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On the Periodic Solutions of a Generalized Nonlinear Van der Pol Oscillator

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Abstract

In this paper a generalized nonlinear Van der Pol oscillator equation $\ddot{X}+X^{(2m+1)/(2n+1)} = \epsilon\left(1-X^2\right)\dot{X}$ with $m,\, n \in \mathbb{N}$ will be studied. It will be shown that the recently developed perturbation method based on integrating vectors can be used to approximate first integrals and periodic solutions. The existence, uniqueness and stability of time-periodic solutions are obtained by using the approximations for the first integrals.

Keywords: *Integrating vector, first integral, perturbation method, nonlinear oscillator, periodic solution.*

1 Introduction

We consider a generalized nonlinear Van der Pol oscillator equation

$$\ddot{X} + X^{\frac{2m+1}{2n+1}} = \epsilon\left(1-X^2\right)\dot{X}, \tag{1.1}$$

where $X = X(t)$, $m,\, n \in \mathbb{N}$, and where ϵ is a small parameter satisfying $0 < \epsilon \ll 1$. The dot represents differentiation with respect to t. Many researchers studied the unperturbed nonlinear oscillator equation

$$\ddot{X} + f(X) = 0. \tag{1.2}$$

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For instance, Awrejcewicz and Andrianov [1, 2] studied (1.2) using the so-called small and large δ-method. Using a generalized harmonic balance method Mickens and his co-authors [3]-[6] also studied equation (1.2). For a particular case of equation (1.2) with $f(X) = X^{1/(2n+1)}$ some results have been presented in [2, 3, 5, 7]. The periods of the periodic solutions for this particular case have been approximated by Mickens in [3, 5]. Moreover, exact expressions for the periods of the periodic solutions for this particular equation (1.2) have been given by Van Horssen in [7]. Equation (1.1) with $m = n = 0$ is the well-known Van der Pol equation. Recently (1.1) with $m = 0$ and $n = 1$ has been studied in [6]. Approximations of the periodic solution are constructed in [6] by using the method of harmonic balance. In this paper the recently developed perturbation method based on integrating factors (see [8]-[12]) is used to approximate first integrals and periodic solutions for the generalized nonlinear Van der Pol oscillator (1.1). In this paper not only asymptotic approximations of first integrals are constructed but also asymptotic approximations of the periodic solutions and their periods are determined. The presented results include existence, uniqueness, and stability properties of the periodic solutions. In this paper we show that straightforward expansions in ϵ can be used to construct asymptotic results on long time-scales. This paper is organized as follows. In section 2 of this paper it is shown how approximations of first integrals can be constructed. It will be shown in section 3 of this paper how the existence, the stability, and the period of time-periodic solutions can be determined from the constructed approximations of the first integrals. Finally in section 4 of this paper some conclusions will be drawn and some remarks will be made.

2 Approximations of First Integrals

In this section we will show how the perturbation method based on integrating factors can be applied to approximate first integrals for a generalized nonlinear Van der Pol oscillator. Consider a generalized nonlinear Van der Pol oscillator equation

$$\ddot{X} + X^{\frac{2m+1}{2n+1}} = \epsilon\left(1 - X^2\right)\dot{X}. \tag{2.1}$$

The unperturbed solutions of (2.1) with $\epsilon = 0$ form a family of periodic orbits. This family covers the entire "phase plane" $(X, \dot{X})$. Each periodic orbit corresponds to a constant energy level $E = \frac{1}{2}\dot{X}^2 + \frac{(2n+1)}{(2m+2n+2)}X^{(2m+2n+2)/(2n+1)}$. To a constant energy level E a phase angle ψ can be defined by $\psi = \int_0^X \frac{dr}{\sqrt{2E - \frac{(2n+1)}{(m+n+1)}r^{(2m+2n+2)/(2n+1)}}}$. We use the transformation $(X, \dot{X}) \longmapsto (E, \psi)$, and then obtain

$$\begin{cases} \dot{E} = \epsilon\dot{X}f & = \quad g_1(E,\psi), \\ \dot{\psi} = 1 - \epsilon\int_0^X \frac{dr}{(2E - \frac{(2n+1)}{(m+n+1)}r^{(2m+2n+2)/(2n+1)})^{\frac{3}{2}}}\dot{X}f & = \quad g_2(E,\psi), \end{cases} \tag{2.2}$$

where $f = (1 - X^2)\,\dot{X}$. By multiplying the first and the second equation in (2.2) by the integrating factors μ_1 and μ_2 respectively it follows from the theory of integrating factors as presented in [8]-[10] that μ_1 and μ_2 have to satisfy

$$\begin{cases} \frac{\partial \mu_1}{\partial \psi} = \frac{\partial \mu_2}{\partial E}, \\ \\ \frac{\partial \mu_1}{\partial t} = -\frac{\partial}{\partial E}\left(\mu_1 g_1 + \mu_2 g_2\right), \frac{\partial \mu_2}{\partial t} = -\frac{\partial}{\partial \psi}\left(\mu_1 g_1 + \mu_2 g_2\right). \end{cases} \tag{2.3}$$

By expanding μ_1 and μ_2 in powers series in ϵ and by substituting g_1, g_2, and the expansions for the integrating factors into (2.3), and by taking together terms of equal powers in ϵ, we finally obtain the usual $\mathcal{O}(\epsilon^n)$-problems, for n=0,1,2,... (see also [9]-[12]). The $\mathcal{O}(\epsilon^0)$-problem is

$$\begin{cases} \frac{\partial \mu_{1,0}}{\partial \psi} = \frac{\partial \mu_{2,0}}{\partial E}, \\ \\ \frac{\partial \mu_{1,0}}{\partial t} = -\frac{\partial \mu_{2,0}}{\partial E}, \\ \\ \frac{\partial \mu_{2,0}}{\partial t} = -\frac{\partial \mu_{2,0}}{\partial \psi}, \end{cases} \tag{2.4}$$

and for $n \geq 1$ the $\mathcal{O}(\epsilon^n)$-problems are

$$\begin{cases} \frac{\partial \mu_{1,n}}{\partial \psi} = \frac{\partial \mu_{2,n}}{\partial E}, \\ \\ \frac{\partial \mu_{1,n}}{\partial t} = -\frac{\partial}{\partial E}\left(\mu_{1,n-1} g_{1,1} + \mu_{2,n-1} g_{2,1} + \mu_{2,n}\right), \\ \\ \frac{\partial \mu_{2,n}}{\partial t} = -\frac{\partial}{\partial \psi}\left(\mu_{1,n-1} g_{1,1} + \mu_{2,n-1} g_{2,1} + \mu_{2,n}\right), \end{cases} \tag{2.5}$$

where $\epsilon g_{1,1} = g_1\,, \epsilon g_{2,1} = g_2 - 1$. The $\mathcal{O}(\epsilon^0)$-problem (2.4) can readily be solved, yielding $\mu_{1,0} = h_{1,0}(E, \psi - t)$ and $\mu_{2,0} = h_{2,0}(E, \psi - t)$ with $\frac{\partial h_{1,0}}{\partial \psi} = \frac{\partial h_{2,0}}{\partial E}$. The functions $h_{1,0}$ and h_{20} are still arbitrary and will now be chosen as simple as possible. We choose $h_{1,0} \equiv 1$ and $h_{2,0} \equiv 0$, and so (see also [8]-[12])

$$\mu_{1,0} = 1,\ \mu_{2,0} = 0. \tag{2.6}$$

Then, from the order ϵ-problem (2.5) $\mu_{1,1}$ and $\mu_{2,1}$ can be obtained, yielding

$$\begin{cases} \mu_{1,1} = -\frac{\partial}{\partial E}\left(\int^t \left(\dot{X}^2 - \dot{X}^2 X^2\right) d\bar{t}\right), \\ \\ \mu_{2,1} = -\frac{\partial}{\partial \psi}\left(\int^t \left(\dot{X}^2 - \dot{X}^2 X^2\right) d\bar{t}\right). \end{cases} \tag{2.7}$$

An approximation F_1 of a first integral $F = \textit{constant}$ of system (2.2) can now be obtained from (2.6), (2.7), and the theory of integrating factors as presented in [8]-[12], yielding

$$F_1 \;=\; E - \epsilon \int^t \left(\dot{X}^2 - \dot{X}^2 X^2\right) d\bar{t}, \tag{2.8}$$

where

$$\dot{X} = \pm\sqrt{2E - \frac{2n+1}{m+n+1}X^{\frac{2m+2n+2}{2n+1}}}. \tag{2.9}$$

The elementary procedure to construct F_1 using the integrating factors is for instance given in [8]-[12]. How well F_1 approximates F in a first integral $F = constant$ follows from the theorems as presented in [9]-[12]. In this case it can be shown that (using the theory as presented in [9]-[12])

$$\frac{dF_1}{dt} = \epsilon\mu_{1,1}g_1 + \epsilon\mu_{2,1}(g_2 - 1) = \epsilon^2\mathcal{R}_1(E,\psi), \tag{2.10}$$

where g_1 and g_2, and $\mu_{1,1}$ and $\mu_{2,1}$ are given by (2.2) and (2.7) respectively. In a similar way we can construct a second (functionally independent) approximations of a first integral by taking

$$\mu_{2,0} = 1,\ \mu_{1,0} = 0, \tag{2.11}$$

instead of (2.6). The $\mathcal{O}(\epsilon)$-problem (2.5) can now again be solved, yielding

$$\begin{cases} \mu_{1,1} = \frac{\partial}{\partial E}\left(\int^t \left(\int_0^X \frac{dr}{(2E - \frac{(2n+1)}{(m+n+1)}r^{(2m+2n+2)/(2n+1)})^{\frac{3}{2}}}\left(\dot{X}^2 - \dot{X}^2X^2\right)\right)d\bar{t}\right), \\ \mu_{2,1} = \frac{\partial}{\partial \psi}\left(\int^t \left(\int_0^X \frac{dr}{(2E - \frac{(2n+1)}{(m+n+1)}r^{(2m+2n+2)/(2n+1)})^{\frac{3}{2}}}\left(\dot{X}^2 - \dot{X}^2X^2\right)\right)d\bar{t}\right). \end{cases} \tag{2.12}$$

An approximation F_2 of a first integral $F = constant$ of system (2.2) can now be obtained from (2.11), (2.12), and the theory of integrating factors as presented in [8]-[12], yielding

$$F_2(E,\psi,t) = (\psi - t) + \epsilon\left[\int^t \left(\int_0^X \frac{dr}{(2E - \frac{(2n+1)}{(m+n+1)}r^{(2m+2n+2)/(2n+1)})^{\frac{3}{2}}}\left(\dot{X}^2 - \dot{X}^2X^2\right)\right)d\bar{t}\right]. \tag{2.13}$$

How well F_2 approximates a first integral $F = constant$ follows from the theorems as presented in [9]-[12]. In this case we have

$$\frac{dF_2}{dt} = \epsilon\mu_{1,1}g_1 + \epsilon\mu_{2,1}(g_2 - 1) = \epsilon^2\mathcal{R}_2(E,\psi), \tag{2.14}$$

where g_1 and g_2, and $\mu_{1,1}$ and $\mu_{2,1}$ are given by (2.2) and (2.12) respectively.

3 Approximations of time-periodic solutions

In section 2 we constructed asymptotic approximations of first integrals. In this section we will show how the existence, the stability, and the approximations of non-trivial, time-periodic solutions can be determined from these asymptotic approximations of the first integrals. Let $T < \infty$ be the period of a periodic solution and let c_1 be a constant in the first integral $F(E, \psi, t; \epsilon) = \textit{constant}$ for which a periodic solution exists. Consider $F = c_1$ for $t = 0$ and $t = T$. Approximating F by F_1 (given by (2.8)), eliminating c_1 by subtraction, we then obtain (using the fact that $E(0) = E(T)$ for a periodic solution)

$$\epsilon\left(\int_0^T \left(\dot{X}^2 - \dot{X}^2X^2\right) d\bar{t}\right) = \mathcal{O}(\epsilon^2) \Leftrightarrow \epsilon\left(\int_{X(0)}^{X(T)} \left(\dot{X} - \dot{X}X^2\right) dX\right) = \mathcal{O}(\epsilon^2). \quad (3.1)$$

Without loss of generality it can be assumed that at $t = 0$ in $(X(0), \dot{X}(0)) = (A, 0)$ with $A > 0$. Because of the symmetry of the unperturbed orbits in the phase plane it follows that $\left(X\left(\frac{T}{2}\right), \dot{X}\left(\frac{T}{2}\right)\right) = (-A, 0)$. From (3.1) it then follows that

$$\epsilon I(E) = \mathcal{O}(\epsilon^2), \text{ where } I(E) = 4\int_0^A \left(\dot{X} - \dot{X}X^2\right) dX. \quad (3.2)$$

To have a periodic solution for (2.1) we have to find an energy E such that $I(E)$ is equal to zero (see also [11, 14, 15]). It should be observed that the same problem (that is, find zeros of $I(E)$) is obtained when the Poincaré return map technique or the Melnikov method is applied (see also [13]-[16]). To find this energy E we rewrite $I(E)$ in (using (2.9))

$$I(E) = 4I_1(E)\left(1 - \frac{I_2(E)}{I_1(E)}\right), \text{ where} \quad (3.3)$$

$$\begin{cases} I_1(E) = \int_0^A \left(2E - \frac{2n+1}{m+n+1}X^{\frac{2m+2n+2}{2n+1}}\right)^{\frac{1}{2}} dX, \\ \\ I_2(E) = \int_0^A X^2\left(2E - \frac{2n+1}{m+n+1}X^{\frac{2m+2n+2}{2n+1}}\right)^{\frac{1}{2}} dX. \end{cases} \quad (3.4)$$

Now it should be observed that $E(t) = \frac{1}{2}\dot{X(t)}^2 + \frac{(2n+1)}{(2m+2n+2)}X(t)^{(2m+2n+2)/(2n+1)}$, and $E(0) = \frac{(2n+1)}{(2m+2n+2)}A^{(2m+2n+2)/(2n+1)}$. From (2.2) we can see that E is constant up to $\mathcal{O}(\epsilon)$ on time-scales of $\mathcal{O}(1)$. By using the transformation $X = Au$ in (3.4) and by using the fact that $E = E(0) + \mathcal{O}(\epsilon)$ for $0 \le t \le T$ it is easy to see from (3.2)-(3.4) that (3.2) can be rewritten in

$$4\epsilon I_1(E)(1 - Q) = \mathcal{O}(\epsilon^p) \text{ with } p > 1, \quad (3.5)$$

where

$$Q = \left(2E\frac{m+n+1}{2n+1}\right)^{\frac{2n+1}{m+n+1}} \frac{J_2(m,n)}{J_1(m,n)}, \quad (3.6)$$

and where

$$\begin{cases} J_1(m,n) &= \int_0^1 \sqrt{1-u^{\frac{2m+2n+2}{2n+1}}}\,du, \\ \\ J_2(m,n) &= \int_0^1 \sqrt{u^4-u^{\frac{2m+10n+6}{2n+1}}}\,du. \end{cases} \tag{3.7}$$

It is easy to see that $J_1(m,n) > 0$ and $J_2(m,n) > 0$ for all values of m, $n \in \mathbb{N}$. It is also easy to see from (3.6) that $\frac{dQ}{dE} > 0$. This implies that Q is strictly monotonically increasing. Since Q is strictly monotonically increasing in E we can conclude that there exists a unique, nontrivial E-value such that $I(E) = 0$. From these results it can be concluded (see also for instance [[11], section 4.2]) that there exists a unique, nontrivial, stable time-periodic solution for (2.1). Suppose that at $t = 0$ $X(0) = A_0$ and $\dot{X}(0) = 0$ for the periodic solution. Then,

$$\frac{1}{2}\dot{X}^2 + \frac{2n+1}{2m+2n+2}X^{\frac{2m+2n+2}{2n+1}} = \frac{2n+1}{2m+2n+2}A_0^{\frac{2m+2n+2}{2n+1}} \equiv E_0, \tag{3.8}$$

where E_0 is the energy such that we have a periodic solution. Obviously E_0 satisfies (see also (3.3) and (3.5))

$$\left(2E_0\frac{m+n+1}{2n+1}\right)^{\frac{2n+1}{m+n+1}} \frac{J_2(m,n)}{J_1(m,n)} = 1, \tag{3.9}$$

up to $\mathcal{O}(\epsilon^{p-1})$ with $p > 1$. The period of the periodic solution can be calculated up to $\mathcal{O}(\epsilon^p)$ with $p > 1$ from (3.8), yielding

$$\frac{dX}{dt} = \pm\sqrt{\frac{2n+1}{m+n+1}}\sqrt{A_0^{\frac{2m+2n+2}{2n+1}} - X^{\frac{2m+2n+2}{2n+1}}}, \tag{3.10}$$

or equivalently

$$\sqrt{\frac{m+n+1}{2n+1}}\frac{dX}{dt}\frac{1}{\sqrt{A_0^{\frac{2m+2n+2}{2n+1}} - X^{\frac{2m+2n+2}{2n+1}}}} = \pm 1, \tag{3.11}$$

Then, integrating (3.11) with respect to t from $t = 0$ to $\frac{T}{2}$ yields

$$T_{m,n} = 4\sqrt{\frac{m+n+1}{2n+1}}A_0^{\frac{n-m}{2n+1}}\int_0^1 \frac{du}{\sqrt{1-u^{\frac{2m+2n+2}{2n+1}}}}. \tag{3.12}$$

Using a standard numerical integration routine the period (3.12) of the periodic solution can easily be approximated numerically (up to $\mathcal{O}(\epsilon^{p-1})$ with $p > 1$).

4 Conclusions and remarks

In this paper it has been shown that the perturbation method based on integrating factors can be used efficiently to approximate first integrals for a generalized nonlinear Van der Pol oscillator. In section 2 of this paper it has been shown how approximations of first integrals for this oscillator equation can be obtained. It has been shown in section 3 how the existence, the stability, and the period of the time-periodic solution can be deduced from the approximations of the first integrals.

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