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Self-similar solutions for the foam drainage equation

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Abstract

The travelling wave solutions of the equation for foam drainage in porous media are developed taking into account the mass conservation criterion. The existence of traveling wave solutions is also discussed. Finally, numerical solutions are obtained using a finite difference scheme together with the Van Leer flux limiter, to reduce numerical dispersion. An excellent match is obtained between the analytical and the numerical solutions.

1 Introduction

When foam develops in strongly water wet medium, the gas bubbles occupy a large part of the volume while the liquid is contained in a network formed by bubble-pore wall and bubble-bubble interstices (i.e., Plateau borders, foam films, etc.). The network is connected throughout the pore space so that, when the liquid is subject to a force field (e.g. gravity force) it is set in flow through the network while the gas remains motionless from the macroscopic point of vue. This process is called foam drainage.

An simple model was recently developed for this foam drainage phenomenon in porous media taking into account the gravity and capillary effects (Zitha [1]). In terms of the reduced saturation s, it was found that the foam drainage process is described by the following general equation

$$\left\{ \begin{array}{c} \dfrac{\partial s}{\partial t} + \dfrac{\partial}{\partial x}\left[s^n - D\left(s\right)\dfrac{\partial s}{\partial x}\right] = 0 \text{ for } x \in \mathbb{R} \\ D\left(s\right) = \dfrac{\alpha}{m} s^{a(m,n)}, \\ a\left(m,n\right) = \dfrac{nm-(m+1)}{m}, \end{array} \right. \tag{1}$$

where n and m are exponents characteristic of the porous system and foam texture and α expresses the ratio of capillary and gravity forces. Several solutions of the above equation were developed considering several limiting cases, depending upon the parameter α, and various boundary conditions. The purpose of this letter is to elaborate on the travelling

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wave solutions for the equation (1) whereas

$$\alpha > 0 \text{ and } s(x,0) = \begin{cases} s_1, x < 0 \\ s_0, x > 0 \end{cases} \text{ with } s_1 > s_0, \tag{2}$$

since, unfortunately, the solutions obtained previously for this particular case (see Table 2 of Zitha [1]) contain errors and do not satisfy the mass conservation principle. Similarly to [1], we use the following change of variable:

$$s(x,t) = \overline{s}(\eta)\,;\eta = x - Vt, \tag{3}$$

where V denotes the speed of the wave, which is to be determined as part of the solution. From equation (1) we can therefore derive

$$-V\overline{s}' + \left[\overline{s}^n - D(\overline{s})\,\overline{s}'\right]' = 0, \tag{4}$$

which can be integrated immediately to

$$-V\overline{s} + \overline{s}^n - D(\overline{s})\,\overline{s}' = C, \tag{5}$$

where C is an integration constant. Since we are interested in bounded solutions, we require that $\overline{s}$ has horizontal asymptotes for $|\eta| \to \infty$. Hence, from the initial condition, we use $\overline{s}(-\infty) = s_1$ and $\overline{s}(+\infty) = s_0$. Further, we require the solution to be monotonical, which is reasonable since it satisfies the maximum principle. This implies that $\overline{s}' \to 0$ as $|\eta| \to +\infty$. Then, it follows from the initial conditions that the constants V and C are obtained from

$$V = \frac{s_1^n - s_0^n}{s_1 - s_0}, C = -s_0 s_1 \frac{s_1^{n-1} - s_0^{n-1}}{s_1 - s_0} \tag{6}$$

Considering (6) in (5) gives a non-linear first order differential equation for $\overline{s}$, which can be integrated by a separation of variables, giving

$$\int \frac{\overline{s}^{a(m,n)} d\overline{s}}{\overline{s}^n - V\overline{s} - C} + K = \frac{m}{\alpha}\eta = \frac{m}{\alpha}(x - Vt). \tag{7}$$

The evaluation of the integral in the above equation gives a relation between $\overline{s}$ and η, since C and V are defined by equation (6). It should be emphasized that K has not been determined and hence we have a family of self-similar solutions. It is also important to note that if $a(m,n) > 0$ the diffusion term is degenerate since it becomes identically zero for $\overline{s} = 0$.

2 Mass conservation

To determine the constant K an additional argument is needed. We locate the travelling wave so that its center of mass coincides with the location of the shock when $\alpha = 0$. This amounts to the following: the difference between the initial liquid fraction and the actual liquid fraction for $\eta < 0$ should balance the difference between the actual concentration and the initial concentration for $\eta > 0$. This can be expressed as

$$\int_{-\infty}^{0} (s_1 - \overline{s})\, d\overline{s} = \int_{0}^{+\infty} (\overline{s} - s_0)\, d\overline{s}. \tag{8}$$

As depicted in Fig. 1, this means physically that the loss of mass in the region $\eta < 0$ is equal to the gain of mass in region $\eta > 0$. This sets a requirement for K. From equation (7) the solution is determined for any value of K and subsequently by the use of equation (8), the constant K is defined. Since the evaluation of the integrals can be very elaborate, we choose to rewrite the condition (8)

$$\int_{s_0}^{s_1} \eta\,(\overline{s})\, d\overline{s} = 0. \tag{9}$$

Then from equation (7) an explicit relation follows between $\overline{s}$ and η. This can be written as

$$\begin{cases} K = -\dfrac{1}{s_1 - s_0} \displaystyle\int_{s_0}^{s_1} F\,(\overline{s})\, d\overline{s} \\ F\,(\overline{s}) = \displaystyle\int \dfrac{\overline{s}^{a(m,n)} d\overline{s}}{\overline{s}^n - V\overline{s} - C}, \end{cases} \tag{10}$$

where $F\,(\overline{s})$ is the anti-derivative of the expression under the integration sign. Hence the integration constant K is uniquely determined by the above equation. To illustrate the solution behaviour, we consider several cases with $m = 2$, $s_0 = 0$ and $s_1 = 1$. Then $V = 1$ and $C = 0$ and equations (7) and (10) reduce to

$$\begin{cases} F_n\,(\overline{s}) + K_n = \frac{m}{\alpha}\eta \\ K_n = -\displaystyle\int_{0}^{1} F\,(\overline{s})\, d\overline{s} \\ F_n\,(\overline{s}) = \displaystyle\int \dfrac{\overline{s}^{a(m,n)} d\overline{s}}{\overline{s}^n - \overline{s}}, \end{cases} \tag{11}$$

where the index n is introduced to distinguish the different cases to be treated below.

2.1 Analytical solutions

In this section we consider several cases. We give the travelling wave solutions for cases of non-degenerate and degenerate diffusion, where in the latter case $D(s) = 0$ for $s = 0$. Since the different types of the degeneracy gives rise to different qualitative solution behaviour, we present several cases of degenerate diffusion.

2.1.1 Non-degenerate diffusion

$n = 3/2, a(m,n) = 0$

This case implies a linear (non-degenerate) diffusion and hence the solution satisfies $0 < s < 1$, with strict inequalities. From equation (11) we have

$$F_1(\overline{s}) = \int \frac{d\overline{s}}{\overline{s}^{3/2} - \overline{s}} = 2\ln\left|\frac{1-\sqrt{\overline{s}}}{\overline{s}}\right|, \tag{12}$$

which is then used to determine K_1, giving

$$K_1 = -2\int_0^1 \ln\left|\frac{1-\sqrt{\overline{s}}}{\overline{s}}\right| ds = -2. \tag{13}$$

Hence, for this case the solution, reads as

$$\eta = \alpha\left[\ln\left(\frac{1-\sqrt{\overline{s}}}{\overline{s}}\right) - 1\right], \tag{14}$$

which is inverted to obtain $\overline{s}$ as an explicit function of η,

$$\overline{s} = \overline{s}(\eta) = \frac{1}{\left[1 + e^{\left(\frac{\eta-\alpha}{\alpha}\right)}\right]^2}. \tag{15}$$

This implies

$$s = s(x,t) = \frac{1}{\left[1 + e^{\left(\frac{x-t-\alpha}{\alpha}\right)}\right]^2}. \tag{16}$$

It can be seen that $\underset{\eta\to-\infty}{lim}\ \overline{s}(\eta) = 1$ and that $\underset{\eta\to+\infty}{lim}\ \overline{s}(\eta) = 0$ and that then the limits are asymptotes, which characterises linear diffusion.

2.1.2 Degenerate diffusion

As already mentioned, the next cases ($n = 2, 3, 4$) imply $a(m,n) > 0$, which in turn implies a degenerate diffusion since $D(s) = 0$ if $s = 0$. However, for $s = 1$ the diffusion is not degenerate and hence we expect that $s = 1$ represents an asymptotic value, i.e. the solution satisfies $0 \leq s < 1$. The solutions will now be derived for each case.

Case 1 (**$n = 2, a(m,n) = 1/2$**)

We have

$$F_2(\overline{s}) = \int \frac{\overline{s}^{1/2} d\overline{s}}{\overline{s}^2 - \overline{s}} = \ln\left|\frac{1-\sqrt{\overline{s}}}{1+\sqrt{\overline{s}}}\right|, \tag{17}$$

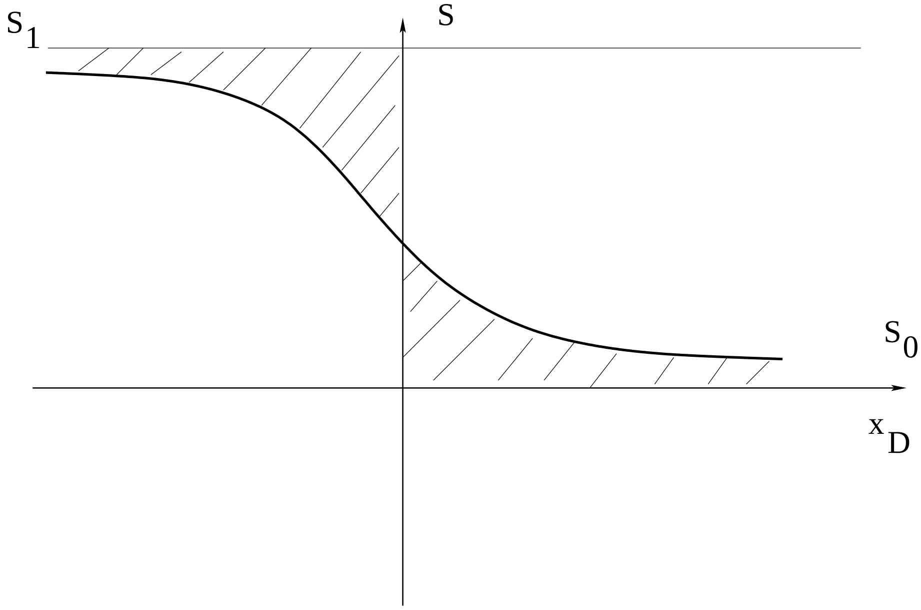

Figure 1: Mass conservation requires that the two hatched areas left and right of the shock position be equal.

which is used together with equation (11) to determine the constant K_2, giving

$$K_2 = -\int_0^1 \ln\left|\frac{1-\sqrt{\overline{s}}}{1+\sqrt{\overline{s}}}\right| ds = +2. \tag{18}$$

Hence the solution reads as

$$\eta = \frac{\alpha}{2}\left[\ln\left|\frac{1-\sqrt{\overline{s}}}{1+\sqrt{\overline{s}}}\right| + 2\right]. \tag{19}$$

From this expression we obtain the explicit solution for $\overline{s}$, which can be written

$$\overline{s} = \overline{s}\,(\eta) = \left[\frac{e^{\left(2\frac{\eta-\alpha}{\alpha}\right)}-1}{e^{\left(2\frac{\eta-\alpha}{\alpha}\right)}+1}\right]^2 = \tanh^2\left(\frac{\eta-\alpha}{\alpha}\right). \tag{20}$$

It can be seen, as depicted in Figure 2 (solid line), that the above function is even with respect to $\eta = \alpha$ and that

$$\lim_{|\eta|\to+\infty} \overline{s}\,(\eta) = 1. \tag{21}$$

which does not satisfy the boundary condition for $\eta \to +\infty$. We note, however, that $\overline{s} = 0$ is a solution of equation (4) for this particular choice of $n = 2$. This motivates the construction of the solution consisting of a part that is given by equation (20) and another part for which $\overline{s} = 0$. These two states are separated by the zero of $\overline{s}\,(\eta)$ in equation (20),

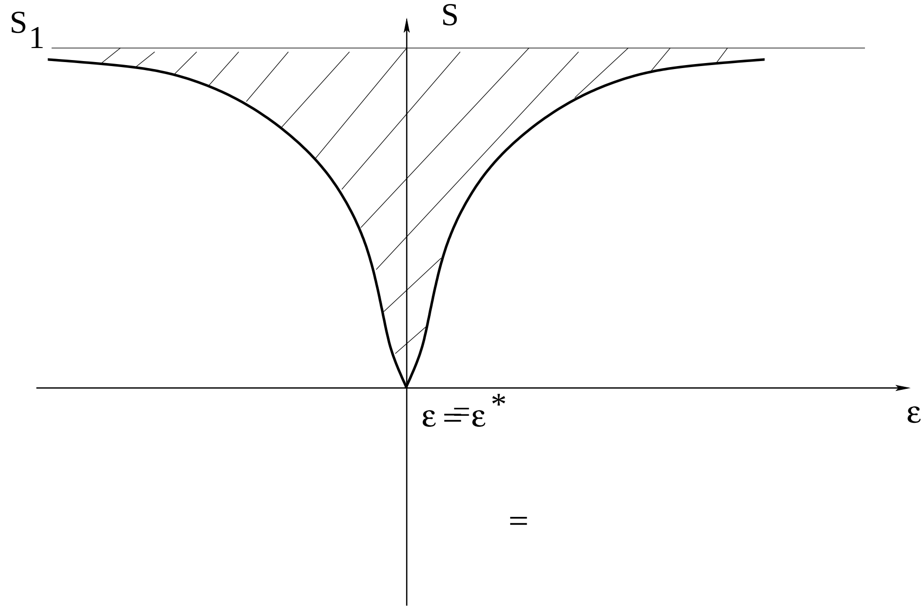

Figure 2: The solid line satisfies the mass conservation for case of degenerate diffusion.

which is given by

$$e^{\left(2\frac{\eta^*-\alpha}{\alpha}\right)} = 1 \Longleftrightarrow \eta^* = \alpha. \tag{22}$$

Hence the overall solution is given by

$$\overline{s}\left(\eta\right) = \begin{cases} \tanh^2\left(\frac{\eta-\alpha}{\alpha}\right) \text{ for } \eta \leq \alpha \\ 0 \text{ for } \eta > \alpha \end{cases}. \tag{23}$$

The solution in equation (23) satisfies both the differential equation (4) and the conditions for $|\eta| \to +\infty$. Further, it can be seen that the first derivative of (23) is continuous.

Case 2 *(*$n = 5/2, a(m,n) = 1$*)*

In this case, we have

$$F_3\left(\overline{s}\right) = \int \frac{\overline{s}d\overline{s}}{\overline{s}^{5/2} - \overline{s}} = \frac{1}{3}\ln\left|\frac{\sqrt{\overline{s}}-1}{\overline{s}+\overline{s}^{1/2}+1}\right| + 2\sqrt{3}\arctan\left[\frac{\sqrt{3}}{3}\left(2\sqrt{\overline{s}}+1\right)\right], \tag{24}$$

and from equation (11)

$$K_3 = -\int_0^1 \left\{\frac{1}{3}\ln\left|\frac{\sqrt{\overline{s}}-1}{\overline{s}+\overline{s}^{1/2}+1}\right| + 2\sqrt{3}\arctan\left[\frac{\sqrt{3}}{3}\left(2\sqrt{\overline{s}}+1\right)\right]\right\} d\overline{s} = 2 - \frac{\sqrt{3}}{3}\pi. \tag{25}$$

Hence the solution reads

$$\eta = \frac{\alpha}{2}\left\{\frac{1}{3}\ln\left|\frac{\sqrt{\overline{s}}-1}{\overline{s}+\overline{s}^{1/2}+1}\right| + 2\sqrt{3}\arctan\left[\frac{\sqrt{3}}{3}\left(2\sqrt{\overline{s}}+1\right)\right] + 2 - \frac{\sqrt{3}}{3}\pi\right\}. \tag{26}$$

From the above expression $\overline{s}$ can be solved by a zero point method for given η, or η can be evaluated as function of $\overline{s}$. Further it can be seen that $\overline{s}=0$ coincides with

$$\eta^* = \frac{\alpha}{6}\left[\pi\left(\frac{1}{6}-\frac{\sqrt{3}}{3}\right)+2\right] \Rightarrow \overline{s}\left(\eta^*\right)=0. \tag{27}$$

We recall that $\overline{s}=0$ is a solution of the differential equation (4). We also note that since

$$\frac{\overline{s}}{\overline{s}^{5/2}-\overline{s}} < 0$$

for $\overline{s} < 1$, the solution is monotonically decreasing for $\overline{s} < 1$. The solution can be constructed by setting $\overline{s}\left(\eta^*\right) = 0$ for $\eta > \eta^*$, to fulfill the condition for $\eta \to \infty$. Since the derivative of $\overline{s}$ with respect to η is discontinuous on η^*, we are faced with a free boundary, which travels with speed $V=1$. The equation of motion becomes, using equation (27)

$$x^* = t + \frac{\alpha}{6}\left[\pi\left(\frac{1}{6}-\frac{\sqrt{3}}{3}\right)+2\right] \text{ with } s(x^*,t)=0. \tag{28}$$

The overal solution can then be expressed by

$$\begin{cases} \eta = \frac{\alpha}{2}\left\{\frac{1}{3}\ln\left|\frac{\sqrt{\overline{s}}-1}{\overline{s}+\overline{s}^{1/2}+1}\right| + 2\sqrt{3}\arctan\left[\frac{\sqrt{3}}{3}\left(2\sqrt{\overline{s}}+1\right)\right] + 2 - \frac{\sqrt{3}}{3}\pi\right\} \text{ for } \eta \leq \eta^* \\ s = 0 \text{ for } \eta > \eta^* \end{cases}. \tag{29}$$

Case 3 *($n=3, a(m,n)=3/2$)*

The argument for this case is in all points similar to the previous case. We have

$$F_4\left(\overline{s}\right) = \int \frac{\overline{s}^{3/2}d\overline{s}}{\overline{s}^3-\overline{s}} = \frac{1}{2}\ln\left|\frac{\sqrt{\overline{s}}-1}{\sqrt{\overline{s}}+1}\right| + \arctan\left(\sqrt{\overline{s}}\right), \tag{30}$$

from we obtain the constant K_4

$$K_4 = -\int_0^1 \left[\frac{1}{2}\ln\left|\frac{\sqrt{\overline{s}}-1}{\sqrt{\overline{s}}+1}\right| + \arctan\left(\sqrt{\overline{s}}\right)\right] d\overline{s} = 2 - \frac{\pi}{2}. \tag{31}$$

Therefore the solution reads

$$\eta = \frac{\alpha}{2}\left[\frac{1}{2}\ln\left|\frac{\sqrt{\overline{s}}-1}{\sqrt{\overline{s}}+1}\right| + \arctan\left(\sqrt{\overline{s}}\right) + 2 - \frac{\pi}{2}\right]. \tag{32}$$

Like in the previous case, $\overline{s}$ can be solved by a zero point method for given η from equation (32) or η can be evaluated as function of $\overline{s}$. Further it can be seen that $\overline{s} = 0$ coincides with

$$\eta^* = \frac{\alpha}{2}\left(2 - \frac{\pi}{2}\right) \Leftrightarrow \overline{s}\left(\eta^*\right) = 0. \tag{33}$$

Again it can be shown that the solution $\overline{s}(\eta)$ is monotonical and therefore the solution is obtained by construction by setting $\overline{s}(\eta^*) = 0$ for $\eta > \eta^*$, to fulfill the condition for $\eta \to \infty$. Since $d\overline{s}/d\eta$ is discontinuous on η^*, we have a free boundary with the following equation of motion, using equation (27)

$$x^* = t + \frac{\alpha}{2}\left(2 - \frac{\pi}{2}\right) \text{ with } s(x^*, t) = 0. \tag{34}$$

The overal solution can then be expressed

$$\begin{cases} \eta = \frac{\alpha}{2}\left[\frac{1}{2}\ln\left|\frac{\sqrt{\overline{s}}-1}{\sqrt{\overline{s}}+1}\right| + \arctan\left(\sqrt{\overline{s}}\right) + 2 - \frac{\pi}{2}\right] \text{ for } \eta \leq \eta^* \\ s = 0 \text{ for } \eta > \eta^* \end{cases}. \tag{35}$$

A few remarks should be inserted at this point. First, we expect the same qualitative behavior for other values of s_1 and s_0. Nevertheless, for $s_0 > 0$, we will have that s$\to s_0$ as $\eta \to \infty$ and hence no construction will be needed to solve the equation. The other procedures to obtain the solution remain similar. Only the integrals change and equation (11) needs to be replaced by equation (10). Finally we would like to stress again that the solutions provided in this letter not only satisfy the equation (4) and the boundary conditions $\overline{s}(-\infty) = s_1$ and $\overline{s}(+\infty) = s_0$ but also comply with the mass conservation principle. They should therefore replace those derived previously.

Concluding this section we plot the various solutions for the travelling waves at the $t = 2$ in Figure 3. It can be seen that the non-degenerate case produces a smooth profile as expected. Furthermore, case 2 consists of a degenerate diffusion part and needs a construction of the solution where the solution is continuous and differentiable over the whole domain. Further, cases 2 and 3 show continuous solutions where the derivative is no longer continuous, i.e. a free boundary develops for $s = 0$ at $\eta = \eta^*$.

3 Existence of travelling waves

In the preceeding it was assumed implicitly that travelling wave solutions exist and the discussion of the existence of such solutions has been omitted. This issue will now be adressed considering the change of variables given by equation (3). Taking for convenience

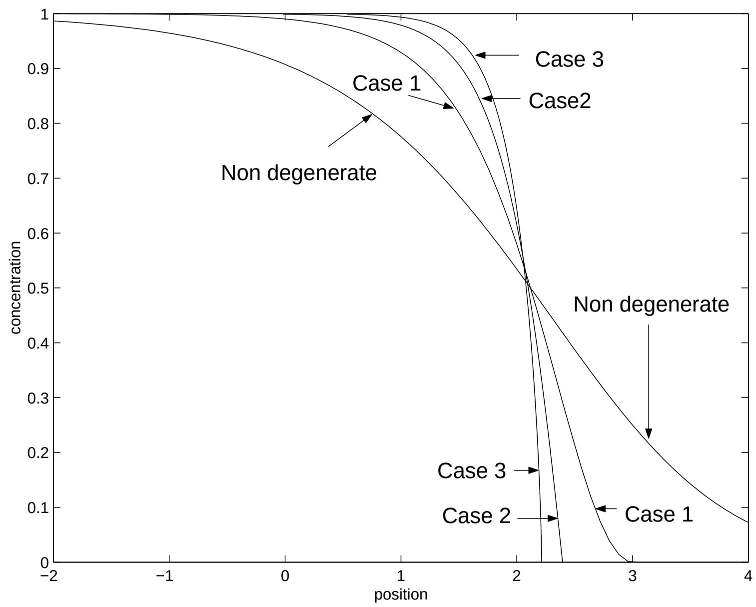

Figure 3: Several travelling wave solutions at $t = 2$ corresponding to the cases that were discussed earlier.

$s_0, s_1 \in \{0, 1\}$ (other cases can be dealt with similarly) we have $C = 0$ and $V = 1$. Hence equation (4) changes into

$$D\left(\overline{s}\right)\overline{s}' = \overline{s}\left(\overline{s}^{n-1} - 1\right) \tag{36}$$

where $D\left(\overline{s}\right) > 0$ since $\alpha > 0$ for $\overline{s} > 0$.

First we deal with case $s_0 = 1$ and $s_1 = 0$. This implies $\overline{s}' > 0$ for a certain η. Equation (36) gives $\overline{s}' > 0$ for $\overline{s} \in (0, 1)$ whenever $n < 1$, whereas $\overline{s}' < 0$ for $\overline{s} \in (0, 1)$ whenever $n > 1$. The latter case leads to a contradiction, since $\overline{s}' < 0$ and $\overline{s}\left(-\infty\right) = 0$ and $\overline{s}\left(+\infty\right) = 1$ is impossible. This implies that no travelling wave solution can exist whenever $s_0 = 1$ and $s_1 = 0$. Besides, $n = 1$ implies $\overline{s}' = 0$ which means that only a constant state solution occurs. This too contradicts the boundary conditions. Accordingly, this case was previously treated using a different change of variables, in the limit $\alpha \to 0$, for which 1 reduces to a hyperbolic problem (see Zitha [1]).

Subsequently we consider the case $s_0 = 0$ and $s_1 = 1$. This implies $\overline{s}' < 0$ for a certain η. Equation (36) gives $\overline{s}' < 0$ for $\overline{s} \in (0, 1)$ whenever $n > 1$, whereas $\overline{s}' > 0$ for $\overline{s} \in (0, 1)$ whenever $n < 1$. The latter case leads to a contradiction, and no travelling wave can exist whenever $s_0 = 0$ and $s_1 = 1$ and $n < 1$. Since $n = 1$ implies a constant solution, it follows that the travelling wave solutions can only exist whenever $n > 1$, $s_0 = 0$ and $s_1 = 1$. We summarise these observations in the following proposition:

Proposition 1 *Given the following problem*

$$\begin{cases} \dfrac{\partial s}{\partial t}+\dfrac{\partial}{\partial x}\left[s^{n}-D\left(s\right)\dfrac{\partial s}{\partial x}\right]=0,\ \textit{for}\ \left(x\in R\right) \\ D\left(s\right)=\dfrac{\alpha}{m}s^{a(m,n)}, \\ a\left(m,n\right)=\dfrac{nm-\left(m+1\right)}{m}, \\ \alpha>0\ \textit{and}\ s\left(x,0\right)=\begin{cases} s_{1}, x<0 \\ s_{0}, x>0 \end{cases}, \end{cases} \tag{37}$$

with $s_0, s_1 \in \{0, 1\}$*, then travelling wave solutions can only exist in the following cases:*
(i) if $s_0 = 1$ *and* $s_1 = 0$ *and* $n < 1$
(ii) if $s_0 = 0$ *and* $s_1 = 1$ *and* $n > 1$.
The travelling wave solution is implicitly defined by

$$\int \frac{\overline{s}^{a(m,n)} d\overline{s}}{\overline{s}^{n}-\overline{s}}+K=\frac{m}{\alpha}\eta. \tag{38}$$

Remark that if $a(m, n) > 0$, then $D(\overline{s}) = 0$ whenver $\overline{s} = 0$ and $D(\overline{s})$ is continuous. According to equation (36) this implies that $\overline{s} = 0$, which gives that both the left- and right handsides vanish. Hence, $\overline{s}'$ is not zero by necessity, which explains that $\overline{s}'$ is not necessity zero. This in turn implies that $\overline{s}'$ is not necessarily continuous for $\overline{s} = 0$.

3.1 Numerical solutions

Finally we remark that it is not always possible to construct (semi) explicit solutions. Only when simplififying assumptions are adopted, some qualitative remarks can be made. Otherwise the solutions rely on numerical methods. To solve the convective (first-order spatial derivative) we use a finite volume method (see for instance Helmig [2] and Leveque [3]). To avoid numerical dispersion as much as possible, especially when $\alpha \to 0$, we apply the Van Leer limiter (Van Leer [4]). See also Hunsdorfer et al. [5] for a comprehensive survey. The time integration is done with an explicit trapezium rule as in Hunsdorfer et al. [5]. Further the non-linear diffusive part is tackled by the use of a semi-implicit Euler time integration, where we use intersections between consecutive gridnodes. The values of the intersections are obtained by taking the averages of s at consecutive nodes. Figure 3 shows both the analytical and numerical solutions for $n = 2$ and $m = 2$ at different times. Some differences exist, however, between the two solutions especially in the short time limit, where we expect the numerical solution to reproduce more realistically the behaviour of the liquid fraction. At long times the two solutions match almost perfectly, which reflects the well known property that the travelling waves correspond to a developed flow. The figure shows this development at the various stages.

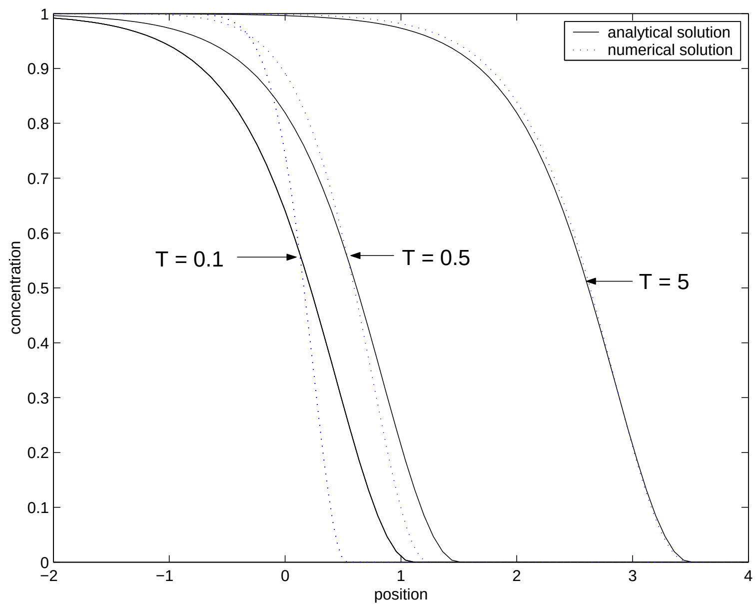

Figure 4: Numerical and analytical solutions of the drainage equation for the case $n = 2$ and $m = 2$. A good match between the two solutions is obtained in the long time limit.

4 Conclusions

A travelling wave solution and a numerical solution of the foam drainage process in porous media has been developed. Travelling wave solutions that satisfy the mass conservation principle were derived for various cases. A good match is found between the analytical and numerical solutions for sufficient longtimes. This analysis corrects and completes the one undertaken earlier and should serve in further studies of the foam drainage phenomenon.

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