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On the vibrations of a simply supported square plate on a weakly nonlinear elastic foundation.

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Abstract

In this paper an initial-boundary value problem for a weakly nonlinear plate equation with a quadratic nonlinearity will be studied. This initial-boundary value problem can be regarded as a simple model describing free oscillations of a simply supported square plate on an elastic foundation. It is assumed that the foundation has a different behavior for compression and for expansion. An approximation for the solution of the initial-boundary value problem will be constructed using a two-timescales perturbation method. The existence and uniqueness of the solution of the problem will be proved. Also the asymptotic validity of the constructed approximations will be shown on long time-scales. For specific parameter values it turns out that complicated internal resonances occur.

Key words: weakly nonlinear plate equation, asymptotics, two-timescales perturbation method, internal resonanses.

AMS subject classifications: 35B20, 35B40, 35Q72, 74K20.

1 Introduction

It is well known that flexible structures like suspension bridges or overhead power transmission lines can be subjected to oscillations due to various causes. Simple models for such oscillations are described with second- and fourth-order partial differential equations as can be seen for example in [1]-[7]. Usually asymptotic methods can be used to construct approximations for solutions of these second- and fourth-order partial differential equations. For a long time initial-boundary value problems for weakly nonlinear wave equations have been studied, for example, in [7]-[9] and in [14], [16], [17]. In [7]-[9] a two-timescales perturbation method has been used to construct approximations. In [7]-[9], [16], [17] asymptotic theories which justify these approximations are presented. The analysis becomes more complicated for beam equations. These equations are discussed for instance in [3], [4], [5], [6], [10], [11], [18] and [19]. Only a little is known about weakly nonlinear plate equations. In this paper an initial-boundary value problem for a simply supported square plate on a weakly nonlinear, elastic foundation will be studied in detail. The analysis includes the well-possedness of the problem in classical sense, the construction of approximations of the solution, and a justification of the obtained asymptotic results on long timescales. The presented analysis in this paper can also be seen as a starting point to study more complicated problems such as wind-induced oscillations of suspension bridges.

A square plate with side length l will be considered. The displacement of the plate in z direction (see also Figure 1(a)) is denoted by $w = w(x, y, t)$, in which t is time. The following symbols will be used: μ is the mass of the plate per unit area perpendicular to z-axis, ρ is the mass density of the plate, A is the area of the cross section of the plate perpendicular to the x-axis, (so $A = lh$, where h is the thickness of the plate), E is the elasticity modulus, I is the moment of inertia of the cross section with respect to the x-axis, and F is the force (in z-direction) per unit area acting on the plate due to the elastic foundation. We neglect internal damping and consider the weight W of the plate per unit area to be constant ($W = \mu g$, g is the gravitational acceleration). No other external forces are assumed to be present. The equation of motion for the vertical displacement of the plate is given by

$$\mu w_{tt} + EI(w_{xxxx} + 2w_{xxyy} + w_{yyyy}) + F(w) = -\mu g. \tag{1}$$

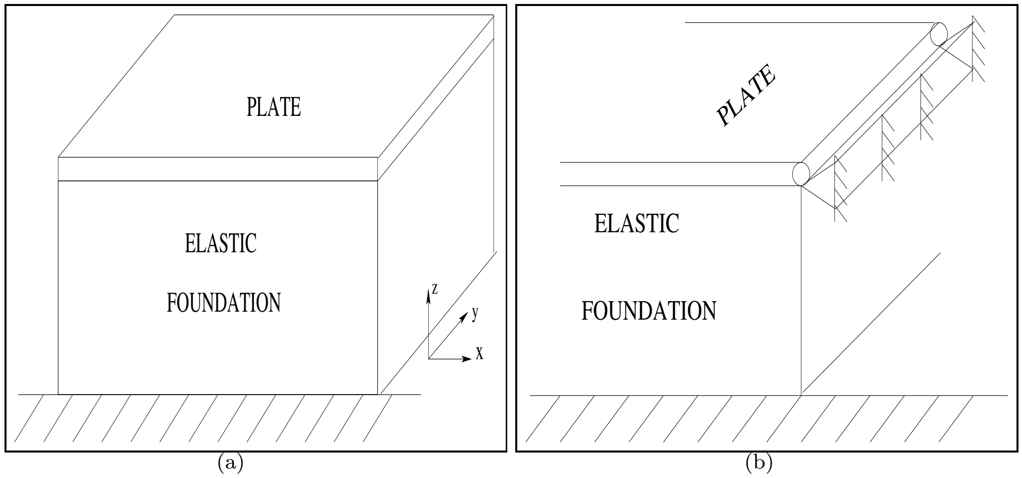


Figure 1: Sketch of (a) a square plate on an elastic foundation and (b) a detail of a simply support on an edge of the plate.

It is assumed that the force F can be expanded in a Taylor series with $F(0) = 0 : F(w) = kw + bw^2 + ...$, where k and b are spring constants. It is assumed that the elastic foundation has a different behaviour for compression and for expansion, i.e., for $w < 0$ and $w > 0$. It is also assumed that the vertical displacements of the plate are small compared to the length l, and that terms of degree three and higher in F can be neglected. Equation (1) then becomes

$$w_{tt} + \frac{EI}{\mu}(w_{xxxx} + 2w_{xxyy} + w_{yyyy}) + \frac{k}{\mu}w + \frac{b}{\mu}w^2 = -g. \quad (2)$$

To simplify (2) the term $-g$ will be removed from (2) by introducing the transformation $w(x, y, t) = \tilde{w}(x, y, t) + \frac{\mu g}{k}s(x, y)$, where $s(x, y)$ satisfies the following time-independent linear boundary value problem

$$\begin{aligned} s_{xxxx}(x, y) + 2s_{xxyy}(x, y) + s_{yyyy}(x, y) + \frac{k}{EI}s(x, y) = -\frac{k}{EI}, \quad 0 < x < l, \quad 0 < y < l, \\ s(0, y) = s(l, y) = s_{xx}(0, y) = s_{xx}(l, y) = 0, \quad 0 < y < l, \\ s(x, 0) = s(x, l) = s_{yy}(x, 0) = s_{yy}(x, l) = 0, \quad 0 < x < l. \end{aligned} \quad (3)$$

In fact the term $\frac{\mu g}{k}s(x, y)$ represents the deflection of the plate in static state due to gravity. The solution of the boundary value problem (3) can be constructed by using the method of separation of variables (see [12], or [13]) or by just assuming that

$$s(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} s_{nm} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi y}{l}\right), \quad (4)$$

where the s_{mn} are constants. These constants can then be determined by substituting (4) into the PDE in (3), by multiplying the so-obtained equation with $\sin\left(\frac{p\pi y}{l}\right) \sin\left(\frac{q\pi y}{l}\right)$, and then by integrating the equation with respect to x and y (from x is 0 to l, and from y is 0 to l), yielding

$$s_{nm} = \frac{-4k}{\pi^2 EInm}(1 - (-1)^n)(1 - (-1)^m).$$

Using the dimensionless variables $\bar{w} = \frac{l}{A}\tilde{w}$, $\bar{x} = \frac{\pi}{l}x$, $\bar{y} = \frac{\pi}{l}y$, $\bar{t} = (\frac{\pi}{l})^2(\frac{EI}{\mu})^{1/2}t$, (2) becomes

$$\begin{aligned} &\bar{w}_{\bar{t}\bar{t}} + \bar{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} + 2\bar{w}_{\bar{x}\bar{x}\bar{y}\bar{y}} + \bar{w}_{\bar{y}\bar{y}\bar{y}\bar{y}} + \\ &\frac{l^4}{\pi^4 EI}\left(k\bar{w} + \frac{bA}{l}\bar{w}^2 + 2\frac{b\mu g}{k}s\left(\frac{l}{\pi}\bar{x}, \frac{l}{\pi}\bar{y}\right)\bar{w} + \frac{bl}{A}\left(\frac{\mu g}{k}s\left(\frac{l}{\pi}\bar{x}, \frac{l}{\pi}\bar{y}\right)\right)^2\right) = 0. \end{aligned} \quad (5)$$

Assuming that the area A of the cross section is small compared to the plate side length l, we put $\tilde{\epsilon} = \frac{A}{l}$ with $\tilde{\epsilon}$ a small parameter. Furthermore, we assume that the deflection of the plate in static state due to gravity (that is $\frac{\mu g}{k}s(x,y)$) is small with respect to the vertical displacement $\tilde{w}$, which is of order $\tilde{\epsilon}$. This means we assume that $\frac{\mu g}{k}s(x,y)$ is $\mathcal{O}(\tilde{\epsilon}^n)$ with $n > 1$. Setting $\epsilon = -b\tilde{\epsilon}(\frac{l}{\pi})^4\frac{1}{EI}$, (5) becomes $\bar{w}_{\bar{t}\bar{t}} + \bar{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} + 2\bar{w}_{\bar{x}\bar{x}\bar{y}\bar{y}} + \bar{w}_{\bar{y}\bar{y}\bar{y}\bar{y}} + p^2\bar{w} = \epsilon\bar{w}^2 + \mathcal{O}(\epsilon^n)$, with $n > 1$, $\quad p^2 = (\frac{l}{\pi})^4\frac{k}{EI}$, and ϵ is a small dimensionless parameter. We can now formulate the following initial-boundary value problem, which describes up to $\mathcal{O}(\epsilon^n)$, $n > 1$, the vertical displacement of a plate on a weakly nonlinear, elastic foundation: (for convenience all bars will be dropped)

$$w_{tt} + w_{xxxx} + 2w_{xxyy} + w_{yyyy} + p^2 w = \epsilon w^2, \quad 0 < x < \pi, \quad 0 < y < \pi, \quad t > 0, \tag{6}$$

$$w(0,y,t) = w(\pi,y,t) = w_{xx}(0,y,t) = w_{xx}(\pi,y,t) = 0, \quad t \geq 0, \tag{7}$$

$$w(x,0,t) = w(x,\pi,t) = w_{yy}(x,0,t) = w_{yy}(x,\pi,t) = 0, \quad t \geq 0, \tag{8}$$

$$w(x,y,0) = w_0(x,y;\epsilon), \quad w_t(x,y,0) = w_1(x,y;\epsilon), \quad 0 < x < \pi, \quad 0 < y < \pi, \tag{9}$$

where ϵ and p are parameters with $0 < |\epsilon| \ll 1$ and $p > 0$, $\quad w = w(x,y,t)$ is the vertical displacement of the plate, $w_0(x,y)$ is the initial displacement of the plate in vertical direction, and $w_1(x,y)$ is the initial velocity of the plate in vertical direction. All functions are assumed to be sufficiently smooth. The first four terms in the left-hand side of (6) are the linear part of the plate equation and $p^2w - \epsilon w^2$ represents the restoring force due to the elastic foundation. The boundary conditions describe a simply supported plate. Since no other external forces are considered, the initial boundary value problem (6)-(9) can be considered as a simple model to describe the free oscillations of a simply supported plate on an elastic, weakly nonlinear foundation.

In this paper an asymptotic theory will be presented for a more general case of (6)-(9), where the nonlinearity in the right-hand side of (6) is of the form $\epsilon f(x,y,t,w;\epsilon)$. We will show that the initial-boundary value problem is well-posed in classical sense, i.e., we will show that there exists a unique, classical solution for this initial-boundary value problem. We will also show the asymptotic validity of approximations, which are constructed by using formal perturbation methods. We will construct order ϵ approximations for the solution of the initial-boundary value problem (6)-(9) by using a Fourier series expansion and a two-timescales perturbation method. We will consider the energy exchange between different oscillation modes for different values of the parameter p^2. For almost all values of p^2 only an interaction of order ϵ between different oscillation modes occurs on a timescales of order $1/\epsilon$, but for certain specific values of p^2 mode interactions of order 1 occur on a timescale of order $1/\epsilon$, i.e., energy transfer of order 1 occurs between two or more modes on an $1/\epsilon$ timescale. We will show that, for instance, for $p^2 = \frac{68}{3}$ an energy transfer of order 1 occurs between modes 2-2 and 3-3 (that is, for oscillation modes described by $\sin(2x)\sin(2y)$ and $\sin(3x)\sin(3y)$). For $p^2 \approx 35.40$ an energy transfer of order 1 occurs between the modes 1-9, 9-1, 2-2 and 6-6 (that is, between the oscillation modes described by $\sin(x)\sin(9y)$, $\sin(9x)\sin(y)$, $\sin(2x)\sin(2y)$, and $\sin(6x)\sin(6y)$).

The outline of this paper is as follows. In section 2 the well-possedness of the initial-boundary value problem (6)-(9) is considered and established on a timescale of order $1/\epsilon$. In section 3 the asymptotic validity of approximations of the solution of this initial-boundary value problem is studied. In sections 4 and 5 the asymptotic theory is applied. On a timescale of order $1/\epsilon$ an order ϵ asymptotic approximation, as $\epsilon \to 0$, for the solution of (6)-(9) will be constructed using a two-timescales perturbation method. For some specific values of p^2, modes with zero initial energy will also be excited. For different values of p^2 these mode interactions will be studied in section 5. In section 6 some conclusions will be drawn and some remarks will be made.

2 The well-possedness of the problem.

In this section we consider the well-possedness in classical sense for the following class of weakly nonlinear, initial-boundary value problems for a real valued function $w(x, y, t)$:

$$w_{tt}+w_{xxxx}+2w_{xxyy}+w_{yyyy}+p^2w=\epsilon f(x,y,t,w;\epsilon), \tag{10}$$
$$0 < x < \pi, \quad 0 < y < \pi, \quad t > 0,$$
$$w(0,y,t) = w(\pi,y,t) = w_{xx}(0,y,t) = w_{xx}(\pi,y,t) = 0, \quad t \geq 0, \tag{11}$$

$$w(x,0,t) = w(x,\pi,t) = w_{yy}(x,0,t) = w_{yy}(x,\pi,t) = 0, \quad t \geq 0, \tag{12}$$
$$w(x,y,0) = w_0(x,y;\epsilon), \quad w_t(x,y,0) = w_1(x,y;\epsilon), \quad 0 < x < \pi, \quad 0 < y < \pi, \tag{13}$$

where ϵ, p are constants, $\epsilon \in [-\epsilon_0, \epsilon_0]$, and $p \geq 0$, and where f, w_0, w_1 satisfy

f and all first-, second-, and third-order partial derivatives of f with respect to x, y, t, w are $\in C([0,\pi] \times [0,\pi] \times [0,\infty] \times \mathbb{R} \times [-\epsilon_0, \epsilon_0], \mathbb{R})$, and $f(0,0,t,0;\epsilon) = f(\pi,\pi,t,0;\epsilon) \equiv 0$ for $t \geq 0$, (14)

$$w_0, \frac{\partial w_0}{\partial x}, \frac{\partial^2 w_0}{\partial x^2}, \frac{\partial^3 w_0}{\partial x^3}, \frac{\partial^4 w_0}{\partial x^4}, \frac{\partial w_0}{\partial y}, \frac{\partial^2 w_0}{\partial y^2}, \frac{\partial^3 w_0}{\partial y^3}, \frac{\partial^4 w_0}{\partial y^4}, \frac{\partial^2 w_0}{\partial x \partial y}, \frac{\partial^3 w_0}{\partial x^2 \partial y}, \frac{\partial^3 w_0}{\partial x \partial y^2},$$
$$\frac{\partial^4 w_0}{\partial x^2 \partial y^2}, \frac{\partial^4 w_0}{\partial x^3 \partial y}, \frac{\partial^4 w_0}{\partial x \partial y^3}, w_1, \frac{\partial w_1}{\partial x}, \frac{\partial^2 w_1}{\partial x^2}, \frac{\partial w_1}{\partial y}, \frac{\partial^2 w_1}{\partial x \partial y}, \frac{\partial^2 w_1}{\partial y^2}, \frac{\partial^3 w_1}{\partial x^3}, \frac{\partial^3 w_1}{\partial x^2 \partial y}, \frac{\partial^2 w_1}{\partial x \partial y^2},$$
$$\frac{\partial^2 w_1}{\partial y^3} \in C([0,\pi] \times [0,\pi] \times [-\epsilon_0, \epsilon_0], \mathbb{R}), \quad \text{with} \quad w_0(0,y;\epsilon) = w_0(\pi,y;\epsilon) =$$

$$\frac{\partial^2 w_0}{\partial x^2}(0,y;\epsilon) = \frac{\partial^2 w_0}{\partial x^2}(\pi,y;\epsilon) \equiv 0, \tag{15}$$

and $w_0(x,0;\epsilon) = w_0(x,\pi;\epsilon) = \dfrac{\partial^2 w_0}{\partial y^2}(x,0;\epsilon) = \dfrac{\partial^2 w_0}{\partial y^2}(x,\pi;\epsilon) \equiv 0,$ and

$w_1(0,y;\epsilon) = w_1(\pi,y;\epsilon) = \dfrac{\partial^2 w_1}{\partial x^2}(0,y;\epsilon) = \dfrac{\partial^2 w_1}{\partial x^2}(\pi,y;\epsilon) \equiv 0,$ and $w_1(x,0;\epsilon) =$

$w_1(x,\pi;\epsilon) = \dfrac{\partial^2 w_1}{\partial y^2}(x,0;\epsilon) = \dfrac{\partial^2 w_1}{\partial y^2}(x,\pi;\epsilon) \equiv 0,$

f and all first-, second-, and third-order partial derivatives of f with respect to x, y, t are uniformly bounded for all x, y, t, ϵ as considered. (16)

We define a classical solution as a function that is three times continuously differentiable on $[0,\pi] \times [0,\pi] \times [0,\infty]$, for which the fourth order partial derivatives with respect to x, y are continuous on $[0,\pi] \times [0,\pi] \times [0,\infty]$, and that satisfies (10)-(13), where f, w_0, w_1 satisfy (14)-(16). In order to prove existence and uniqueness of a classical solution of the initial-boundary value problem (10)-(13) an equivalent integral equation will be used. We obtain this integral equation by using the Green's function G for the linear operator $\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} + \frac{\partial^2}{\partial t^2} + p^2$ and the simply supported boundary conditions (see Appendix A):

$$w(x,y,t) = \epsilon \int_0^t \int_0^\pi \int_0^\pi G(\xi,\eta,\tau;x,y,t) f(\xi,\eta,\tau,w;\epsilon) d\xi d\eta d\tau + w_l(x,y,t;\epsilon) \equiv (Tw)(x,y,t), \tag{17}$$

where

$$G(\xi,\eta,\tau;x,y,t) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\sqrt{(n^2+m^2)^2+p^2}} \sin\left[\sqrt{(n^2+m^2)^2+p^2}(t-\tau)\right] \tag{18}$$
$$\times H(t-\tau) \sin(n\xi) \sin(m\eta) \sin(nx) \sin(my)$$

for $\xi, x, \eta, y \in [0, \pi]$, $\tau, t \geq 0$, where the Heaviside function $H(a)$ is equal to 1 for $a > 0$ and equal to 0 for $a < 0$. The solution of the initial-boundary value problem (10)-(13) with $f \equiv 0$ is

$$w_l(x, y, t; \epsilon) = \int_0^\pi \int_0^\pi \big(G(\xi, \eta, 0; x, y, t) w_1(\xi, \eta; \epsilon) - G_\tau(\xi, \eta, 0; x, y, t) w_0(\xi, \eta; \epsilon)\big) d\xi d\eta. \tag{19}$$

It can be shown elementarily that the integral equation (17) and the initial-boundary value problem (10)-(13) are equivalent when three times continuously differentiable functions with continuous fourth order partial derivatives with respect to x, y (on $[0, \pi] \times [0, \pi] \times [0, \infty]$) are considered. This means that if $w(x, y, t)$ is a three times continuously differentiable solution of the initial-boundary value problem (10)-(13) and w_{xxxx}, w_{yyyy}, w_{xxyy} are continuous, then $w(x, y, t)$ is also a solution of the integral equation (17) and that, if $v(x, y, t)$ is a three times continuously differentiable solution of the integral equation (17) and v_{xxxx}, v_{yyyy}, v_{xxyy} are continuous, then $v(x, y, t)$ is also a solution of the initial-boundary value problem (10)-(13). We will start with some definitions. Let

$$\Omega_L = \Big\{(x, y, t)\Big|\ 0 \leq x \leq \pi, \quad 0 \leq y \leq \pi, \quad 0 \leq t \leq L|\epsilon^{-1}|\Big\}, \tag{20}$$

with L a sufficiently small, positive constant independent of ϵ. Let the Banach space B of all real-valued continuous functions w on Ω_L be given and let $C_M(\Omega_L)$ be the closed subset $C_M(\Omega_L) = \{w \in B \big| \parallel w \parallel = max_{(x,y,t) \in \Omega_L} |w(x, y, t)| \leq M\}$. We now state the following theorem.

THEOREM 1. *Suppose* f, w_0, w_1 *satisfy* (14)-(17). *Then for every* ϵ *and* p *satisfying* $0 < |\epsilon| \ll 1$ *and* $p \geq 0$ *the initial-boundary value problem* (10)-(13) *has a unique and three times continuously differentiable solution with continuous fourth order partial derivatives with respect to* x, y *for* $(x, y, t) \in \Omega_L$, *with* L *a sufficiently small, positive constant independent of* ϵ. *This unique solution depends continuously on the initial values.*

Proof. As stated above the initial-boundary value problem (10)-(13) is equivalent to the integral equation (17). To prove existence and uniqueness of the solution of (17) a fixed point theorem will be used. Using the fact that w_0, $\frac{\partial^2 w_0}{\partial x^2}$, $\frac{\partial^2 w_0}{\partial x \partial y}$, $\frac{\partial^2 w_0}{\partial y^2}$, and w_1 are continuous on the closed and bounded interval $[0, \pi] \times [0, \pi] \times [-\epsilon_0, \epsilon_0]$ and therefore uniformly bounded on the interval, and using the estimates (141), (142) as obtained in Appendix B, it follows that there is a constant M_1 independent of ϵ such that, for fixed w_0 and w_1,

$$\parallel w_l \parallel \leq \frac{1}{2} M_1, \tag{21}$$

i.e., w_l as given by (19) is bounded. Since f and $\frac{\partial f}{\partial w}$ are assumed to be continuous and uniformly bounded for those values of x, y, t, ϵ under consideration, there are constants M_2 and M_3 independent of ϵ such that

$$|f(x, y, t, w; \epsilon)| \leq M_2, \tag{22}$$

$$|f(x, y, t, v_1; \epsilon) - f(x, y, t, v_2; \epsilon)| \leq M_3 \parallel v_1 - v_2 \parallel \tag{23}$$

for all $(x, y, t) \in \Omega_L$, $\quad \epsilon \in [-\epsilon_0, \epsilon_0]$, and $w, v_1, v_2 \in C_{M_1}(\Omega_L)$. Using (22), (23), (142), and the fact that $(x, y, t) \in \Omega_L$, we can show that $Tw \in C_{M_1}(\Omega_L)$, i.e., the integral operator T maps $C_{M_1}(\Omega_L)$ into itself:

$$\begin{aligned} |(Tw)(x, y, t)| &\leq \Big|\epsilon \int_0^t \int_0^\pi \int_0^\pi G(\xi, \eta, \tau; x, y, t) f(\xi, \eta, \tau, w; \epsilon) d\xi d\eta d\tau\Big| + |w_l(x, y, t; \epsilon)| \\ &\leq \Big|\epsilon \int_0^t \pi^2 \max_{0 \leq x, y \leq \pi} |f(x, y, \tau, w(x, y, \tau); \epsilon)| d\tau\Big| + \frac{1}{2} M_1 \\ &\leq |\epsilon| t \pi^2 M_2 + \frac{1}{2} M_1. \end{aligned}$$

If the maximum of the left-hand side is taken for $(x, y, t) \in \Omega_L$, we obtain

$$\| Tw \| \leq \pi^2 L M_2 + \frac{1}{2} M_1.$$

If the constant L is taken such that $\pi^2 L M_2 \leq \frac{1}{2} M_1$, it follows that

$$\| Tw \| \leq M_1 \quad \text{for all} \quad w \in C_{M_1}(\Omega_L).$$

Hence it follows that $T : C_{M_1}(\Omega_L) \longrightarrow C_{M_1}(\Omega_L)$. In a similar way it can be shown that the integral operator T is a contraction on $C_{M_1}(\Omega_L)$. Let $v_1, v_2 \in C_{M_1}(\Omega_L)$, then

$$\begin{aligned}
|(Tv_1)(x,y,t) - (Tv_2)(x,y,t)| \quad &\leq \quad |\epsilon| \Big| \int_0^t \int_0^\pi \int_0^\pi G(\xi, \eta, \tau; x, y, t)\{f(\xi, \eta, \tau, v_1; \epsilon) \\
&\quad -f(\xi, \eta, \tau, v_2; \epsilon)\} d\xi d\eta d\tau \Big| \\
&\leq \quad \Big| \epsilon \int_0^t \pi^2 \max_{0 \leq x, y \leq \pi} |f(x, y, \tau, v_1(x, y, \tau); \epsilon) \\
&\quad -f(x, y, \tau, v_2(x, y, \tau); \epsilon)| d\tau \Big| \\
&\leq \quad \pi^2 M_3 L \max_{(x,y,t) \in \Omega_L} |v_1(x, y, t) - v_2(x, y, t)| \\
&\quad \leq \pi^2 M_3 L \| v_1 - v_2 \|
\end{aligned}$$

If the constant L is taken such that $\pi^2 L M_2 \leq \frac{1}{2} M_1$ and $\pi^2 L M_3 \leq k$ with $0 < k < 1$, it follows that

$$\| Tv_1 - Tv_2 \| \leq k \| v_1 - v_2 \| \quad \text{with} \quad 0 < k < 1 \quad \text{for all} \quad v_1, v_2 \in C_{M_1}(\Omega_L),$$

i.e., T is a contraction on $C_{M_1}(\Omega_L)$. Then, Banach's fixed point theorem implies that the integral operator T has a unique fixed point $w \in C_{M_1}(\Omega_L)$, i.e., a continuous function w on Ω_L satisfying the integral equation (17). It can be shown elementarily that T maps C^i functions into C^i functions for $i = 1, 2, 3$. Furthermore, it can be shown that T maps C^3 functions into functions that have continuous fourth order derivatives with respect to x, y. So, the unique solution of the integral equation (17) is a three times continuously differentiable function, with continuous fourth order derivatives with respect to x, y. Since the integral equation (17) and the initial-boundary value problem (10)-(13) are equivalent, it follows that the initial-boundary value problem has a unique and three times continuously differentiable solution w, with w_{xxxx}, w_{yyyy}, w_{xxyy} continuous. This proves the first part of Theorem 1.

Next it will be shown that the solution of the initial-boundary value problem depends continuously on the initial values. Let $w(x, y, t)$ satisfy (10)-(13) and let $\tilde{w}(x, y, t)$ satisfy (10)-(12) with $\tilde{w}(x, y, 0) = \tilde{w}_0(x, y; \epsilon)$, $\quad \tilde{w}_t(x, y, 0) = \tilde{w}_1(x, y; \epsilon)$, where $\tilde{w}_0$ and $\tilde{w}_1$ satisfy the same properties (15) as for w_0 and w_1. Using the equivalent integral equation (17), (23), (141), the fact that $(x, y, t) \in \Omega_L$, and taking

$w, \tilde{w} \in C_{M_1}(\Omega_L)$, we obtain

$$
\begin{aligned}
&|w(x,y,t) - \tilde{w}(x,y,t)| \\
&\leq |\epsilon| \Big| \int_0^t \int_0^\pi \int_0^\pi G(\xi,\eta,\tau;x,y,t)\{f(\xi,\eta,\tau,w;\epsilon) - \\
&\qquad f(\xi,\eta,\tau,\tilde{w};\epsilon)\} d\xi d\eta d\tau \Big| \\
&+ \Big| \int_0^\pi \int_0^\pi G(\xi,\eta,0;x,y,t)\{w_1(\xi,\eta;\epsilon) - \tilde{w}_1(\xi,\eta;\epsilon)\} d\xi d\eta \Big| \\
&+ \Big| \int_0^\pi \int_0^\pi G_\tau(\xi,\eta,0;x,y,t)\{w_0(\xi,\eta;\epsilon) - \tilde{w}_0(\xi,\eta;\epsilon)\} d\xi d\eta \Big|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Big| \epsilon \int_0^t \pi^2 \max_{0 \leq x,y \leq \pi} |f(x,y,\tau,w(x,y,\tau);\epsilon) - f(x,y,\tau,\tilde{w}(x,y,\tau);\epsilon)| d\tau \Big| + \\
&\qquad \pi^2 \max_{0 \leq x,y \leq \pi} \parallel w_1(x,y;\epsilon) - \tilde{w}_1(x,y;\epsilon) \parallel \\
&+ \pi^2 \max_{0 \leq x,y \leq \pi} \Big\{ p^2 |w_0(x,y;\epsilon) - \tilde{w}_0(x,y;\epsilon)| + \Big| \frac{\partial^2}{\partial x^2} w_0(x,y;\epsilon) - \frac{\partial^2}{\partial x^2} \tilde{w}_0(x,y;\epsilon) \Big| \\
&\qquad + 2 \Big| \frac{\partial^2}{\partial x \partial y} w_0(x,y;\epsilon) - \frac{\partial^2}{\partial x \partial y} \tilde{w}_0(x,y;\epsilon) \Big| + \Big| \frac{\partial^2}{\partial y^2} w_0(x,y;\epsilon) - \frac{\partial^2}{\partial y^2} \tilde{w}_0(x,y;\epsilon) \Big| \Big\} \\
&\leq \pi^2 L M_3 \parallel w - \tilde{w} \parallel + \pi^2 \parallel w_1 - \tilde{w}_1 \parallel + \pi^2 p^2 \parallel w_0 - \tilde{w}_0 \parallel \\
&\qquad + \pi^2 \Big\| \frac{\partial^2 w_0}{\partial x^2} - \frac{\partial^2 \tilde{w}_0}{\partial x^2} \Big\| + 2\pi^2 \Big\| \frac{\partial^2 w_0}{\partial x \partial y} - \frac{\partial^2 \tilde{w}_0}{\partial x \partial y} \Big\| + \pi^2 \Big\| \frac{\partial^2 w_0}{\partial y^2} - \frac{\partial^2 \tilde{w}_0}{\partial y^2} \Big\| \\
&\leq k \parallel w - \tilde{w} \parallel + \pi^2 \Big\{ \parallel w_1 - \tilde{w}_1 \parallel + p^2 \parallel w_0 - \tilde{w}_0 \parallel \\
&\quad + \Big\| \frac{\partial^2 w_0}{\partial x^2} - \frac{\partial^2 \tilde{w}_0}{\partial x^2} \Big\| + 2 \Big\| \frac{\partial^2 w_0}{\partial x \partial y} - \frac{\partial^2 \tilde{w}_0}{\partial x \partial y} \Big\| + \Big\| \frac{\partial^2 w_0}{\partial y^2} - \frac{\partial^2 \tilde{w}_0}{\partial y^2} \Big\| \Big\}
\end{aligned}
$$

for all $(x,y,t) \in \Omega_L$ and with $0 < k < 1$. If the maximum of $|w - \tilde{w}|$ on the left-hand side is taken for $(x,y,t) \in \Omega_L$, we obtain

$$
\begin{aligned}
\parallel w - \tilde{w} \parallel \leq \frac{\pi^2}{1-k} \Big\{ p^2 \parallel w_0 - \tilde{w}_0 \parallel + \Big\| \frac{\partial^2 w_0}{\partial x^2} - \frac{\partial^2 \tilde{w}_0}{\partial x^2} \Big\| + 2 \Big\| \frac{\partial^2 w_0}{\partial x \partial y} - \frac{\partial^2 \tilde{w}_0}{\partial x \partial y} \Big\| + \\
\Big\| \frac{\partial^2 w_0}{\partial y^2} - \frac{\partial^2 \tilde{w}_0}{\partial y^2} \Big\| + \parallel w_1 - \tilde{w}_1 \parallel \Big\}.
\end{aligned}
$$

This means that small differences between the initial values cause small differences between the solutions w and $\tilde{w}$ on Ω_L. This completes the proof.

3 On the asymptotic validity of formal approximations.

In section 4 an approximation of the solution of the initial-boundary value problem (10)-(13) will be constructed for $f(x,y,t,w;\epsilon) = w^2$. This approximation is a formal approximation, i.e., a function which satisfies the partial differential equation (10) and the initial conditions (13) up to some order depending on the small parameter ϵ. The formal approximation in section 4 satisfies (10) and (13) up to $\mathcal{O}(\epsilon^2)$. In this section it will be shown that a formal approximation of the solution of the initial-boundary value problem (10)-(13) is also an asymptotic approximation, i.e., the difference between the formal approximation and the exact solution $\to 0$ as $\epsilon \to 0$, on a timescale of order $1/\epsilon$.

Suppose a three times continuously differentiable function $v(x,y,t;\epsilon)$ is constructed on Ω_L with v_{xxxx}, v_{yyyy}, v_{xxyy} continuous, and which satisfies

$$v_{tt}+v_{xxxx}+2v_{xxyy}+v_{yyyy}+p^2v = \epsilon f(x,y,t,v;\epsilon)+|\epsilon|^m R_1(x,y,t;\epsilon), \quad 0<x<\pi,\ 0<y<\pi,\ t>0, \tag{24}$$

$$v(0,y,t) = v(\pi,y,t) = v_{xx}(0,y,t) = v_{xx}(\pi,y,t) = 0, \quad t\geq 0, \tag{25}$$

$$v(x,0,t) = v(x,\pi,t) = v_{yy}(x,0,t) = v_{yy}(x,\pi,t) = 0, \quad t\geq 0, \tag{26}$$

$$v(x,y,0;\epsilon) = w_0(x,y;\epsilon)+|\epsilon|^{m-1}R_2(x,y,t;\epsilon) = v_0(x,y;\epsilon), \quad 0<x,y<\pi, \tag{27}$$

$$v_t(x,y,0;\epsilon) = w_1(x,y;\epsilon)+|\epsilon|^{m-1}R_3(x,y,t;\epsilon) = v_1(x,y;\epsilon), \quad 0<x,y<\pi, \tag{28}$$

with $m>1$, where ϵ, p, f, w_0, w_1 satisfy the same conditions as in section 2 (that is, (14)-(16)), and where R_1, R_2, R_3 satisfy

$$\begin{aligned} &R_1 \quad \text{and all first-, second-, and third-order partial derivatives of} \quad R_1 \\ &\text{with respect to} \quad x,y,t, \quad \text{are} \quad \in C([0,\pi]\times[0,\pi]\times[0,\infty]\times[-\epsilon_0,\epsilon_0],\mathbb{R}), \\ &\text{and} \quad R_1(0,0,t,0;\epsilon) = R_1(\pi,\pi,t,0;\epsilon)\equiv 0 \quad \text{for} \quad t\geq 0, \end{aligned} \tag{29}$$

$$\begin{aligned} &R_2, \frac{\partial R_2}{\partial x}, \frac{\partial^2 R_2}{\partial x^2}, \frac{\partial^3 R_2}{\partial x^3}, \frac{\partial^4 R_2}{\partial x^4}, \frac{\partial R_2}{\partial y}, \frac{\partial^2 R_2}{\partial y^2}, \frac{\partial^3 R_2}{\partial y^3}, \frac{\partial^4 R_2}{\partial y^4}, \frac{\partial^2 R_2}{\partial x\partial y}, \frac{\partial^4 R_2}{\partial x^2\partial y^2}, \frac{\partial^3 R_2}{\partial x^2\partial y}, \\ &\frac{\partial^3 R_2}{\partial x\partial y^2}, \frac{\partial^4 R_2}{\partial x^3\partial y}, \frac{\partial^4 R_2}{\partial x\partial y^3}, R_3, \frac{\partial R_3}{\partial x}, \frac{\partial^2 R_3}{\partial x^2}, \frac{\partial R_3}{\partial y}, \frac{\partial^2 R_3}{\partial y^2}, \frac{\partial^2 R_3}{\partial x\partial y}, \frac{\partial^3 R_3}{\partial x^3}, \frac{\partial^2 R_3}{\partial x^2\partial y}, \frac{\partial^2 R_3}{\partial x\partial y^2}, \end{aligned}$$

$$\begin{aligned} &\frac{\partial^2 R_3}{\partial y^3} \in C([0,\pi]\times[0,\pi]\times[-\epsilon_0,\epsilon_0],\mathbb{R}), \quad \text{with} \quad R_2(0,y;\epsilon) = R_2(\pi,y;\epsilon) = \\ &\frac{\partial^2 R_2}{\partial x^2}(0,y;\epsilon) = \frac{\partial^2 R_2}{\partial x^2}(\pi,y;\epsilon)\equiv 0 \quad \text{and} \quad R_2(x,0;\epsilon) = R_2(x,\pi;\epsilon) = \frac{\partial^2 R_2}{\partial y^2}(x,0;\epsilon) = \\ &\frac{\partial^2 R_2}{\partial y^2}(x,\pi;\epsilon)\equiv 0 \quad \text{and} \quad R_3(0,y;\epsilon) = R_3(\pi,y;\epsilon) = \frac{\partial^2 R_3}{\partial x^2}(0,y;\epsilon) = \frac{\partial^2 R_3}{\partial x^2}(\pi,y;\epsilon)\equiv 0 \\ &\text{and} \quad R_3(x,0;\epsilon) = R_3(x,\pi;\epsilon) = \frac{\partial^2 R_3}{\partial y^2}(x,0;\epsilon) = \frac{\partial^2 R_3}{\partial y^2}(x,\pi;\epsilon)\equiv 0 \end{aligned} \tag{30}$$

$$\begin{aligned} &R_1 \quad \text{and all first-, second-, and third-order partial derivatives of} \quad R_1 \\ &\text{with respect to} \quad x,y,t, \quad \text{are uniformly bounded for all} \quad x,y,t,\epsilon \quad \text{considered.} \end{aligned} \tag{31}$$

We now formulate the following theorem.

Theorem 2. *Let v satisfy (24)-(28), where f, w_0 and w_1 satisfy (14)-(17) and R_1, R_2 and R_3 satisfy (29)-(32). Then for $m>1$ the formal approximation v is an asymptotic approximation (as $\epsilon\to 0$) of the solution w of the nonlinear initial-boundary value problem (10)-(13) for $(x,y,t)\in\Omega_L$. This means that, as $\epsilon\to 0$,*

$$|w(x,y,t)-v(x,y,t;\epsilon)| = \mathcal{O}(|\epsilon|^{m-1}) \quad \text{for} \quad 0\leq x\leq\pi, \quad 0\leq y\leq\pi \quad \text{and} \quad 0\leq t\leq L|\epsilon|^{-1},$$

in which L is a sufficiently small, positive constant independent of ϵ.

Proof. Let $\hat{f}(x,t,v;\epsilon) = f(x,t,v;\epsilon)+|\epsilon|^{m-1}R_1(x,t;\epsilon)$, and let v_l be given by

$$v_l(x,y,t;\epsilon) = \int_0^\pi\int_0^\pi \{G(\xi,\eta,0;x,y,t)v_1(\xi,\eta;\epsilon) - G_\tau(\xi,\eta,0;x,y,t)v_0(\xi,\eta;\epsilon)\}d\xi d\eta.$$

Suppose v_l satisfies $\| v_l \| \leq \frac{1}{2}M_1$ and $\hat{f}$ satisfies (22)-(23). It then follows from Theorem 1 that (24)-(28) has a unique, three times continuously differentiable solution $v(x,y,t;\epsilon)$ on Ω_L, with v_{xxxx}, v_{yyyy}, v_{xxyy} continuous, v is also a solution of the equivalent integral equation

$$v(x,y,t;\epsilon) = \epsilon \int_0^t \int_0^\pi \int_0^\pi G(\xi,\eta,\tau;x,y,t)\hat{f}(\xi,\eta,\tau,v;\epsilon)d\xi d\eta d\tau + v_l(x,y,t;\epsilon) \equiv (Tv)(x,y,t;\epsilon). \tag{32}$$

Since the functions R_1, R_2, R_3 satisfy (29)-(32), it follows that there are constants M_5, M_6, M_7, M_8, M_9, and M_{10} such that

$$|R_1(x,y,t;\epsilon)| \leq M_5, \quad |R_2(x,y,t;\epsilon)| \leq M_6, \quad |R_3(x,y,t;\epsilon)| \leq M_7, \tag{33}$$

$$\left|\frac{\partial^2 R_2(x,y;\epsilon)}{\partial x^2}\right| \leq M_8, \quad \left|\frac{\partial^2 R_2(x,y;\epsilon)}{\partial y^2}\right| \leq M_9, \quad \left|\frac{\partial^2 R_2(x,y;\epsilon)}{\partial x \partial y}\right| \leq M_{10} \tag{34}$$

for all $(x,y,t;\epsilon) \in \Omega_L, \epsilon \in [-\epsilon_0, \epsilon_0]$, and $w, v_1, v_2 \in C_{M_1}(\Omega_L)$. Subtracting the integral equation (32) from the integral equation (17), using (23), (33), (34), (141), the fact that $w, v \in C_{M_1}(\Omega_L)$, and the fact that $(x,y,t;\epsilon) \in \Omega_L$, it follows that

$$\begin{aligned}
&|w(x,y,t) - v(x,y,t;\epsilon)| \\
\leq &|\epsilon|\Big|\int_0^t \int_0^\pi \int_0^\pi G(\xi,\eta,\tau;x,y,t)\Big\{f(\xi,\eta,\tau,w;\epsilon) - \hat{f}(\xi,\eta,\tau,v;\epsilon)\Big\}d\xi d\eta d\tau\Big| \\
&\qquad + |w_l(x,y,t;\epsilon) - v_l(x,y,t;\epsilon)| \\
\leq &|\epsilon|\Big|\int_0^t \int_0^\pi \int_0^\pi G(\xi,\eta,\tau;x,y,t)\{f(\xi,\eta,\tau,w;\epsilon) - f(\xi,\eta,\tau,v;\epsilon)\}d\xi d\eta d\tau\Big| \\
&+ |\epsilon|^m\Big|\int_0^t \int_0^\pi \int_0^\pi G(\xi,\eta,\tau;x,y,t)R_1(\xi,\eta,\tau;\epsilon)d\xi d\eta d\tau\Big| \\
&+ \Big|\int_0^\pi \int_0^\pi G(\xi,\eta,0;x,y,t)\{w_1(\xi,\eta;\epsilon) - v_1(\xi,\eta;\epsilon)\}d\xi d\eta\Big| \\
&+ \Big|\int_0^\pi \int_0^\pi G_\tau(\xi,\eta,0;x,y,t)\{w_0(\xi,\eta;\epsilon) - v_0(\xi,\eta;\epsilon)\}d\xi d\eta\Big| \\
\leq &|\epsilon| \int_0^t \pi^2 M_3 \| w - v \| d\tau + |\epsilon|^m \int_0^t \pi^2 M_5 d\tau + |\epsilon|^{m-1} \int_0^\pi \pi^2 M_7 \\
&+ |\epsilon|^{m-1}\pi^2 \max_{0 \leq x,y \leq \pi}\Big\{p^2|R_2(x,y;\epsilon)| + \Big|\frac{\partial^2 R_2(x,y;\epsilon)}{\partial x^2}\Big| + \Big|\frac{\partial^2 R_2(x,y;\epsilon)}{\partial y^2}\Big| + \Big|\frac{\partial^2 R_2(x,y;\epsilon)}{\partial x \partial y}\Big|\Big\} \\
&\leq |\epsilon| t\pi^2 M_3 \| w - v \| + |\epsilon|^m t\pi^2 M_5 + |\epsilon|^{m-1}\pi^2(M_7 + p^2 M_6 + M_8 + M_9 + M_{10}) \\
&\leq \pi^2 L M_3 \| w - v \| + |\epsilon|^{m-1}\pi^2(LM_5 + M_7 + p^2 M_6 + M_8 + M_9 + M_{10}) \\
&\leq k \| w - v \| + |\epsilon|^{m-1}\pi^2(LM_5 + M_7 + p^2 M_6 + M_8 + M_9 + M_{10})
\end{aligned}$$

for all $(x,y,t;\epsilon) \in \Omega_L$ and with $0 < k < 1$. If the maximum of $|w - v|$ on the left-hand side is taken for $(x,y,t;\epsilon) \in \Omega_L$, we obtain

$$\| w - v \| \leq |\epsilon|^{m-1}\frac{\pi^2}{1-k}(LM_5 + M_7 + p^2 M_6 + M_8 + M_9 + M_{10}).$$

So, for $(x, y, t; \epsilon) \in \Omega_L, |w(x, y, t) - v(x, y, t; \epsilon)| = O(|\epsilon|^{m-1})$ as $\epsilon \to 0$. Hence, for $m > 1$ the function v is an asymptotic approximation (as $\epsilon \to 0$) of the solution w of the initial-boundary value problem (10)-(13). This completes the proof.

4 The construction of asymptotic approximations: general case.

In this section and in the following section asymptotic approximations of the solution of the initial-boundary value problem (6)-(9) will be constructed. When straightforward $\epsilon-$expansions are used to approximate solutions, secular terms can occur for specific values of p^2. To avoid these secular terms a two-timescales perturbation method is used. The initial-boundary value problem (6)-(9) is extended to an initial value problem by extending all functions in x and y. The boundary conditions imply that w should be extended as an odd 2π-periodic function in x and as an odd 2π-periodic function in y. Let us write w as a Fourier sine-series in x and y:

$$w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn}(t) \sin(mx) \sin(ny). \tag{35}$$

This extension implies that all terms in (6) should be extended as odd, 2π-periodic function in x and in y. For the nonlinearity on the right-hand side of (6) this means that (6) has to be rewritten as

$$w_{tt} + w_{xxxx} + 2w_{xxyy} + w_{yyyy} + p^2 w = \epsilon h(x)g(y)w^2, \tag{36}$$

where the functions h and g, defined on $\mathbb{R}$, are given by $h(x) = 1$ for $0 < x < \pi$, $h(0) = h(\pi) = 0$, and h is an odd and 2π-periodic function in x; $g(y) = 1$ for $0 < y < \pi$, $g(0) = g(\pi) = 0$, and g is an odd and 2π-periodic function in y. The functions $h(x)$ and $g(y)$ can then be written in the following Fourier sine-series:

$$h(x) = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\sin((2j+1)x)}{2j+1}, \quad g(y) = \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{\sin((2i+1)y)}{2i+1}. \tag{37}$$

By substituting (35) and (37) into (36) we obtain

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Big(\ddot{q}_{mn}(t) + (m^4 + 2m^2n^2 + n^4 + p^2) q_{mn} \Big) \sin(mx) \sin(ny) =$$
$$\epsilon \frac{16}{\pi^2} \sum_{m,h=1}^{\infty} \sum_{n,s=1}^{\infty} \sum_{i,j=0}^{\infty} \frac{q_{mn}(t) q_{hs}(t)}{(2i+1)(2j+1)} \sin(mx) \sin(hx) \sin(ny) \sin(sy) \sin((2j+1)x) \sin((2i+1)y). \tag{38}$$

The right-hand side of (38) can be rewritten by using the goniometric formula $\sin(mx)\sin(hx)\sin((2j+1)x) = \frac{1}{4}\big(\sin((m+h-(2j+1))x) - \sin((m-h-(2j+1))x) - \sin((m+h+(2j+1))x) + \sin((m-h+(2j+1))x)\big)$. The equations for q_{kl} is obtained by multiplying (38) with $\frac{4}{\pi^2} \sin(kx) \sin(ly)$ and then by integrating the so-obtained equation with respect to both x and y from 0 to π. Using the orthogonality relations of the sin-functions and the symmetry in m, n and h, s the following equation for each q_{kl} for $k, l = 1, 2, 3, \ldots$ is obtained:

$$\begin{aligned} \ddot{q}_{kl} + d_{kl_p}^2 q_{kl} = & \\ & \frac{\epsilon}{\pi^2} \Bigg(2 \sum_{k=m-h+(2j+1)} -2 \sum_{k=m-h-(2j+1)} + \sum_{k=m+h-(2j+1)} - \\ & \sum_{k=m+h+(2j+1)} - \sum_{k=-m-h+(2j+1)} \Bigg) \Bigg(2 \sum_{l=n-s+(2i+1)} -2 \sum_{l=n-s-(2i+1)} + \\ & \sum_{l=n+s-(2i+1)} - \sum_{l=n+s+(2i+1)} - \sum_{l=-n-s+(2i+1)} \Bigg) \frac{q_{mn}(t) q_{hs}(t)}{(2i+1)(2j+1)}, \end{aligned} \tag{39}$$

where $d_{kl_p} = \sqrt{(k^2+l^2)^2+p^2}$ and q_{kl} has to satisfy the following initial conditions

$$q_{kl}(0) = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} w_0(x,y)\sin(kx)\sin(ly)dxdy,\ \dot{q}_{kl}(0) = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} w_1(x,y)\sin(kx)\sin(ly)dxdy.$$

It should be observed that (39) also can be obtained by substituting (35) into (6), and by multiplying the so-obtained equation with $\sin(kx)\sin(ly)$, and then by integrating with respect to x and y from 0 to π. Equation (39) can be rewritten in the following form

$$\begin{aligned}
&\ddot{q}_{kl} + d_{kl_p}^2 q_{kl} = \\
&\frac{\epsilon}{\pi^2}\Bigg(4\sum_{\substack{k=m-h+(2j+1)\\ l=n-s+(2i+1)}} -4\sum_{\substack{k=m-h-(2j+1)\\ l=n-s+(2i+1)}} +2\sum_{\substack{k=m+h-(2j+1)\\ l=n-s+(2i+1)}} - \\
&2\sum_{\substack{k=m+h+(2j+1)\\ l=n-s+(2i+1)}} -2\sum_{\substack{k=-m-h+(2j+1)\\ l=n-s+(2i+1)}} -4\sum_{\substack{k=m-h+(2j+1)\\ l=n-s-(2i+1)}} +4\sum_{\substack{k=m-h-(2j+1)\\ l=n-s-(2i+1)}} - \\
&2\sum_{\substack{k=m+h-(2j+1)\\ l=n-s-(2i+1)}} +2\sum_{\substack{k=m+h+(2j+1)\\ l=n-s-(2i+1)}} +2\sum_{\substack{k=-m-h+(2j+1)\\ l=n-s-(2i+1)}} +2\sum_{\substack{k=m-h+(2j+1)\\ l=n+s-(2i+1)}} - \\
&2\sum_{\substack{k=m-h-(2j+1)\\ l=n+s-(2i+1)}} +\sum_{\substack{k=m+h-(2j+1)\\ l=n+s-(2i+1)}} -\sum_{\substack{k=m+h+(2j+1)\\ l=n+s-(2i+1)}} -\sum_{\substack{k=-m-h+(2j+1)\\ l=n+s-(2i+1)}} - \\
&2\sum_{\substack{k=m-h+(2j+1)\\ l=n+s+(2i+1)}} +2\sum_{\substack{k=m-h-(2j+1)\\ l=n+s+(2i+1)}} -\sum_{\substack{k=m+h-(2j+1)\\ l=n+s+(2i+1)}} +\sum_{\substack{k=m+h+(2j+1)\\ l=n+s+(2i+1)}} + \\
&\sum_{\substack{k=-m-h+(2j+1)\\ l=n+s+(2i+1)}} -2\sum_{\substack{k=m-h+(2j+1)\\ l=-n-s+(2i+1)}} +2\sum_{\substack{k=m-h-(2j+1)\\ l=-n-s+(2i+1)}} -\sum_{\substack{k=m+h-(2j+1)\\ l=-n-s+(2i+1)}} + \\
&\sum_{\substack{k=m+h+(2j+1)\\ l=-n-s+(2i+1)}} +\sum_{\substack{k=-m-h+(2j+1)\\ l=-n-s+(2i+1)}}\Bigg)\frac{q_{mn}q_{hs}}{(2i+1)(2j+1)}.
\end{aligned} \tag{40}$$

In the literature, for example in [15], systems similar to (40) ara analyzed using averaging methods. In this paper, however, as in [11] we use the method of multiple scales for its efficiency and wider applicability. Terms that give rise to secular terms may occur in the right-hand side of (40). To eliminate these terms we introduce two time scales, $t_0 = t$ and $t_1 = \epsilon t$, and assume that q_{kl} can be expanded in a formal power series in ϵ, that is, $q_{kl}(t) = q_{kl,0}(t_0,t_1) + \epsilon q_{kl,1}(t_0,t_1) + \epsilon^2 q_{kl,2}(t_0,t_1) + \ldots$. We substitute this into (40) and collect equal powers in ϵ. The $\mathcal{O}(\epsilon^0)$-problem becomes

$$\frac{\partial^2 q_{kl,0}}{\partial t_0^2} + d_{kl_p}^2 q_{kl,0} = 0, \quad t > 0, \tag{41}$$

$$q_{kl,0}(0,0) = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} w_0(x,y)\sin(kx)\sin(ly)dxdy, \tag{42}$$

$$\frac{\partial}{\partial t_0}q_{kl,0}(0,0) = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} w_1(x,y)\sin(kx)\sin(ly)dxdy$$

for $k,l = 1,2,3,...$ The general solution of (41)-(42) is

$$q_{kl,0}(t_0,t_1) = A_{kl,0}(t_1)\cos(d_{kl_p}t_0) + B_{kl,0}(t_1)\sin(d_{kl_p}t_0), \tag{43}$$

where $A_{kl,0}, B_{kl,0}$ satisfy the following initial conditions:

$$A_{kl,0}(0) = q_{kl,0}(0,0), \quad B_{kl,0}(0) = \frac{1}{d_{kl_p}}\frac{\partial}{\partial t_0}q_{kl,0}(0,0). \tag{44}$$

Next we consider the $\mathcal{O}(\epsilon^1)$-problem

$$\frac{\partial^2 q_{kl,1}}{\partial t_0^2} + d_{kl_p}^2 q_{kl,1} = -2\frac{\partial^2 q_{kl,0}}{\partial t_0 \partial t_1} + \frac{1}{\pi^2}R, \tag{45}$$

$$q_{kl,1}(0,0) = 0, \quad \frac{\partial}{\partial t_0} q_{kl,1}(0,0) = -\frac{\partial}{\partial t_1} q_{kl,0}(0,0), \tag{46}$$

for $k, l = 1, 2, 3,...$, where

$$\begin{aligned} R = \Bigg(& 4 \sum_{\substack{k=m-h+(2j+1)\\ l=n-s+(2i+1)}} -4 \sum_{\substack{k=m-h-(2j+1)\\ l=n-s+(2i+1)}} +2 \sum_{\substack{k=m+h-(2j+1)\\ l=n-s+(2i+1)}} -2 \sum_{\substack{k=m+h+(2j+1)\\ l=n-s+(2i+1)}} - \\ & 2 \sum_{\substack{k=-m-h+(2j+1)\\ l=n-s+(2i+1)}} -4 \sum_{\substack{k=m-h+(2j+1)\\ l=n-s-(2i+1)}} +4 \sum_{\substack{k=m-h-(2j+1)\\ l=n-s-(2i+1)}} -2 \sum_{\substack{k=m+h-(2j+1)\\ l=n-s-(2i+1)}} + \\ & 2 \sum_{\substack{k=m+h+(2j+1)\\ l=n-s-(2i+1)}} +2 \sum_{\substack{k=-m-h+(2j+1)\\ l=n-s-(2i+1)}} +2 \sum_{\substack{k=m-h+(2j+1)\\ l=n+s-(2i+1)}} -2 \sum_{\substack{k=m-h-(2j+1)\\ l=n+s-(2i+1)}} + \\ & \sum_{\substack{k=m+h-(2j+1)\\ l=n+s-(2i+1)}} - \sum_{\substack{k=m+h+(2j+1)\\ l=n+s-(2i+1)}} - \sum_{\substack{k=-m-h+(2j+1)\\ l=n+s-(2i+1)}} -2 \sum_{\substack{k=m-h+(2j+1)\\ l=n+s+(2i+1)}} +2 \sum_{\substack{k=m-h-(2j+1)\\ l=n+s+(2i+1)}} - \\ & \sum_{\substack{k=m+h-(2j+1)\\ l=n+s+(2i+1)}} + \sum_{\substack{k=m+h+(2j+1)\\ l=n+s+(2i+1)}} + \sum_{\substack{k=-m-h+(2j+1)\\ l=n+s+(2i+1)}} -2 \sum_{\substack{k=m-h+(2j+1)\\ l=-n-s+(2i+1)}} + \\ & 2 \sum_{\substack{k=m-h-(2j+1)\\ l=-n-s+(2i+1)}} - \sum_{\substack{k=m+h-(2j+1)\\ l=-n-s+(2i+1)}} + \sum_{\substack{k=m+h+(2j+1)\\ l=-n-s+(2i+1)}} + \sum_{\substack{k=-m-h+(2j+1)\\ l=-n-s+(2i+1)}} \Bigg) \frac{q_{mn,0}q_{hs,0}}{(2i+1)(2j+1)}. \end{aligned} \tag{47}$$

By substituting (43) into (45) the following equation is obtained

$$\frac{\partial^2 q_{kl,1}}{\partial t_0^2} + d_{kl_p}^2 q_{kl,1} = 2d_{kl_p}\Big(\frac{dA_{kl,0}}{dt_1}\sin(d_{kl_p}t_0) - \frac{dB_{kl,0}}{dt_1}\cos(d_{kl_p}t_0)\Big) + \frac{1}{\pi^2}H, \tag{48}$$

where

$$\begin{aligned} H = \Bigg(& 4 \sum_{\substack{k=m-h+(2j+1)\\ l=n-s+(2i+1)}} -4 \sum_{\substack{k=m-h-(2j+1)\\ l=n-s+(2i+1)}} +2 \sum_{\substack{k=m+h-(2j+1)\\ l=n-s+(2i+1)}} -2 \sum_{\substack{k=m+h+(2j+1)\\ l=n-s+(2i+1)}} -2 \sum_{\substack{k=-m-h+(2j+1)\\ l=n-s+(2i+1)}} - \\ & 4 \sum_{\substack{k=m-h+(2j+1)\\ l=n-s-(2i+1)}} +4 \sum_{\substack{k=m-h-(2j+1)\\ l=n-s-(2i+1)}} -2 \sum_{\substack{k=m+h-(2j+1)\\ l=n-s-(2i+1)}} +2 \sum_{\substack{k=m+h+(2j+1)\\ l=n-s-(2i+1)}} +2 \sum_{\substack{k=-m-h+(2j+1)\\ l=n-s-(2i+1)}} + \\ & 2 \sum_{\substack{k=m-h+(2j+1)\\ l=n+s-(2i+1)}} -2 \sum_{\substack{k=m-h-(2j+1)\\ l=n+s-(2i+1)}} + \sum_{\substack{k=m+h-(2j+1)\\ l=n+s-(2i+1)}} - \sum_{\substack{k=m+h+(2j+1)\\ l=n+s-(2i+1)}} - \sum_{\substack{k=-m-h+(2j+1)\\ l=n+s-(2i+1)}} - \\ & 2 \sum_{\substack{k=m-h+(2j+1)\\ l=n+s+(2i+1)}} +2 \sum_{\substack{k=m-h-(2j+1)\\ l=n+s+(2i+1)}} - \sum_{\substack{k=m+h-(2j+1)\\ l=n+s+(2i+1)}} + \sum_{\substack{k=m+h+(2j+1)\\ l=n+s+(2i+1)}} + \sum_{\substack{k=-m-h+(2j+1)\\ l=n+s+(2i+1)}} - \\ & 2 \sum_{\substack{k=m-h+(2j+1)\\ l=-n-s+(2i+1)}} +2 \sum_{\substack{k=m-h-(2j+1)\\ l=-n-s+(2i+1)}} - \sum_{\substack{k=m+h-(2j+1)\\ l=-n-s+(2i+1)}} + \sum_{\substack{k=m+h+(2j+1)\\ l=-n-s+(2i+1)}} + \sum_{\substack{k=-m-h+(2j+1)\\ l=-n-s+(2i+1)}} \Bigg) \frac{1}{2i+1}\frac{1}{2j+1} \\ & \Big((A_{mn,0}(t_1)\cos(d_{mn_p}t_0) + B_{mn,0}(t_1)\sin(d_{mn_p}t_0))(A_{hs,0}(t_1)\cos(d_{hs_p}t_0) + B_{hs,0}(t_1)\sin(d_{hs_p}t_0)\Big). \end{aligned} \tag{49}$$

Since $\cos(d_{kl_p}t_0)$ and $\sin(d_{kl_p}t_0)$ are part of the homogeneous solution of the equation for $q_{kl,1}$, it follows that the coefficients of $\cos(d_{kl_p}t_0)$ and $\sin(d_{kl_p}t_0)$ in the right-hand side of (48) have to be equal to zero (elimination of secular terms). This gives us differential equations for $A_{kl,0}$ and $B_{kl,0}$. Now we show that to find the equations for $A_{kl,0}$ and $B_{kl,0}$, the terms in (48) that give rise to secular terms in the approximation have to be determined. Using some goniometric formulas H can be rewritten as:

$$H = \frac{1}{2}\Bigg(\Big(4 \sum_{\substack{k=m-h+\lambda \\ l=n-s+\beta}} -4 \sum_{\substack{k=m-h-\lambda \\ l=n-s+\beta}} +2 \sum_{\substack{k=m+h-\lambda \\ l=n-s+\beta}} -2 \sum_{\substack{k=m+h+\lambda \\ l=n-s+\beta}} -2 \sum_{\substack{k=-m-h+\lambda \\ l=n-s+\beta}} -$$
$$4 \sum_{\substack{k=m-h+\lambda \\ l=n-s-\beta}} +4 \sum_{\substack{k=m-h-\lambda \\ l=n-s-\beta}} -2 \sum_{\substack{k=m+h-\lambda \\ l=n-s-\beta}} +2 \sum_{\substack{k=m+h+\lambda \\ l=n-s-\beta}} +2 \sum_{\substack{k=-m-h+\lambda \\ l=n-s-\beta}} +$$
$$2 \sum_{\substack{k=m-h+\lambda \\ l=n+s-\beta}} -2 \sum_{\substack{k=m-h-\lambda \\ l=n+s-\beta}} + \sum_{\substack{k=m+h-\lambda \\ l=n+s-\beta}} - \sum_{\substack{k=m+h+\lambda \\ l=n+s-\beta}} - \sum_{\substack{k=-m-h+\lambda \\ l=n+s-\beta}} -$$
$$2 \sum_{\substack{k=m-h+\lambda \\ l=n+s+\beta}} +2 \sum_{\substack{k=m-h-\lambda \\ l=n+s+\beta}} - \sum_{\substack{k=m+h-\lambda \\ l=n+s+\beta}} + \sum_{\substack{k=m+h+\lambda \\ l=n+s+\beta}} + \sum_{\substack{k=-m-h+\lambda \\ l=n+s+\beta}} - \tag{50}$$
$$2 \sum_{\substack{k=m-h+\lambda \\ l=-n-s+\beta}} +2 \sum_{\substack{k=m-h-\lambda \\ l=-n-s+\beta}} - \sum_{\substack{k=m+h-\lambda \\ l=-n-s+\beta}} + \sum_{\substack{k=m+h+\lambda \\ l=-n-s+\beta}} + \sum_{\substack{k=-m-h+\lambda \\ l=-n-s+\beta}} \Big)\frac{1}{\lambda}\frac{1}{\beta}$$

$$\Big[(A_{mn,0}A_{hs,0} - B_{mn,0}B_{hs,0})\cos([\sqrt{(m^2+n^2)^2+p^2} + \sqrt{(h^2+s^2)^2+p^2}]t_0) +$$
$$(A_{mn,0}A_{hs,0} + B_{mn,0}B_{hs,0})\cos([\sqrt{(m^2+n^2)^2+p^2} - \sqrt{(h^2+s^2)^2+p^2}]t_0)$$
$$(B_{mn,0}A_{hs,0} + A_{mn,0}B_{hs,0})\sin([\sqrt{(m^2+n^2)^2+p^2} + \sqrt{(h^2+s^2)^2+p^2}]t_0) +$$
$$(B_{mn,0}A_{hs,0} - A_{mn,0}B_{hs,0})\sin([\sqrt{(m^2+n^2)^2+p^2} - \sqrt{(h^2+s^2)^2+p^2}]t_0)\Big]\Bigg),$$

where $\lambda = 2j+1$ and $\beta = 2i+1$.The terms given in (50) can cause secular terms if

$$\pm\sqrt{(k^2+l^2)^2+p^2} = \pm\sqrt{(m^2+n^2)^2+p^2} \pm \sqrt{(h^2+s^2)^2+p^2}.$$

To determine the contribution of the summations in (48) to the coefficients of $\cos(d_{kl_p}t_0)$ and $\sin(d_{kl_p}t_0)$ in the right-hand side of (48), the following equations have to be examined:

$$\begin{aligned} \sqrt{(k^2+l^2)^2+p^2} &= \sqrt{(m^2+n^2)^2+p^2} + \sqrt{(h^2+s^2)^2+p^2}, \\ \sqrt{(k^2+l^2)^2+p^2} &= \sqrt{(m^2+n^2)^2+p^2} - \sqrt{(h^2+s^2)^2+p^2}, \\ -\sqrt{(k^2+l^2)^2+p^2} &= \sqrt{(m^2+n^2)^2+p^2} - \sqrt{(h^2+s^2)^2+p^2}, \end{aligned} \tag{51}$$

It should be noted that the three cases in (51) are in fact equivalent. For that reason only the first case is considered. By putting: $k^2+l^2 = N, \quad m^2+n^2 = K, \quad h^2+s^2 = S$, the first equation in (51) can be written in the following form

$$\sqrt{N^2+p^2} = \sqrt{K^2+p^2} + \sqrt{S^2+p^2}. \tag{52}$$

For $p = 0$ it follows directly from (52) that $N = K+S$. To solve the problem for $p > 0$ the following inequality will be used:

$$\sqrt{j^2+p^2} \le j - 1 + \sqrt{1+p^2} \quad \text{for all p, and all} \quad j \in \mathbb{Z}^+. \tag{53}$$

Using (52) and (53) we obtain for $p > 0$

$$N < \sqrt{N^2+p^2} = \sqrt{K^2+p^2} + \sqrt{S^2+p^2} \le K + S + 2(\sqrt{1+p^2} - 1).$$

By squaring (52) it follows for $p > 0$ that :

$$\begin{aligned} N^2+p^2 &= K^2+p^2+S^2+p^2+2\sqrt{K^2+p^2}\sqrt{S^2+p^2} \Leftrightarrow \\ N^2 &= (K+S)^2 + 2(\sqrt{K^2+p^2}\sqrt{S^2+p^2} - KS) + p^2 \ge (K+S)^2 + p^2. \end{aligned} \tag{54}$$

Since $K, S, N \in \mathbb{Z}^+$ it follows that $N > K + S$ for $p > 0$. So, if (52) is considered, we conclude that secular terms can occur for

$$N = K + S + \lambda^*, \tag{55}$$

where $K, S, N, \lambda^* \in \mathbb{Z}^+$. From (50) we know that $k = \pm m \pm h \pm \lambda$ and $l = \pm n \pm s \pm \beta$, where both λ and β are odd, and where $k^2 + l^2 = N, \quad m^2 + n^2 = K$, and $h^2 + s^2 = S$. So, if the expressions for K, N, S are substituted into equation (55), we obtain

$$\lambda^* = \lambda^2 + \beta^2 \pm 2(mh + m\lambda + h\lambda) \pm 2(ns + n\beta + s\beta)$$

and it is clear that λ^* is always even. By substituting (55) into (52) and by squaring the so-obtained equation it follows that

$$KS + \lambda^*(K + S) + \frac{(\lambda^*)^2 - p^2}{2} = \sqrt{K^2 + p^2}\sqrt{K^2 + p^2}. \tag{56}$$

By squaring (56) and after rearranging terms we obtain

$$K^2(2\lambda^* S + (\lambda^*)^2 - p^2) + K(S + \lambda^*)(2\lambda^* S + (\lambda^*)^2 - p^2) + \left(\lambda^* S + \frac{(\lambda^*)^2 - p^2}{2}\right)^2 - p^2 S^2 - p^4 = 0. \tag{57}$$

It should be observed that the discriminant of this quadratic equation in K is negative for $p^2 > 2\lambda^* S + (\lambda^*)^2$, and so $K \notin \mathbb{Z}^+$. If $p^2 = 2\lambda^* S + (\lambda^*)^2$ then (57) also does not have a solution $K \in \mathbb{Z}^+$. For $p^2 < 2\lambda^* S + (\lambda^*)^2$ the solution $K = K_{\lambda^*}(S; p^2)$ of (57) is given by $K \in \mathbb{Z}^+$ and

$$K_{\lambda^*}(S; p^2) = -\frac{S + \lambda^*}{2} + \frac{\sqrt{2\lambda^* S^3 + (\lambda^*)^2 S^2 + 2\lambda^* S p^2 + (\lambda^*)^2 p^2 + 3p^2 S^2 + 3p^4}}{2\sqrt{2\lambda^* S + (\lambda^*)^2 - p^2}}. \tag{58}$$

By implicitly differentiating (56) it can be shown for a fixed value of λ^*, and $p^2 < 2\lambda^* S + (\lambda^*)^2$ that $\partial K_{\lambda^*}/\partial p^2 > 0$ and $\partial K_{\lambda^*}/\partial S < 0$.

Table 1. "Secular modes" and corresponding value of p^2 for $0 \le p^2 \le 50$

Values of p^2 for which secular terms occur	S, K, N
$0 \le p^2 < 12$	
(a) $p^2 = -\frac{2}{3}(K^2 + 4K + 10) + \frac{2}{3}\sqrt{(K^2 + 4K + 10)^2 + 36(K + 1)(K + 3)}$	2, K, K+4
$12 \le p^2 < 24$	
(a) $p^2 = -\frac{2}{3}(K^2 + 7K + 37) + \frac{2}{3}\sqrt{(K^2 + 7K + 37)^2 + 72(K + 1)(K + 6)}$	5, K, K+7
(b) $p^2 = -\frac{2}{3}(K^2 + 6K + 20) + \frac{2}{3}\sqrt{(K^2 + 6K + 20)^2 + 96(K + 2)(K + 4)}$, $K \in \{2, 5\}$	2, K, K+6
(c) $p^2 = 22\frac{2}{3}$	8, 8, 18
$24 \le p^2 < 35.5$	
(a) $p^2 = -\frac{2}{3}(K^2 + 6K + 20) + \frac{2}{3}\sqrt{(K^2 + 6K + 20)^2 + 96(K + 2)(K + 4)}$, $K \in \{8, 10, 13, 17...\}$	2, K, K+6
(b) $p^2 = -\frac{2}{3}(K^2 + 10K + 82) + \frac{2}{3}\sqrt{(K^2 + 10K + 82)^2 + 108(K + 1)(K + 9)}$	8, K, K+10
(c) $p^2 = -\frac{2}{3}(K^2 + 12K + 122) + \frac{2}{3}\sqrt{(K^2 + 12K + 122)^2 + 132(K + 1)(K + 11)}$, $K \in \{8, 10, 13, 17, 18\}$	10, K, K+12
(d) $p^2 = 32$	5, 5, 14
(e) $p^2 = 28$	2, 2, 10

Extension of Table 1.

Values of p^2 for which secular terms occur	S, K, N
$35.5 \le p^2 < 41$	
(a) $p^2 = -\frac{2}{3}(K^2+12K+122)+\frac{2}{3}\sqrt{(K^2+12K+122)^2+132(K+1)(K+11)}$, $K \in \{20, 25...\}$	10, K, K+12
(b) $p^2 = -\frac{2}{3}(K^2+15K+197)+\frac{2}{3}\sqrt{(K^2+15K+197)^2+168(K+1)(K+14)}$ $K \in \{13, 17\}$	13, K, K+15
(c) $p^2 = 38.54787$	5, 8, 17
(d) p^2=37.71114	2, 5, 13
$41 \le p^2 \le 50$	
(a) $p^2 = -\frac{2}{3}(K^2+15K+197)+\frac{2}{3}\sqrt{(K^2+15K+197)^2+168(K+1)(K+14)}$ $K \in \{18, 20, 25, 26...\}$	13, K, K+15
(b) $p^2 = -\frac{2}{3}(K^2+9K+53)+\frac{2}{3}\sqrt{(K^2+9K+53)^2+168(K+2)(K+7)}$ $K \in \{10, 13, 17, 18, 20\}$	5, K, K+9
(c) $p^2 = -\frac{2}{3}(K^2+8K+34)+\frac{2}{3}\sqrt{(K^2+8K+34)^2+180(K+5)(K+3)}$, $K \in \{8, 10\}$	2, K, K+8
(d) $p^2 = -\frac{2}{3}(K^2+19K+325)+\frac{2}{3}\sqrt{(K^2+19K+325)^2+216(K+1)(K+18)}$, $K \in \{17, 18\}$	17, K, K+19
(e) $p^2 = -\frac{2}{3}(K^2+20K+362)+\frac{2}{3}\sqrt{(K^2+20K+362)^2+228(K+1)(K+19)}$, $K \in \{13, 17, 18\}$	18, K, K+20
(f) p^2=48	8, 8, 20
(g) p^2=$42\frac{2}{3}$	2, 2, 12

In Figure 2 $K_2(S; p^2)$, $K_4(S; p^2)$ and $K_6(S; p^2)$ are given for some $S \in \mathbb{Z}^+$ and $0 \le p^2 \le 50$. Furthermore, in Table 1 all values of p^2 with $0 \le p^2 \le 50$ for which secular terms occur are given as well as the corresponding "secular modes" expressed in (S,K,N). For these values of p^2 we will now determine the functions $A_{kl,0}$ and $B_{kl,0}$, $k, l = 1, 2, 3, ...$ such that these secular terms do not occur.

It is known that for every curve in Figure 2 both λ^* and S are constant. Hence to find for a given value of p^2 as listed in Table 1 all secular modes the complete set of all integer-valued solutions have to be found for the equation

$$N = K + S + \lambda^*,$$

with $N = k^2 + l^2$, $K = m^2 + n^2$, and $S = h^2 + s^2$, and where $k, l, m, n, h, s = 1, 2, 3, ...$ For instance for $p^2 = 22\frac{2}{3}$ we have to solve $N = 18 = k^2 + l^2$, $K = 8 = m^2 + n^2$, and $S = 8 = h^2 + s^2$, yielding $k = l = 3$, $m = n = h = s = 2$. And for instance for $p^2 = 35.40$ we have to solve $N = 82 = k^2 + l^2$, $K = 72 = m^2 + n^2$, and $S = 8 = h^2 + s^2$, yielding $k = 9, l = 1$ or $k = 1, l = 9$ and $m = n = 6$, $h = s = 2$. For $p^2 \ne p^2_{cr}$ the summations in (48) do not give any contributions and the only secular terms in the right-hand side of (48) are $2d_{kl_p}\frac{dA_{kl,0}}{dt_1}\sin(d_{kl_p}t_0)$ and $-2d_{kl_p}\frac{dB_{kl,0}}{dt_1}\cos(d_{kl_p}t_0)$. This means that the equations for $A_{kl,0}$, $B_{kl,0}$ are

$$\frac{dA_{kl,0}}{dt_1} = \frac{dB_{kl,0}}{dt_1} \equiv 0 \quad for \quad k, l = 1, 2, 3, ..., \tag{59}$$

which means $A_{kl,0}(t_1) \equiv A_{kl,0}(0)$ and $B_{kl,0}(t_1) \equiv B_{kl,0}(0)$ for all k, l. So, if we start with zero initial energy, there will be no energy present up to $\mathcal{O}(\epsilon)$ on a timescale of order ϵ^{-1}. For $p^2 \ne p^2_{cr}$ this allows truncation to those modes that have nonzero initial energy.

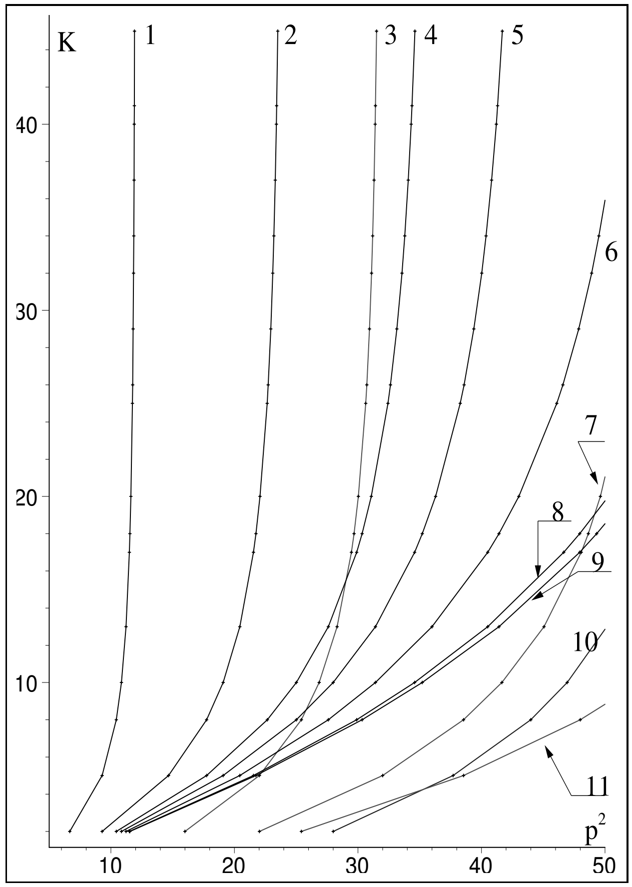


Figure 2: The functions $K_2(S;p^2)$, $K_4(S;p^2)$ and $K_6(S;p^2)$ given by (58) for $S \in \mathbb{Z}^+$ and $0 \le p^2 \le 50$, where the curves 1, 2, 4, 5, 6, 8 and 9 represent the functions $K_2(2;p^2)$, $K_2(5;p^2)$,$K_2(8;p^2)$, $K_2(10;p^2)$, $K_2(13;p^2)$, $K_2(17;p^2)$, and $K_2(18;p^2)$ respectively, the curves 3, 7 and 11 represent the functions $K_4(2;p^2)$, $K_4(5;p^2)$ and $K_4(8;p^2)$respectively, and the curve 10 represents the function $K_6(2;p^2)$.

5 The construction of asymptotic approximations for some specific p^2-values.

In the previous section it has been shown that for most p^2-values (with $0 \le p^2 \le 50$) no mode-interactions will occur, that is, if initially no energy is present in the $k-l$th mode (that is, the mode described by $\sin(kx)\sin(ly)$) then no energy will be present in this mode up to $\mathcal{O}(\epsilon)$ on a time-scale order of ϵ^{-1}. However, for some specific values of p^2 as listed in Table 1 additional mode interactions can occur, and an energy transfer of $\mathcal{O}(1)$ can occur to another mode (with initially zero energy) on a time-scale of order ϵ^{-1}. In this section three such p^2 cases will be studied (including detuning). These three cases are $p^2 \approx 22\frac{2}{3}$, $p^2 \approx 28$, and $p^2 \approx 35.40$. In the first case there will be an interaction between two modes, in the second case an interaction between three modes will occur, and in the third case four modes will interact. It is expected that all other cases can be treated similarly.

5.1 The case $p^2 = p^2_{cr} + \epsilon\alpha$ with $p^2_{cr} = 22\frac{2}{3}$.

In this case $p^2 = 22\frac{2}{3} + \epsilon\alpha$ (where α is an $\mathcal{O}(1)$ detuning parameter) we have interactions between the modes 2-2 and 3-3, that is, interactions between the modes described by $\sin(2x)\sin(2y)$ and $\sin(3x)\sin(3y)$. Extra terms in the equations (48) for $A_{kl,0}, B_{kl,0}$ occur for k=l=2 and k=l=3. The equations for

$A_{22,0}, B_{22,0}, A_{33,0}, B_{33,0}$ now become (as follows from (48)):

$$
\begin{aligned}
2d_{22}\frac{dA_{22,0}}{dt_1} - \alpha B_{22,0} + \frac{610}{441\pi^2}(B_{33,0}A_{22,0} - A_{33,0}B_{22,0}) = 0,\\
-2d_{22}\frac{dB_{22,0}}{dt_1} + \alpha A_{22,0} + \frac{610}{441\pi^2}(A_{33,0}A_{22,0} + B_{33,0}B_{22,0}) = 0,\\
2d_{33}\frac{dA_{33,0}}{dt_1} - \alpha B_{33,0} + \frac{1296}{441\pi^2}A_{22,0}B_{22,0} = 0,\\
-2d_{33}\frac{dB_{33,0}}{dt_1} + \alpha A_{33,0} + \frac{648}{441\pi^2}(A_{22,0}^2 - B_{22,0}^2) = 0.
\end{aligned}
\tag{60}
$$

For $(k,l) \neq (2,2)$ and $(k,l) \neq (3,3)$, $A_{kl,0}, B_{kl,0}$ satisfy

$$
\frac{dA_{kl,0}}{dt_1} = \frac{\alpha}{d_{kl}}B_{kl,0}, \quad \frac{dB_{kl,0}}{dt_1} = -\frac{\alpha}{d_{kl}}A_{kl,0}. \tag{61}
$$

From (61) it can be seen that if $A_{kl,0}(0) = B_{kl,0}(0) = 0$, then for all $t_1 > 0$ $A_{kl,0}(t_1) = B_{kl,0}(t_1) \equiv 0$ for $(k,l) \neq (2,2)$ and $(k,l) \neq (3,3)$. So if we start with zero initial energy in the $k-l$th mode with $(k,l) \neq (2,2)$ and $(k,l) \neq (3,3)$, then there will be no energy present up to $\mathcal{O}(\epsilon)$ on a timescale of order ϵ^{-1}. We say that the coupling between the modes with $(k,l) \neq (2,2)$ and $(k,l) \neq (3,3)$ is of $\mathcal{O}(\epsilon)$. This means that modes with zero initial energy do not have to be taken into account (for $(k,l) \neq (2,2)$ and $(k,l) \neq (3,3)$). On the other hand there is an $\mathcal{O}(1)$ coupling in this case between the modes 2-2 and 3-3. This means that if there is initially energy present in mode 2-2 an energy transfer occurs between the modes 2-2 and 3-3. Truncation to one mode is not valid: both modes 2-2 and 3-3 have to be taken into account, even if mode 3-3 has zero initial energy. Now let us consider equation (60), so we consider only the equations for $k = l = 2$ and $k = l = 3$. We assume that all other modes have zero initial energy. Let

$$
a = \frac{-305}{441 d_{22}\pi^2}, \quad \text{and} \quad b = \frac{-648}{441 d_{33}\pi^2}
$$

then (60) becomes

$$
\begin{aligned}
\dot{A}_{22,0} &= \frac{\alpha}{2d_{22}}B_{22,0} + a(A_{22}B_{33} - A_{33}B_{22}),\\
\dot{B}_{22,0} &= -\frac{\alpha}{2d_{22}}A_{22,0} - a(A_{22}A_{33} - B_{22}B_{33}),
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
\dot{A}_{33,0} &= \frac{\alpha}{2d_{22}}B_{33,0} + 2bA_{22}B_{22},\\
\dot{B}_{33,0} &= -\frac{\alpha}{2d_{22}}A_{33,0} - b(A_{22}^2 - B_{22}^2),
\end{aligned}
$$

where the dot represents differentiation with respect to t_1. To study (62) polar coordinates are introduced

$$
A_{nn} = r_n \cos(\phi_n), \quad B_{nn} = r_n \sin(\phi_n). \tag{63}
$$

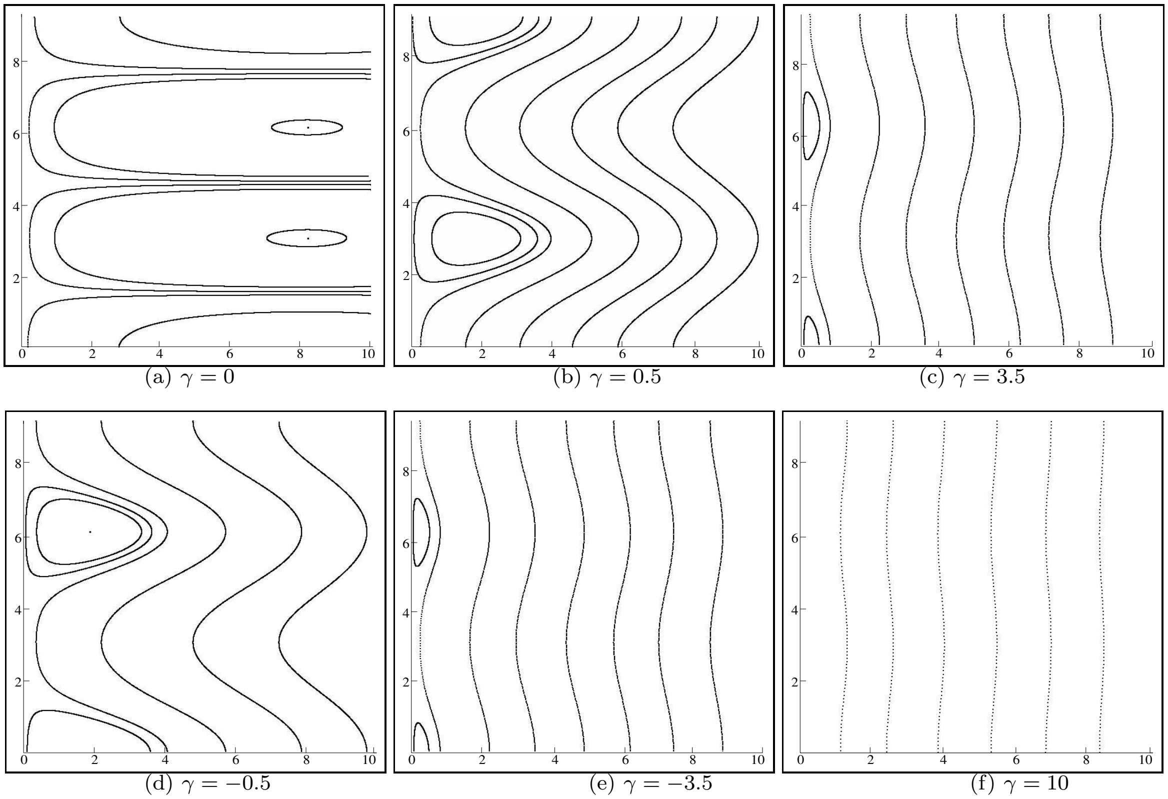
In the polar coordinates (63) for $(k,l) = (2,2)$, and $(3,3)$ equation (62) becomes

$$
\dot{r}_2 = a r_2 r_3 \sin(\phi_3 - 2\phi_2), \tag{64}
$$

$$
\dot{\phi}_2 = \gamma_2 - a r_3 \cos(\phi_3 - 2\phi_2), \tag{65}
$$

$$
\dot{r}_3 = -b r_2^2 \sin(\phi_3 - 2\phi_2), \tag{66}
$$

$$
\dot{\phi}_3 = \gamma_3 - b\frac{r_2^2}{r_3}\cos(\phi_3 - 2\phi_2), \tag{67}
$$



(a) $\gamma = 0$ (b) $\gamma = 0.5$ (c) $\gamma = 3.5$

(d) $\gamma = -0.5$ (e) $\gamma = -3.5$ (f) $\gamma = 10$

Figure 3: The phase space for $p^2 = 22\frac{2}{3} + \epsilon\alpha$ with $\gamma = \alpha\left(\frac{2}{\sqrt{64+p_{cr}^2}} - \frac{1}{\sqrt{324+p_{cr}^2}}\right)$, with r_3 (horizontal) from 0 to 10, ψ (vertical) from 0 to 3π.

where $\gamma_2 = -\frac{\alpha}{2d_{22}}$, $\gamma_3 = -\frac{\alpha}{2d_{33}}$. Multiplying (64) with br_2 and (66) with ar_3 and adding both equations we obtain $br_2\dot{r}_2 + ar_3\dot{r}_3 = 0$, which implies

$$br_2^2 + ar_3^2 = c_1, \tag{68}$$

where c_1 is a constant of integration. Using (68) it follows that (64), and (67) can be analyzed in the (r_3, ψ) phase space with $\psi = \phi_3 - 2\phi_2$:

$$\dot{r}_3 = (ar_3^2 - c_1)\sin(\psi), \tag{69}$$

$$\dot{\psi} = \gamma + \frac{1}{r_3}(3ar_3^2 - c_1)\cos(\psi), \tag{70}$$

where $\gamma = \gamma_3 - 2\gamma_2$. First let us rescale (69), and (70) by introducing the new variables $r_3 = \sqrt{\frac{c_1}{a}}\bar{r}_3$, $t = \sqrt{\frac{1}{c_1 a}}\bar{t}$. In the new variables (69) and (70) become:

$$\bar{r}_3' = (\bar{r}_3^2 - 1)\sin(\psi), \tag{71}$$

$$\psi' = \bar{\gamma} + \frac{1}{\bar{r}_3}(3\bar{r}_3^2 - 1)\cos(\psi), \tag{72}$$

where $\bar{\gamma} = \frac{\alpha}{\sqrt{c_1 a}}\left(\frac{1}{d_{22}} - \frac{1}{2d_{33}}\right)$, and where the prime $'$ denotes differentiation with respect to $\bar{t}$. For $\bar{r}_3 = 0$ (71)-(72) do not hold. In this case the original differential equations (62) have to be analyzed. We will determine the critical points of (69)-(70) analytically for different values of γ. Let us start with $\gamma = 0$. System (69)-(70) is analyzed in the (r_3, ψ) phase space. This system is 2π-periodic in ψ. Four critical points are found for $0 \le \psi \le 2\pi$: $(1,0)$ and $(\sqrt{\frac{1}{3}}, \pi)$, both centers, and $(1, \frac{\pi}{2})$ and $(1, \frac{3\pi}{2})$, both saddles. The behavior of the solution of (71)-(72) with $\gamma = 0$ in the $(\bar{r}_3, \psi)$ phase space is given in Figure 3(a). In the exceptional case, when $\bar{r}_3$ becomes 0, the original differential equations (62) impose a phasejump from $\psi = 0$, to $\psi = \frac{\pi}{2}$. The periods of the orbits in Figure 3(a) also can be found. From system (71)-(72) with $\gamma = 0$ it follows that

$$\frac{(3\bar{r}_3^2 - 1)}{\bar{r}_3(\bar{r}_3^2 - 1)} d\bar{r}_3 = \tan\psi d\psi. \tag{73}$$

After integration we get the trajectories of the closed orbits

$$\bar{r}_3(\bar{r}_3^2 - 1) = \frac{c_2}{\cos\psi}, \tag{74}$$

where c_2 is constant of integration. The period of a closed orbit in Figure 3(a) is given by.

$$T = 2\sin\left(\frac{1}{3}\arccos\left(\frac{3\sqrt{3}c_2}{2}\right)\right), \quad -\frac{2}{3\sqrt{3}} < c_2 < \frac{2}{3\sqrt{3}}. \tag{75}$$

A completely similar analysis can be given for $\gamma \neq 0$. Some results in the phase space are given in Figure 3(b)-(f). For large values of the detuning parameter it can be seen in Figure 3 that the trajectories in the phase space are almost straight lines implying that (almost) no mode interactions take place.

5.2 The case $p^2 = p_{cr}^2 + \epsilon\alpha$ with $p_{cr}^2 = 28$.

For the case when $p^2 = 28 + \epsilon\alpha$ (where α is an $\mathcal{O}(1)$ detuning parameter) we have interactions between three modes: the 1-1, the 1-3 and the 3-1 mode. In this case extra terms in the equations (48) for $A_{kt,0}, B_{kt,0}$ occur for k=l=1; k=1, l=3 and k=3,l=1. The equations for $A_{11,0}, B_{11,0}, A_{13,0}, B_{13,0}, A_{31,0}$, $B_{31,0}$ now become (as follows from (48)):

$$\begin{aligned}
2d_{11}\frac{dA_{11,0}}{dt_1} - \alpha B_{11,0} - \frac{96}{45\pi^2}(A_{11,0}(B_{13,0} + B_{31,0}) - B_{11,0}(A_{13,0} + A_{31,0})) &= 0, \\
-2d_{11}\frac{dB_{11,0}}{dt_1} + \alpha A_{11,0} - \frac{96}{45\pi^2}(A_{11,0}(A_{13,0} + A_{31,0}) - B_{11,0}(B_{13,0} + B_{31,0})) &= 0, \\
2d_{13}\frac{dA_{13,0}}{dt_1} - \alpha B_{13,0} - \frac{64}{45\pi^2}A_{11,0}B_{11,0} &= 0,
\end{aligned} \tag{76}$$

$$\begin{aligned}
-2d_{13}\frac{dB_{13,0}}{dt_1} + \alpha A_{13,0} - \frac{32}{45\pi^2}(A_{11,0}^2 - B_{11,0}^2) &= 0, \\
2d_{31}\frac{dA_{31,0}}{dt_1} - \alpha B_{31,0} - \frac{64}{45\pi^2}A_{11,0}B_{11,0} &= 0, \\
-2d_{31}\frac{dB_{31,0}}{dt_1} + \alpha A_{31,0} - \frac{32}{45\pi^2}(A_{11,0}^2 - B_{11,0}^2) &= 0.
\end{aligned}$$

For $(k,l) \neq (1,1)$, $(k,l) \neq (1,3)$, and $(k,l) \neq (3,1)$, $A_{kl,0}, B_{kl,0}$ satisfy (61). From (61) it can be seen that if $A_{kl,0}(0) = B_{kl,0}(0) = 0$, then for all $t_1 > 0$ $A_{kl,0}(t_1) = B_{kl,0}(t_1) \equiv 0$ for $(k,l) \neq (1,1)$, $(k,l) \neq (1,3)$, and $(k,l) \neq (3,1)$. So if we start with zero initial energy in the $k - l$th mode $((k,l) \neq (1,1)$, $(k,l) \neq (1,3)$, $(k,l) \neq (3,1))$, there will be no energy present up to $\mathcal{O}(\epsilon)$ on a timescale of order ϵ^{-1}. We say that the coupling between the modes with $(k,l) \neq (1,1), (k,l) \neq (1,3)$, and $(k,l) \neq (3,1)$ is of $\mathcal{O}(\epsilon)$. This means that modes with zero initial energy do not have to be taken into account (for $(k,l) \neq (1,1)$, $(k,l) \neq (1,3)$, and $(k,l) \neq (3,1)$). On the other hand there is an $\mathcal{O}(1)$ coupling in this case between the modes 1-1, 1-3 and 3-1. This means that an energy transfer occurs between these modes even if the modes 1-3 and 3-1 have zero initial energy. Truncation to one mode is not valid: all modes 1-1, 1-3 and 3-1 have to be taken into account. Now let us consider the equations (76) and let

$$\frac{48}{45d_{11}\pi^2} = a_1, \quad \frac{16}{45d_{13}\pi^2} = b_1.$$

Then, (76) becomes

$$
\begin{aligned}
\dot{A}_{11,0} &= \frac{\alpha}{2d_{11}} B_{11,0} + 2a_1(A_{11,0}(B_{13,0}+B_{31,0}) - B_{11,0}(A_{13,0}+A_{31,0})),\\
\dot{B}_{11,0} &= -\frac{\alpha}{2d_{11}} A_{11,0} - 2a_1(A_{11,0}(A_{13,0}+A_{31,0}) - B_{11,0}(B_{13,0}+B_{31,0})),\\
\dot{A}_{13,0} &= \frac{\alpha}{2d_{13}} B_{13,0} + 2b_1 A_{11,0}B_{11,0},\\
\dot{B}_{13,0} &= -\frac{\alpha}{2d_{13}} A_{13,0} - b(A_{11,0}^2 - B_{11,0}^2),\\
\dot{A}_{31,0} &= \frac{\alpha}{2d_{31}} B_{31,0} + 2b_1 A_{11,0}B_{11,0},\\
\dot{B}_{31,0} &= -\frac{\alpha}{2d_{31}} A_{31,0} - b_1(A_{11,0}^2 - B_{11,0}^2).
\end{aligned}
\tag{77}
$$

First let us rescale the equations (77) by introducing $A_{11,0} = \frac{1}{\sqrt{a_1 b_1}} x_{11}$, $B_{11,0} = \frac{1}{\sqrt{a_1 b_1}} y_{11}$, $A_{13,0} = \frac{1}{a_1} x_{13}$,$B_{13,0} = \frac{1}{a_1} y_{13}$, $A_{31,0} = \frac{1}{a_1} x_{31}$, and $B_{31,0} = \frac{1}{a_1} y_{31}$. Then system (77) becomes:

$$\dot{x}_{11} = \frac{\alpha}{d_{11}} y_{11} + x_{11}(y_{13}+y_{31}) - y_{11}(x_{13}+x_{31}), \tag{78}$$

$$\dot{y}_{11} = -\frac{\alpha}{d_{11}} x_{11} - x_{11}(x_{13}+x_{31}) - y_{11}(y_{13}+y_{31}), \tag{79}$$

$$\dot{x}_{13} = \frac{\alpha}{d_{13}} y_{13} + 2x_{11}y_{11}, \tag{80}$$

$$\dot{y}_{13} = -\frac{\alpha}{d_{13}} x_{13} - (x_{11}^2 - y_{11}^2), \tag{81}$$

$$\dot{x}_{31} = \frac{\alpha}{d_{31}} y_{31} + 2x_{11}y_{11}, \tag{82}$$

$$\dot{y}_{31} = -\frac{\alpha}{d_{31}} x_{31} - (x_{11}^2 - y_{11}^2). \tag{83}$$

From the equations (80)-(83) the following two integrals can easily be obtained (observing that $d_{13} = d_{31}$) in the following way: if equation (80) is subtracted from equation (82), and equation (81) is subtracted from equation (83), then the following sub-system is obtained

$$
\begin{aligned}
\dot{x}_{31} - \dot{x}_{13} &= \frac{\alpha}{2d_{13}}(y_{31} - y_{13}),\\
\dot{y}_{31} - \dot{y}_{13} &= \frac{-\alpha}{2d_{13}}(x_{31} - x_{13}).
\end{aligned}
$$

The solution of this sub-system is given by

$$x_{13} = x_{31} + a\sin(\gamma t_1 + b), \quad y_{13} = y_{31} + a\cos(\gamma t_1 + b), \tag{84}$$

where a, b are constants of integration, and $\gamma = \frac{\alpha}{2d_{13}}$. Substituting (84) into system (78)-(83) then yields

$$\dot{x}_{11} = \frac{\alpha}{2d_{11}} y_{11} + x_{11}(2y_{31} + a\cos(\gamma t_1 + b)) - y_{11}(2x_{31} + a\sin(\gamma t_1 + b)), \tag{85}$$

$$\dot{y}_{11} = -\frac{\alpha}{2d_{11}} x_{11} - x_{11}(2x_{31} + a\sin(\gamma t_1 + b)) - y_{11}(2y_{31} + a\cos(\gamma t_1 + b)), \tag{86}$$

$$\dot{x}_{31} = \gamma y_{31} + 2x_{11}y_{11}, \tag{87}$$

$$\dot{y}_{31} = -\gamma x_{31} - (x_{11}^2 - y_{11}^2). \tag{88}$$

Now let's introduce the following change of variables

$$x_{31}(t_1) = a_{31}(t_1) - \frac{a}{2}\sin(\gamma t_1 + b), \quad y_{31}(t_1) = b_{31}(t_1) - \frac{a}{2}\cos(\gamma t_1 + b).$$

Then,

$$\dot{x}_{31} = \dot{a}_{31} - \frac{a}{2}\gamma\cos(\gamma t_1 + b) = \gamma b_{31} - \frac{a}{2}\gamma\cos(\gamma t_1 + b) + 2x_{11}y_{11},$$
$$\dot{y}_{31} = \dot{b}_{31} + \frac{a}{2}\gamma\sin(\gamma t_1 + b) = -\gamma a_{31} + \frac{a}{2}\gamma\sin(\gamma t_1 + b) + (x_{11}^2 - y_{11}^2)$$

and system (85)-(88) can be rewritten in the form

$$\dot{x}_{11} = \frac{\alpha}{2d_{11}}y_{11} + 2x_{11}b_{13} - 2y_{11}a_{13}, \tag{89}$$
$$\dot{y}_{11} = -\frac{\alpha}{2d_{11}}x_{11} - 2y_{11}b_{13} - 2x_{11}a_{13}, \tag{90}$$
$$\dot{a}_{31} = \gamma b_{31} + 2x_{11}y_{11}, \tag{91}$$
$$\dot{b}_{31} = -\gamma a_{31} - (x_{11}^2 - y_{11}^2). \tag{92}$$

Now observe that $\frac{1}{2}(x_{11}^2 + y_{11}^2 + 2a_{31}^2 + 2b_{31}^2)^{\cdot} = 0$, so

$$x_{11}^2 + y_{11}^2 + 2a_{31}^2 + 2b_{31}^2 = c, \tag{93}$$

where c is a constant of integration. By introducing the polar coordinates

$$x_{11} = r_1\cos(\phi_1), \quad y_{11} = r_1\sin(\phi_1), \quad a_{31} = r_3\cos(\phi_3), \quad b_{31} = r_3\sin(\phi_3)$$

system (89)-(92) becomes

$$\dot{r}_1 = 2r_1r_3\sin(\phi_3 - 2\phi_1), \tag{95}$$
$$\dot{\phi}_1 = -\gamma_1 - 2r_3\cos(\phi_3 - 2\phi_1), \tag{96}$$
$$\dot{r}_3 = \frac{r_1^2}{2}\sin(\phi_3 - 2\phi_1), \tag{97}$$
$$\dot{\phi}_3 = -\gamma - \frac{r_1^2}{r_3}\cos(\phi_3 - 2\phi_1), \tag{98}$$

where $\gamma_1 = \frac{\alpha}{2d_{11}}$. The first integral (93) can now be rewritten as $r_1^2 + r_3^2 = c$, and if we introduce the new variable

$$\psi = \phi_{13} - 2\phi_{11}$$

system (95)-(98) becomes (for $r_3 \neq 0$),

$$\dot{r}_3 = (2r_3^2 - c)\sin\psi, \tag{100}$$
$$\dot{\psi} = \gamma - 2\gamma_1 + \frac{(6r_3^2 - c)}{r_3}\cos\psi. \tag{101}$$

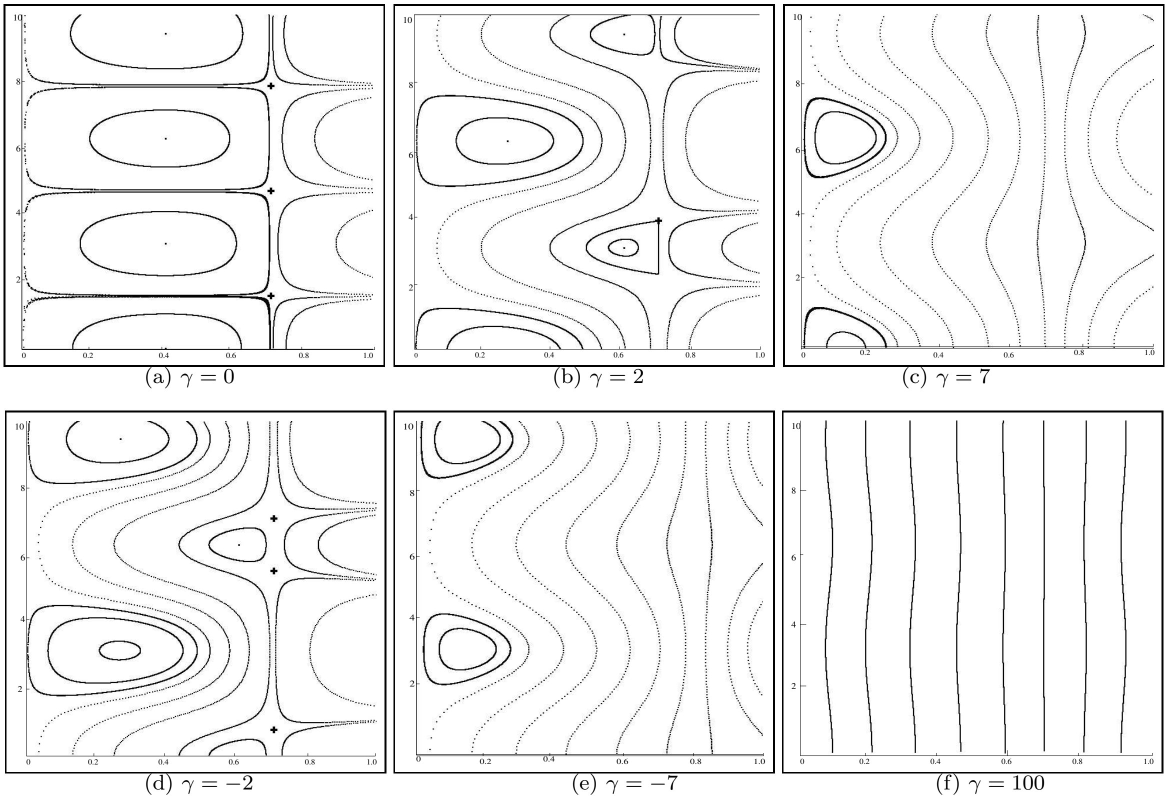
By introducing the rescaling

$$r_3 = \sqrt{c}r, \quad t_1 = \frac{1}{\sqrt{c}}\bar{t},$$

system (100)-(101) can be written as follows

$$\dot{r} = (2r^2 - 1)\sin\psi, \tag{103}$$
$$\dot{\psi} = \bar{\gamma} + \frac{(6r^2 - 1)}{r}\cos\psi, \tag{104}$$

where $\bar{\gamma} = \gamma - 2\gamma_1$. For $r = 0$ (103)-(104) do not hold. In this case the original differential equations (77) have to be analyzed. The critical points of (103)- (104) can be determined analytically. The system is 2π-periodic in ψ. Four critical points are found for $0 \leq \psi \leq 2\pi$ and $\bar{\gamma} = 0$: $\left(\sqrt{\frac{1}{6}}, 0\right)$, $\left(\sqrt{\frac{1}{6}}, \pi\right)$, $\left(\sqrt{\frac{1}{2}}, \frac{\pi}{2}\right)$, and $\left(\sqrt{\frac{1}{2}}, \frac{3\pi}{2}\right)$. The behavior of the solutions of (103)-(104) in the (r_3, ψ) phase space is given in Figure 4. In the exceptional case, when r becomes 0, the original differential equations (77) impose a phasejump from $\psi = 0$ to $\psi = \frac{\pi}{2}$, as can be seen in Figure 4. As in section 5.1 we also can determine the periods of the orbits in Figure 4(a). For $\gamma \neq 0$ a completely similar analysis can be given. The results in the (r, ψ) phase space are given in Figure 4. Again it can be seen in Figure 4 that for large values of the detuning parameter α the function r is (almost) constant resembling the behavior for a "noncritical" value for p^2.



(a) $\gamma = 0$ (b) $\gamma = 2$ (c) $\gamma = 7$

(d) $\gamma = -2$ (e) $\gamma = -7$ (f) $\gamma = 100$

Figure 4: The phase space for $p^2 = 28 + \epsilon\alpha$ with $\gamma = \alpha\left(\frac{1}{2\sqrt{100+p_{cr}^2}} - \frac{2}{\sqrt{4+p_{cr}^2}}\right)$, with r (horizontal) from 0 to 1, and ψ (vertical) from 0 to 3π.

5.3 The case $p^2 = p_{cr}^2 + \epsilon\alpha$ with $p_{cr}^2 \approx 35.40$.

Finally let us consider a more difficult case with interactions between the following four modes: 1-9, 9-1, 2-2 and 6-6. In this case $p_{cr}^2 \approx 35.40$ and α an $\mathcal{O}(1)$ detuning parameter. The extra terms in the equations (48) for $A_{kl,0}, B_{kl,0}$ occur for $k = l = 2$; $k = l = 6$; $k = 1, l = 9$, and $k = 9, l = 1$. The equations for $A_{22}, B_{22}, A_{66}, B_{66}, A_{19}, B_{19}$ and A_{91}, B_{91} are given by (as follows from (48)):

$$2d_{91}\frac{dA_{91,0}}{dt_1} - \alpha B_{91} + \frac{32}{153\pi^2}(A_{22,0}B_{66,0} + A_{66,0}B_{22,0}) = 0,$$
$$-2d_{91}\frac{dB_{91,0}}{dt_1} + \alpha A_{91} + \frac{32}{153\pi^2}(A_{22,0}A_{66,0} + B_{66,0}B_{22,0}) = 0,$$
$$2d_{19}\frac{dA_{19,0}}{dt_1} - \alpha B_{19} + \frac{32}{153\pi^2}(A_{22,0}B_{66,0} + A_{66,0}B_{22,0}) = 0,$$

$$-2d_{19}\frac{dB_{19,0}}{dt_1} + \alpha A_{19} + \frac{32}{153\pi^2}(A_{22,0}A_{66,0} + B_{66,0}B_{22,0}) = 0, \tag{105}$$
$$2d_{66}\frac{dA_{66,0}}{dt_1} - \alpha B_{66} + \frac{158}{1071\pi^2}(A_{22,0}B_{91,0} - A_{91,0}B_{22,0}) + \frac{158}{1071\pi^2}(A_{22,0}B_{19,0} - A_{19,0}B_{22,0}) = 0,$$
$$-2d_{66}\frac{dB_{66,0}}{dt_1} + \alpha A_{66} + \frac{158}{1071\pi^2}(A_{22,0}A_{91,0} + B_{91,0}B_{22,0}) + \frac{158}{1071\pi^2}(A_{22,0}A_{19,0} + B_{19,0}B_{22,0}) = 0,$$
$$2d_{22}\frac{dA_{22,0}}{dt_1} - \alpha B_{22} + \frac{154}{1275\pi^2}(A_{91,0}B_{66,0} - A_{66,0}B_{91,0}) + \frac{154}{1275\pi^2}(A_{19,0}B_{66,0} - A_{66,0}B_{19,0}) = 0,$$
$$-2d_{22}\frac{dB_{22,0}}{dt_1} + \alpha A_{22} + \frac{154}{1275\pi^2}(A_{66,0}A_{91,0} + B_{66,0}B_{22,0}) + \frac{154}{1275\pi^2}(A_{66,0}A_{19,0} + B_{66,0}B_{22,0}) = 0.$$

For $(k,l) \neq (2,2)$, $(k,l) \neq (1,9)$, $(k,l) \neq (9,1)$, and $(k,l) \neq (6,6)$ $A_{kl,0}, B_{kl,0}$ satisfy (61). Observing that $d_{91} = d_{19}$, and let

$$a_1 = \frac{-16}{153 d_{91}\pi^2}, \quad a_2 = \frac{-79}{1071 d_{66}\pi^2}, \quad a_3 = \frac{-77}{1275 d_{221}\pi^2}$$

and then (105) becomes

$$\begin{aligned}
\dot{A}_{91,0} &= \frac{\alpha}{2d_{91}} + a_1(A_{22,0}B_{66,0} + A_{66,0}B_{22,0}),\\
\dot{B}_{91,0} &= -\frac{\alpha}{2d_{91}} - a_1(A_{22,0}A_{66,0} + B_{66,0}B_{22,0}),\\
\dot{A}_{19,0} &= \frac{\alpha}{2d_{19}} + a_1(A_{22,0}B_{66,0} + A_{66,0}B_{22,0}),\\
\dot{B}_{19,0} &= -\frac{\alpha}{2d_{19}} - a_1(A_{22,0}A_{66,0} + B_{66,0}B_{22,0}),\\
\dot{A}_{66,0} &= \frac{\alpha}{2d_{66}} + a_2(A_{22,0}(B_{19,0} + B_{91,0}) - B_{22,0}(A_{19,0} + A_{91,0})),\\
\dot{B}_{66,0} &= -\frac{\alpha}{2d_{66}} - a_2(A_{22,0}(A_{19,0} + A_{91,0}) + B_{22,0}(B_{19,0} + B_{91,0})),\\
\\
\dot{A}_{22,0} &= \frac{\alpha}{2d_{22}} - a_3(A_{66,0}(B_{19,0} + B_{91,0}) - B_{66,0}(A_{19,0} + A_{91,0})),\\
\dot{B}_{22,0} &= -\frac{\alpha}{2d_{22}} + a_3(A_{66,0}(A_{19,0} + A_{91,0}) + B_{66,0}(B_{19,0} + B_{91,0})),
\end{aligned} \tag{106}$$

where the dot represents differentiation with respect to t_1. Then we introduce the following rescalings in (106)

$$A_{91,0} = \frac{1}{\sqrt{a_2a_3}}x_{91}, \quad B_{91,0} = \frac{1}{\sqrt{a_2a_3}}y_{91}, \quad A_{19,0} = \frac{1}{\sqrt{a_2a_3}}x_{19}, \quad B_{19,0} = \frac{1}{\sqrt{a_2a_3}}y_{19},$$
$$A_{66,0} = \frac{1}{\sqrt{a_1a_3}}x_{66}, \quad B_{66,0} = \frac{1}{\sqrt{a_1a_3}}y_{661}, \quad A_{22,0} = \frac{1}{\sqrt{a_1a_2}}x_{22}, \quad B_{22,0} = \frac{1}{\sqrt{a_1a_2}}y_{22},$$

to obtain

$$\dot{x}_{91} = \frac{\alpha}{2d_{91}}y_{91} + x_{22}y_{66} + x_{66}y_{22}, \tag{107}$$
$$\dot{y}_{91} = -\frac{\alpha}{2d_{91}}x_{91} + y_{22}y_{66} - x_{66}x_{22}, \tag{108}$$
$$\dot{x}_{19} = \frac{\alpha}{2d_{19}}y_{19} + x_{22}y_{66} + x_{66}y_{22}, \tag{109}$$
$$\dot{y}_{19} = -\frac{\alpha}{2d_{19}}x_{19} + y_{22}y_{66} - x_{66}x_{22}, \tag{110}$$

$$\dot{x}_{66} = \frac{\alpha}{2d_{66}} y_{66} + x_{22}(y_{19} + y_{91}) - y_{22}(x_{19} + x_{91}), \tag{111}$$

$$\dot{y}_{66} = -\frac{\alpha}{2d_{66}} x_{66} - x_{22}(x_{19} + x_{91}) - y_{22}(y_{19} + y_{91}), \tag{112}$$

$$\dot{x}_{22} = \frac{\alpha}{2d_{22}} y_{22} - x_{66}(y_{19} + y_{91}) + y_{66}(x_{19} + x_{91}), \tag{113}$$

$$\dot{y}_{22} = -\frac{\alpha}{2d_{22}} x_{22} + x_{66}(x_{19} + x_{91}) + y_{66}(y_{19} + y_{91}). \tag{114}$$

Substracting equation (109) from (107) and equation (110) from (108), the following system is obtained

$$\dot{x}_{91} - \dot{x}_{19} = \frac{\alpha}{2d_{19}}(y_{91} - y_{19}),$$
$$\dot{y}_{91} - \dot{y}_{19} = -\frac{\alpha}{2d_{19}}(x_{91} - x_{19}).$$

Solving this so-obtained system two integrals for system (107)-(114) can be obtained

$$x_{91} = x_{19} + a\sin(\gamma t_1 + b), \quad y_{91} = y_{19} + a\cos(\gamma t_1 + b), \tag{115}$$

where a, b are constants of integration. After substituting (115) into system (107)-(114) we obtain the following system of ODEs:

$$\dot{x}_{19} = \gamma_9 y_{19} + x_{22}y_{66} + x_{66}y_{22},$$

$$\begin{aligned}
\dot{y}_{19} &= -\gamma_9 x_{19} + y_{22}y_{66} + x_{66}x_{22}, \\
\dot{x}_{66} &= \frac{\alpha}{2d_{66}} y_{66} + x_{22}(2y_{19} + a\cos(\gamma_9 t_1 + b)) - y_{22}(2x_{19} + a\sin(\gamma_9 t_1 + b)), \\
\dot{y}_{66} &= -\frac{\alpha}{2d_{66}} x_{66} - x_{22}(2x_{19} + a\sin(\gamma_9 t_1 + b)) - y_{22}(2y_{19} + a\cos(\gamma_9 t_1 + b)), \\
\dot{x}_{22} &= \frac{\alpha}{2d_{22}} y_{22} - x_{66}(2y_{19} + a\cos(\gamma_9 t_1 + b)) + y_{66}(2x_{19} + a\sin(\gamma_9 t_1 + b)), \\
\dot{y}_{22} &= -\frac{\alpha}{2d_{22}} x_{22} + x_{66}(2x_{19} + a\sin(\gamma_9 t_1 + b)) + y_{66}(2y_{19} + a\cos(\gamma_9 t_1 + b)),
\end{aligned} \tag{116}$$

where $\gamma_9 = \frac{\alpha}{2d_{19}}$. If the following transformation is used

$$x_{19} = a_{19} - \frac{a}{2}\sin(\gamma_9 t_1 + b), \quad y_{19} = b_{19} - \frac{a}{2}\cos(\gamma_9 t_1 + b),$$

then

$$\dot{x}_{19} = \dot{a}_{19} - \frac{a}{2}\gamma_9\cos(\gamma_9 t_1 + b) = \gamma_9 b_{19} - \frac{a}{2}\gamma_9\cos(\gamma_9 t_1 + b) + x_{22}y_{66} + x_{66}y_{22},$$
$$\dot{y}_{19} = \dot{b}_{19} + \frac{a}{2}\gamma_9\sin(\gamma_9 t_1 + b) = -\gamma_9 a_{19} + \frac{a}{2}\gamma_9\sin(\gamma_9 t_1 + b) + y_{22}y_{66} - x_{66}x_{22}.$$

Using this transformation system (116) can be rewritten in the following autonomous form:

$$\begin{aligned}
\dot{a}_{19} &= \gamma_9 b_{19} + x_{22}y_{66} + x_{66}y_{22}, \\
\dot{b}_{19} &= -\gamma_9 a_{19} + y_{22}y_{66} - x_{66}x_{22}, \\
\dot{x}_{66} &= \frac{\alpha}{2d_{66}} y_{66} + x_{22}b_{19} - y_{22}a_{19}, \\
\dot{y}_{66} &= -\frac{\alpha}{2d_{66}} x_{66} - x_{22}a_{19} - y_{22}b_{19}, \\
\dot{x}_{22} &= \frac{\alpha}{2d_{22}} y_{22} - x_{66}b_{19} + y_{66}a_{19}, \\
\dot{y}_{22} &= -\frac{\alpha}{2d_{22}} x_{22} + x_{66}a_{19} + y_{66}b_{19}.
\end{aligned} \tag{117}$$

Introducing polar coordinates, that is, for $i = 2$ and 6

$$x_{ii} = r_i \cos\phi_i, \quad y_{ii} = r_i \sin\phi_i, \text{ and } a_{19} = r_9 \cos(\phi_9), \quad b_{19} = r_9 \sin(\phi_9),$$

system (117) becomes:

$$\dot{r}_9 = r_2 r_6 \sin(\phi_6 + \phi_2 - \phi_9), \tag{119}$$

$$\dot{\phi}_9 = -\gamma_9 - \frac{-r_2 r_6}{r_9} \cos(\phi_6 + \phi_2 - \phi_9), \tag{120}$$

$$\dot{r}_6 = -r_2 r_9 \sin(\phi_6 + \phi_2 - \phi_9), \tag{121}$$

$$\dot{\phi}_6 = -\frac{\alpha}{2d_{66}} - \frac{r_2 r_9}{r_6} \cos(\phi_6 + \phi_2 - \phi_9), \tag{122}$$

$$\dot{r}_2 = r_9 r_6 \sin(\phi_6 + \phi_2 - \phi_9), \tag{123}$$

$$\dot{\phi}_2 = -\frac{\alpha}{2d_{22}} + \frac{r_9 r_6}{r_2} \cos(\phi_6 + \phi_2 - \phi_9). \tag{124}$$

If we multiply equation (119) with r_9, equation (121) with r_6 and then add, we obtain the following first integral

$$r_6^2 + r_9^2 = c_1. \tag{125}$$

Another first integral can be obtained when we multiply equation (121) with r_6, equation (123) with r_2 and then add, yielding

$$r_6^2 + r_2^2 = c_2. \tag{126}$$

Let us introduce

$$\psi = \phi_6 + \phi_2 - \phi_9. \tag{127}$$

Using the two first integrals (125), (126), and (127) it follows that system (119)-(124) can be reduced to a system of two differential equations:

$$\dot{r}_2 = \sqrt{c_2 - r_2^2}\sqrt{c_1 - c_2 + r_2^2} \sin\psi, \tag{128}$$

$$\dot{\psi} = \gamma_1 + \frac{-3r_2^4 + 4c_2 r_2^2 - 2c_1 r_2^2 + c_2 c_1 - c_2^2}{r_2\sqrt{c_2 - r_2^2}\sqrt{c_1 - c_2 + r_2^2}} \cos\psi, \tag{129}$$

where $\gamma_1 = \gamma_9 - \frac{\alpha}{2d_{66}} - \frac{\alpha}{2d_{22}}$. Finally we introduce the following rescalings in (128)-(129)

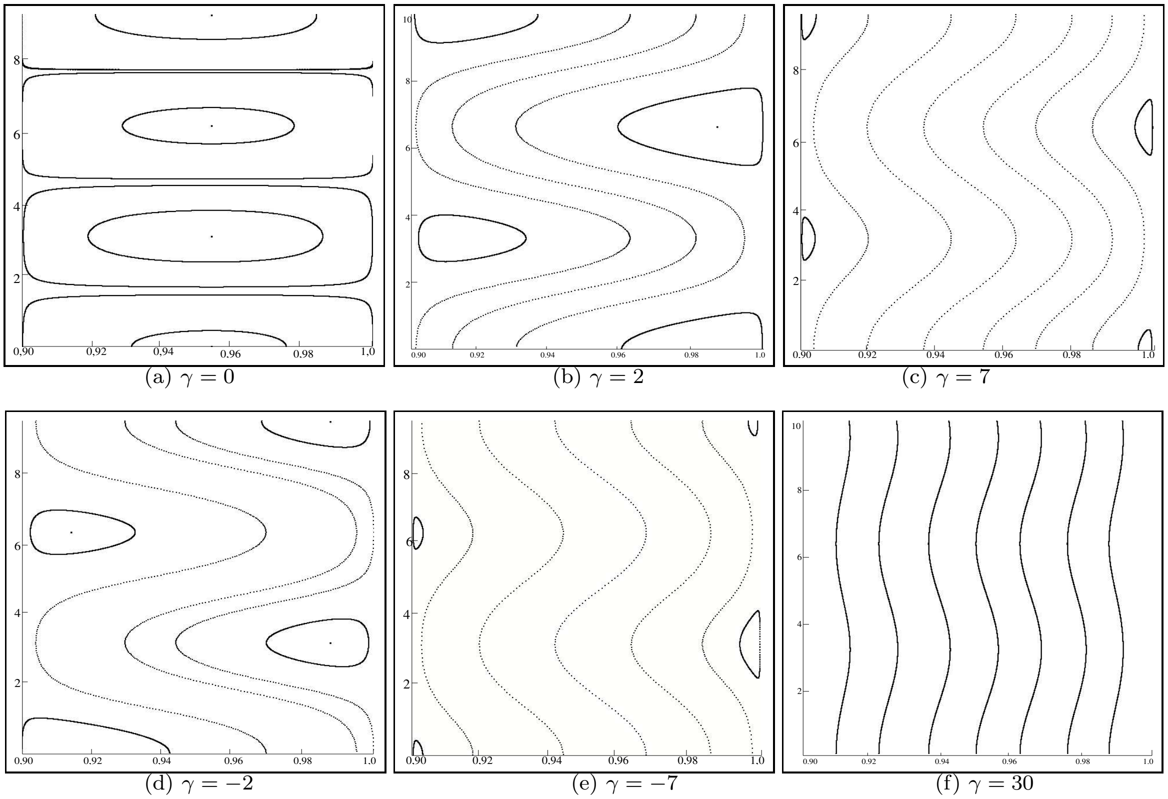
$$r_2 = \sqrt{c_2}\, r, \quad t = \sqrt{c_2}\,\tau, \quad \frac{c_1}{c_2} = c_3,$$

to obtain

$$\dot{r} = \sqrt{1 - r^2}\sqrt{c_3 - 1 + r^2} \sin\psi, \tag{131}$$

$$\dot{\psi} = \gamma + \frac{-3r^4 + 4r^2 - 2c_3 r^2 + c_3 - 1}{r\sqrt{1 - r^2}\sqrt{c_3 - 1 + r^2}} \cos\psi, \tag{132}$$

where $\gamma = \frac{\gamma_1}{\sqrt{c_2}}$. For $r = 0, r = 1$ and $r = \sqrt{1 - c_3}$ (131)-(132) do not hold. In those cases we have to analyze the original differential equations (106). It should also be observed from (131)-(132) that $1 - c_3 \le r^2 \le 1$. From the analysis of these original differential equations it follows for $\gamma = 0$ that the points $\left(1, \frac{\pi}{2}\right)$, $\left(1, \frac{3\pi}{2}\right)$, $\left(\sqrt{1 - c_3}, \frac{\pi}{2}\right)$, and $\left(\sqrt{1 - c_3}, \frac{3\pi}{2}\right)$ are saddles in the phase space as can be seen from Figure 5 (a). These saddles disappear by increasing the detuning parameter α. We will determine the critical points of (131)-(132) analytically. For convenience we take $c_3 = 0.19$, which implies that $0.9 \le r \le 1$. The figures in the phase space are essentially the same for $c_3 \ne 0.19$. The system is 2π-periodic in ψ. The following critical points are found: $\left(\frac{1}{3}\sqrt{6 - 3c_3 + 3C}, 0\right)$, $\left(\frac{1}{3}\sqrt{6 - 3c_3 + 3C}, \pi\right)$, $\left(\frac{1}{3}\sqrt{6 - 3c_3 - 3C}, 0\right)$, $\left(\frac{1}{3}\sqrt{6 - 3c_3 + 3C}, \pi\right)$, where $C = sqrt1 - c_3 + c_3^2$. The behavior of the solutions of (131)-(132) in the (r, ψ) phase space is given in Figure 5.



(a) $\gamma = 0$ (b) $\gamma = 2$ (c) $\gamma = 7$

(d) $\gamma = -2$ (e) $\gamma = -7$ (f) $\gamma = 30$

Figure 5: The phase space for $p^2 = 35.40 + \epsilon\alpha$ with $\gamma = \alpha \left(\frac{1}{2\sqrt{6561+p_{cr}^2}} - \frac{1}{2\sqrt{64+p_{cr}^2}} - \frac{1}{\sqrt{5184+p_{cr}^2}} \right)$, and $\sqrt{1-c_3} \leq r \leq 1$ and $c_3 = 0.19$; r (horizontal) from 0.9 to 1, and ψ (vertical) from 0 to 3π.

6 Conclusions.

In this paper an initial-boundary value problem for a weakly nonlinear plate equation has been considered. Order ϵ asymptotic approximations of the solution have been constructed and interactions between different oscillation modes have been considered. A theory has been presented which shows that the constructed approximations are asymptotically valid on time-scales of order ϵ^{-1}. For most p^2-values it has been shown that no mode interactions occur between different modes up to $\mathcal{O}(\epsilon)$ on time-scales of order ϵ^{-1}, which means that there is no energy transfer between different modes up to $\mathcal{O}(\epsilon)$ on time-scales of order ϵ^{-1}. The coupling between these modes is said to be of order ϵ, and truncation is allowed to those modes that have nonzero initial energy.

However, for some p^2-values interactions between different modes occur, which are caused by complicated internal resonances. Physically this means than in some cases (depending on the value of p^2, which depends on the elasticity-characteristics of the foundation and on certain properties of the plate), when the plate initially oscillates in a high vibration mode, lower vibration modes can be excited, and an energy transfer occurs between the different modes. In Table 1 all these critical p^2-values are given for $0 \leq p^2 \leq 50$. For three different critical values of p_{cr}^2, $p_{cr}^2 \approx \frac{68}{3}$, $p_{cr}^2 \approx 35.40$, $p_{cr}^2 \approx 28$, the equations have been studied in detail.

For $p_{cr}^2 \approx \frac{68}{3}$ it has been shown that an energy transfer occurs between the modes 2-2 and 3-3, even if mode 3-3 has zero initial energy. We call this a coupling between the modes of $\mathcal{O}(1)$. Truncation to one mode will give loss of information, and approximations will not be valid on time-scales of order ϵ^{-1}. Both modes have to be taken into account. Examining the behavior of the oscillations in this case, it follows that the system oscillates around an equilibrium state which is a combination of two modes (energy in both modes). The detuning analysis shows that the system gradually changes from a combined oscillation of two modes with non-constant amplitudes to an oscillation of two modes with constant amplitudes, as p^2 moves away form the critical value $\frac{68}{3}$ (as can be seen in Figure 3). All this holds up to $\mathcal{O}(\epsilon)$ on a timescale of order ϵ^{-1}. For $p_{cr}^2 \approx 28$ it has been shown that an energy transfer occurs between the modes 1-1, 1-3 and 3-1, even if modes 1-3 and 3-1 have zero initial energy. Truncation to one or two modes will give loss of information in this case, and an approximation will not be valid on time-scales of order ϵ^{-1}. All three modes have to be taken into account. For $p_{cr}^2 \approx 35.40$ it has been shown that an energy transfer occurs between the modes 1-9, 9-1, 2-2 and 6-6.

In this paper an asymptotic theory for an initital-boundary value problem for a weakly nonlinear plate equation has been presented. In section 4 and 5 of this paper formal approximations of the solutions have been constructed. These approximations satisfy the original PDE and the original initial values up to $\mathcal{O}(\epsilon^2)$. The asymptotic theory as presented in section 3 of this paper then implies that the approximations are asymptotically valid up to $\mathcal{O}(\epsilon)$ on time-scales of $\mathcal{O}(\epsilon^{-1})$. The theory (as presented in section 2 and 3 of this paper) can also easily be extended to problem for rectangular plates with more complicated nonlinearities in the PDE. It is also possible to extend the theory to problems with other (than simply supported) boundary conditions.

Appendix A. Green's function G and the integral equation.

In this appendix we construct the Green's function G for the linear operator $\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^2}{\partial t^2} + p^2$ and the simply supported boundary conditions. We will also derive the equivalent integral equation as given by (17). The Green's function $G(\xi, \eta, \tau; x, y, t)$ is defined to be the solution of the following problem:

$$G_{tt} + G_{xxxx} + 2G_{xxyy} + G_{yyyy} + p^2 G = \delta(x-\xi, y-\eta, t-\tau), \tag{133}$$
$$x, \xi, y, \eta \in]0, \pi[, \tau > 0,$$
$$G(\xi,\eta,\tau;0,y,t) = G(\xi,\eta,\tau;\pi,y,t) = G_{xx}(\xi,\eta,\tau;0,y,t) = G_{xx}(\xi,\eta,\tau;\pi,y,t) \equiv 0, \tag{134}$$
$$t > 0, \quad \tau > 0,$$
$$G(\xi,\eta,\tau;x,0,t) = G(\xi,\eta,\tau;x,\pi,t) = G_{yy}(\xi,\eta,\tau;x,0,t) = G_{yy}(\xi,\eta,\tau;x,\pi,t) \equiv 0, \tag{135}$$
$$t > 0, \quad \tau > 0,$$
$$G(\xi,\eta,\tau;x,y,t) \equiv 0, \quad \tau \geq t. \tag{136}$$

The boundary conditions imply that G can be written in a Fourier sine series in x and y:

$$G(\xi,\eta,\tau;x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} g_{mn}(\xi,\eta,\tau;t)\sin(nx)\sin(my).$$

Substituting this series into (133) and using the orthogonality properties of the sine functions, we obtain the following set of equations for g_{mn} (where a dot represents differentiation with respect to t):

$$\ddot{g}_{mn} + ((m^2+n^2)^2 + p^2)g_{mn} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} \delta(x-\xi, y-\eta, t-\tau)\sin(nx)\sin(my)dxdy,$$
$$0 < \xi < \pi, \quad 0 < \eta < \pi, \quad \tau > 0, \tag{137}$$
$$g_{mn}(\xi,\eta,\tau;0) = g_{mn}(\xi,\eta,\tau;\tau) \equiv 0, \quad 0 < \xi < \pi, \quad 0 < \eta < \pi, \tag{138}$$

for $m, n = 1, 2, 3....$ The equations (137)-(138) can be solved by using the method of "variation of constants", yielding

$$g_{mn}(\xi,\eta,\tau;t) = \frac{4}{\pi^2\sqrt{(n^2+m^2)^2+p^2}}\sin\left[\sqrt{(n^2+m^2)^2+p^2}(t-\tau)\right]H(t-\tau)\sin(n\xi)\sin(m\eta),$$

for $m, n = 1, 2, 3...$, and therefore

$$G(\xi,\eta,\tau;x,y,t) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\sqrt{(n^2+m^2)^2+p^2}}\sin\left[\sqrt{(n^2+m^2)^2+p^2}(t-\tau)\right] \tag{139}$$
$$\times H(t-\tau)\sin(n\xi)\sin(m\eta)\sin(nx)\sin(my).$$

We find the integral equation (17) by multiplying (133) with $w(\xi,\eta,\tau)$ and integrating over $0 \le \xi \le \pi, \quad 0 \le \eta \le \pi, \quad 0 \le \tau \le t$. By integrating by parts, and by using the boundary conditions (134)-(135) for G and the boundary conditions for w we finally obtain the equivalent integral equation

$$w(x,y,t) = \epsilon \int_0^t \int_0^\pi \int_0^\pi G(\xi,\eta,\tau;x,y,t) f(\xi,\eta,\tau,w(\xi,\eta,\tau);\epsilon) d\xi d\eta d\tau \tag{140}$$

$$+ \int_0^\pi \int_0^\pi \big\{ G(\xi,\eta,0;x,y,t) w_1(\xi,\eta;\epsilon) - G_\tau(\xi,\eta,0;x,y,t) w_0(\xi,\eta;\epsilon) \big\} d\xi d\eta = (Tw)(x,y,t).$$

Appendix B. Integral inequalities.

In this appendix two integral inequalities are derived. These inequalities play an important role in the asymptotic theory presented in this paper. Let $f_1(x,y)$ and its partial derivatives up to order four be continuous on $[0,\pi] \times [0,\pi]$ and let $f_2(x,y)$ and its partial derivatives up to second order be continuous on $[0,\pi] \times [0,\pi]$. The functions f_1 and f_2 also have to satisfy additional boundary conditions like: $f_1(0,y) = f_1(\pi,y) = f_2(0,y) = f_2(\pi,y) = \frac{\partial^2 f_1(0,y)}{\partial x^2} = \frac{\partial^2 f_1(\pi 0,y)}{\partial x^2} = \frac{\partial^2 f_2(0,y)}{\partial x^2} = \frac{\partial^2 f_2(\pi,y)}{\partial x^2} = 0$. We will show that for all $x,y,t \in \Omega_L$ (as defined in (20)) and for $p \ge 0$ the following inequalities hold:

$$\Big| \int_0^\pi \int_0^\pi G_\tau(\xi,\eta,0;x,y,t) f_1(\xi,\eta) d\xi d\eta \Big| \tag{141}$$

$$\le \pi^2 \max_{0 \le x,y \le \pi} \Big\{ p^2 |f_1(x,y)| + \Big|\frac{\partial^2 f_1(x,y)}{\partial x^2}\Big| + 2\Big|\frac{\partial^2 f_1(x,y)}{\partial x \partial y}\Big| + \Big|\frac{\partial^2 f_1(x,y)}{\partial y^2}\Big| \Big\}$$

$$\text{and} \quad \Big| \int_0^\pi \int_0^\pi G(\xi,\eta,0;x,y,t) f_2(\xi,\eta) d\xi d\eta \Big| \le \pi^2 \max_{0 \le x,y \le \pi} |f_2(x,y)|. \tag{142}$$

To prove these inequalities we consider the following linear initial-boundary value problem for a three-times differentiable function $w(x,y,t)$ with w_{xxxx}, w_{yyyy}, w_{xxyy} continuous:

$$w_{tt} + w_{xxxx} + 2w_{xxyy} + w_{yyyy} + p^2 w = 0, \quad 0 < x < \pi, \quad 0 < y < \pi, \quad t > 0, \tag{143}$$

$$w(0,y,t) = w(\pi,y,t) = w_{xx}(0,y,t) = w_{xx}(\pi,y,t) = 0, \quad t \ge 0, \tag{144}$$

$$w(x,0,t) = w(x,\pi,t) = w_{yy}(x,0,t) = w_{yy}(x,\pi,t) = 0, \quad t \ge 0, \tag{145}$$

$$w(x,y,0) = f_1(x,y), \quad w_t(x,y,0) = f_2(x,y), \quad 0 < x < \pi, \quad 0 < y < \pi. \tag{146}$$

Using Green's function G for the linear operator $\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} + \frac{\partial^2}{\partial t^2} + p^2$ and the simply supported boundary conditions it follows elementarily that the unique and three-times continuously differentiable solution of the initial-boundary value problem (143)-(146), which has continuous fourth order derivatives with respect to x,y is given by

$$w(x,y,t) = \int_0^\pi \int_0^\pi (G(\xi,\eta,0;x,y,t) f_2(\xi,\eta) - G_\tau(\xi,\eta,0;x,y,t) f_1(\xi,\eta)) d\xi d\eta. \tag{147}$$

To be able to estimate $|w(x,y,t)|$ and thus the integral given in (147) we will use the following energy equation related to the initial boundary value problem (143)-(146):

$$\int_0^\pi \int_0^\pi \{ w_{tt}^2(x,y,t_0) + w_{xx}^2(x,y,t_0) + 2w_{xy}^2(x,y,t_0) + w_{yy}^2 + p^2 w^2(x,y,t_0) \} dx dx =$$

$$\int_0^\pi \int_0^\pi \Big\{ f_2^2(x,y) + \Big(\frac{\partial^2 f_1(x,y)}{\partial x^2}\Big)^2 + \Big(\frac{\partial^2 f_1(x,y)}{\partial y^2}\Big)^2 + \Big(\frac{\partial^2 f_1(x,y)}{\partial x \partial y}\Big)^2 + p^2 f_1^2(x,y) \Big\} dx dy.$$

We obtain this energy equation by multiplying (143) with w_t and by integrating with respect to x, y and t over $0 \leq x \leq \pi, \quad 0 \leq y \leq \pi, \quad 0 \leq t \leq t_0$, using the initial and boundary conditions (144)-(146). On the other hand, we have

$$w(x,y,t) = \int_0^y \int_0^x w_{\xi\eta}(\xi,\eta,t) d\xi d\eta. \tag{148}$$

Using (148) and Hölder's inequality we now have

$$\begin{aligned}
|w(x,y,t)| &\leq \int_0^y \int_0^x |w_{\xi\eta}(\xi,\eta,t)| d\xi d\eta \leq \int_0^\pi \int_0^\pi |w_{xy}(x,y,t)| dxdy \\
&\leq \Big(\int_0^\pi \int_0^\pi 1^2 dxdy \Big)^{\frac{1}{2}} \Big(\int_0^\pi \int_0^\pi w_{xy}^2 dxdy \Big)^{\frac{1}{2}} \leq \pi \Big(\frac{1}{2} \int_0^\pi \int_0^\pi \{ w_t^2 + w_{xx}^2 + 2w_{xy}^2 + w_{yy}^2 + p^2 w^2 \} dxdy \Big)^{\frac{1}{2}} \\
&\leq \pi \Big(\frac{1}{2} \int_0^\pi \int_0^\pi \Big\{ f_2^2(x,y) + p^2 f_1^2(x,y) + \Big(\frac{\partial^2 f_1(x,y)}{\partial x^2} \Big)^2 + 2 \Big(\frac{\partial^2 f_1(x,y)}{\partial x \partial y} \Big)^2 + \\
&\qquad \Big(\frac{\partial^2 f_1(x,y)}{\partial y^2} \Big)^2 \Big\} dxdy \Big)^{\frac{1}{2}} \\
&\leq \pi \Big(\frac{1}{2} \int_0^\pi \int_0^\pi \max_{0 \leq x,y \leq \pi} \Big\{ f_2^2(x,y) + p^2 f_1^2(x,y) + \Big(\frac{\partial^2 f_1(x,y)}{\partial x^2} \Big)^2 + 2 \Big(\frac{\partial^2 f_1(x,y)}{\partial x \partial y} \Big)^2 + \\
&\qquad \Big(\frac{\partial^2 f_1(x,y)}{\partial y^2} \Big)^2 \Big\} dxdy \Big)^{\frac{1}{2}} \\
&\leq \frac{\pi^2}{\sqrt{2}} \max_{0 \leq x,y \leq \pi} \Big\{ f_2^2(x,y) + p^2 f_1^2(x,y) + \Big(\frac{\partial^2 f_1(x,y)}{\partial x^2} \Big)^2 + 2 \Big(\frac{\partial^2 f_1(x,y)}{\partial x \partial y} \Big)^2 + \Big(\frac{\partial^2 f_1(x,y)}{\partial y^2} \Big)^2 \Big\}^{\frac{1}{2}}.
\end{aligned} \tag{149}$$

Hence it follows from (147) and (149) that if $f_2(x,y) \equiv 0$, then

$$\Big| \int_0^\pi \int_0^\pi G_\tau(\xi,\eta,0;x,y,t) f_1(\xi,\eta) d\xi d\eta \Big|$$
$$\leq \frac{\pi^2}{\sqrt{2}} \max_{0 \leq x,y \leq \pi} \Big\{ p^2 |f_1(x,y)| + \Big| \frac{\partial^2 f_1(x,y)}{\partial x^2} \Big| + 2 \Big| \frac{\partial^2 f_1(x,y)}{\partial x \partial y} \Big| + \Big| \frac{\partial^2 f_1(x,y)}{\partial y^2} \Big| \Big\},$$

and similarly, if we take $f_1(x,y) \equiv 0$, then

$$\Big| \int_0^\pi \int_0^\pi G(\xi,\eta,0;x,y,t) f_2(\xi,\eta) d\xi d\eta \Big| \leq \frac{\pi^2}{\sqrt{2}} \max_{0 \leq x,y \leq \pi} |f_2(x,y)|.$$

In this way it has been shown that the integral inequalities (141) and (142) hold.

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