ-independent constant) we can now estimate the difference between y and $\bar{y}$, that is,

$$\|y - \bar{y}\|_{\mathcal{H}} \le \left\|\varepsilon^3 K\right\|_{\mathcal{H}} + \varepsilon^3 \int_0^t \|\Theta\|_{\mathcal{H}}\, ds, \tag{201}$$

where $\|\Theta\|_{\mathcal{H}}$ satisfies

$$\|\Theta\|_{\mathcal{H}}^2 \le \frac{\beta p^2}{1-\varepsilon\alpha}\left(Const_o \exp\left(-2\frac{\bar{t}-\alpha t}{\beta}\right) + \varepsilon.Const_1 \exp\left(-\frac{\bar{t}-\alpha t}{\beta}\right)\right) + \varepsilon\beta M_0 + M_1, \tag{202}$$

in which

$$Const_o = 2\beta p^2 \left(\beta\left(\phi_1'(\pi) + \alpha\varphi_o(\pi)\right)\right)^2 \left(\frac{\cosh\left(2\frac{(1-\varepsilon\alpha)\pi}{\varepsilon\beta}\right) - 1}{\cosh\left(2\frac{(1-\varepsilon\alpha)\pi}{\varepsilon\beta}\right)}\right),$$

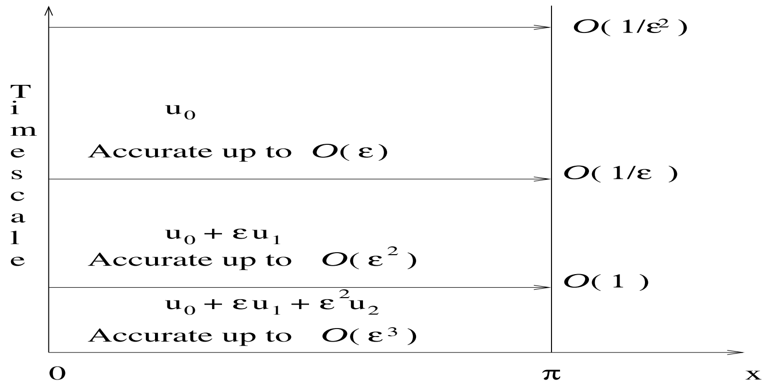
and

$$Const_1 = 2\beta.M.\left|2\beta\left(\phi_1'(\pi) + \alpha\varphi_0(\pi)\right)\right| \left(\frac{\cosh\left(\frac{(1-\varepsilon\alpha)\pi}{\varepsilon\beta}\right) - 1}{\cosh\left(\frac{(1-\varepsilon\alpha)\pi}{\varepsilon\beta}\right)}\right),$$

and in which M_0, M_1, and M are positive constants. From (201) - (202) it follows that there are positive constants M_3 and M_4 such that the following inequality

$$\|y - \bar{y}\|_{\mathcal{H}} \leq \varepsilon^3 \left(M_3 + M_4 t\right) \tag{203}$$

holds. We conclude from (203) that $y - \bar{y} = O(\varepsilon^3)$ on a time scale of order one. It also



*F*igure 3. Validity of the approximations on different time-scales.

follows from (203) that $u - \left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2\right) = O(\varepsilon^3)$ on a time scale of order one, and that $u - \left(u_0 + \varepsilon u_1\right) = O(\varepsilon^2)$ on a time scale of order ϵ^{-1}, and that $u - u_0 = O(\varepsilon)$ on a time scale of order ε^{-2}. We resume the validity of the approximations on different time scales in figure 3.

6. Conclusions and remarks

In this paper an initial-boundary value problem for a string equation has been considered. One end of the string is assumed to be fixed and the other end of the string is attached to a dashpot system, where the damping generated by the dashpot system is assumed to be small, and is assumed to be proportional to the vertical and the angular velocity of the string in the endpoint. A semigroup approach has been used to prove the wellposedness of the singularly perturbed problem. It has been shown how a multiple scales perturbation method can be applied to construct asymptotic approximations of the solution which are valid for large values of time t. It turns out that at least six scales (that is, four time-scales and two space-scales) are necessary to describe the behaviour of the solution sufficiently accurate for these large values of time. It has been shown in section 4 that the presence of the term $-\epsilon\beta u_{xt}$ in the boundary condition at $x = \pi$ gives rise to a singularly perturbed problem, which in fact leads to a characteristic layer problem. This term (that is, $-\epsilon\beta u_{xt}$) is proportional to the angular velocity of the string in the endpoint at $x = \pi$. From the results as presented in section 4 it follows that due to this type of angular velocity damping the angle of the string at $x = \pi$ tends to zero in a very short time (that is, within times of

$O(1)$). The vertical oscillations of the string are hardly influenced by this angular velocity damper. The vertical oscillations of the string also decrease to zero (due to the term $-\epsilon\alpha u_t$ in the boundary condition at $x = \pi$), but this happens on a much larger time-scale, that is, on time-scales of $O(\frac{1}{\epsilon})$ or larger. In section 4.3 we also presented shortly an alternative way to approximate the solution by applying the method of separation of variables to the problem. This approach of course can only be applied to linear problems, whereas in section 4.2 the method as presented and worked out in detail can also be applied to weakly nonlinear problems.

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