On the influence of gravity on the static state of an inclined tensioned string

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Abstract

In this paper the static state of an inclined stretched string due to gravity is considered. The string is stretched between two fixed supports which are situated at two different levels. It is assumed that the tension in the string is sufficiently large such that the sag of the string due to gravity is small. The static displacements due to gravity of the string in the direction along the string and in the direction perpendicular to the string are determined by solving a nonlinearly coupled system of two second order, ordinary differential equations.

1 Introduction

The study of oscillations of stretched strings is not only an interesting subject but also an important subject in the field of dynamical systems. Some examples of physical problems, which can be modeled by stretched strings, are the oscillations of transmission lines, the vibrations of cables supporting TV-towers, or the oscillations of cables in cable-stayed bridges. Sometimes linear models can be used to describe these oscillations, but in most cases nonlinear models have to be used to describe these vibrations sufficiently accurate (see for instance [1]). In general a cable or a string will oscillate around its static or equilibrium state. To investigate a non-linear dynamical system this static or equilibrium state has to be determined first. Recently these equilibrium states for non-linear strings obtained some attention in [2, 3, 4]. It is obvious that if the tension (due to stretching) in the string or cable is very large then the influence of gravity can be neglected, that is, the displacements of the string due to gravity in the direction of the string and in the direction perpendicular to the string are extremely small. These cases of large tension due to stretching have been studied in [5, 6]. In this paper it will be assumed

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that the tension in the string due to stretching is large but not so large that the small sag of the string due to gravity can be neglected. When the stretched string is suspended between two supports at the same level it is well-known (by using linear model equations) that the shape of the string in static state can be approximated by a parabola (see [3, 4, 7, 8]) or by a catenary (see [9, 11, 10]). Also in [8, 12] experiments are described to determine the sags of the cables and the tensions in the cables. The results of these experiments are compared to the results as obtained from the linear model equations, and turn out to be accurate up to 5% when compared to the results as obtained from the linear model equations.

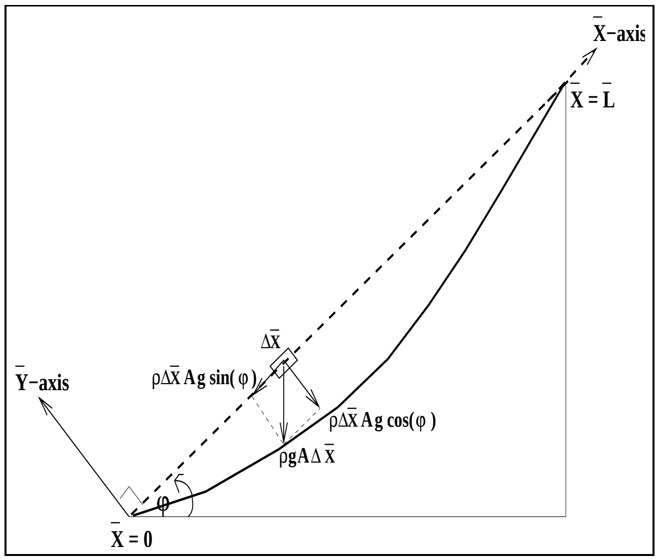

Figure 1: The inclined stretched string in static state due to gravity.

In this paper an inclined stretched string between two fixed supports will be considered (see also Figure 1). In section 2 of this paper a variational method will be used to derive the equations of motion of the string in the direction along the string and in the direction perpendicular to the string. From these equations a system of two nonlinearly coupled, second order ordinary differential equations will be derived, which describes the static state of the string. These ordinary differential equations will be solved exactly in section 3 of this paper. By using these solutions it can be shown that if the tension in the string due to gravity is small compared to the tension in the string due to stretching then the shape of the string in static state can again be approximated by a parabola. Compared to the existing literature (mostly based on linear models or based on only transversal displacements) a nonlinear model for the longitudinal and the transversal displacements will be used in this paper to describe the static shape of the string.

2 The model equations

Consider an inclined, perfectly flexible, elastic unstretched string with length L. Without gravity the string is stretched uniformly by a pretension T_o such that the length of the string is $\bar{L}$. At the end $\bar{X} = 0$ the string is attached to a horizontal plane and at $\bar{X} = \bar{L}$ the end is fixed to a vertical rigid bar (see Figure 1). The gravitational force acting on each material point of the string can be decomposed into two components: a force perpendicular to the string and a force along the string. It is assumed that the pretension T_o is sufficiently large, such that the sag of the string due to gravity is small, and such that the total tension $T(X_o)$ in the string at $X = X_o$ can be well approximated by

$$T(X_o) = T_o + \rho g A \sin(\varphi) X_o, \tag{2.1}$$

where

- T_o is the pretension in the string (in $\frac{kgm}{s^2}$),
- ρ is the mass of the string per unit volume (in $\frac{kg}{m^3}$),
- g is the acceleration due to gravity (in $\frac{m}{s^2}$),
- A is the cross-sectional area of the string (in m^2), and
- φ is the angle between the string and the horizontal plane (in radians).

Let the coordinates $(\bar{X}, \bar{Y})$ of a material point of the unstretched string be $(X, 0)$ with $X \in [0, L]$, where the $\bar{X}$-axis and the $\bar{Y}$-axis are defined in Figure 1. The vector position $\mathbf{r}(X, \bar{\tau})$ of this material point in the dynamic state can be written as:

$$\mathbf{r}(X,\bar{\tau}) = \Big[X + \frac{T_o}{AE}X + \frac{\rho g X^2 \sin(\varphi)}{2E} + U(X,\bar{\tau})\Big]\mathbf{i} + V(X,\bar{\tau})\mathbf{j}, \tag{2.2}$$

where E is Young's modulus (in $\frac{kg}{ms^2}$), $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors along the $\bar{X}$-axis and $\bar{Y}$-axis, U and V are the displacements in $\bar{X}$-direction and $\bar{Y}$-direction, respectively, with respect to the stretched state due to total tension in the string (see (2.2)), and $\bar{\tau}$ is time. It follows from (2.2) that the relative strain per unit length of the stretched string in the dynamic state is:

$$\Omega(X,\bar{\tau}) = \sqrt{\Big[1 + \frac{T_o}{AE} + \frac{\rho g X \sin(\varphi)}{E} + U_X(X,\bar{\tau})\Big]^2 + V_X^2(X,\bar{\tau})} - 1, \tag{2.3}$$

where U_X and V_X represent the derivatives of $U(X,\bar{\tau})$ and $V(X,\bar{\tau})$ with respect to X. The potential and kinetic energy densities of the system are defined by

$$\begin{aligned} \mathbf{P} &= \frac{1}{2}AEL\Omega^2(x,\bar{\tau}) + \rho g AL\Big(U(X,\bar{\tau})\sin(\varphi) + V(X,\bar{\tau})\cos(\varphi)\Big), \quad \text{and} \\ \mathbf{K} &= \frac{1}{2}\rho AL\Big(U_{\bar{\tau}}^2(X,\bar{\tau}) + V_{\bar{\tau}}^2(x,\bar{\tau})\Big), \end{aligned} \tag{2.4}$$

respectively. By applying a variational principle [13] to the Lagrangian density $D = K - P$, it follows that $\frac{\partial}{\partial \bar{\tau}}(\frac{\partial D}{\partial U_{\bar{\tau}}}) + \frac{\partial}{\partial X}(\frac{\partial D}{\partial U_X}) - \frac{\partial D}{\partial U} = 0$ and $\frac{\partial}{\partial \bar{\tau}}(\frac{\partial D}{\partial V_{\bar{\tau}}}) + \frac{\partial}{\partial X}(\frac{\partial D}{\partial V_X}) - \frac{\partial D}{\partial V} = 0$, or equivalently the equations of motion are:

$$\begin{aligned} U_{\bar{\tau}\bar{\tau}}(x,\bar{\tau}) - \frac{E}{\rho}\frac{\partial}{\partial X}\Bigg[& - \frac{1 + \omega_o + \frac{\rho g X \sin(\varphi)}{E} + U_X(X,\bar{\tau})}{\sqrt{\Big[1 + \omega_o + \frac{\rho g X \sin(\varphi)}{E} + U_X(X,\bar{\tau})\Big]^2 + V_X^2(X,\bar{\tau})}} + \\ & U_X(X,\bar{\tau})\Bigg] = 0, \end{aligned} \tag{2.5}$$

$$V_{\bar{\tau}\bar{\tau}}(X,\bar{\tau}) - \frac{E}{\rho}\frac{\partial}{\partial X}\left[- \frac{V_X(X,\bar{\tau})}{\sqrt{\left[1+\omega_o+\frac{\rho g X\sin(\varphi)}{E}+U_X(X,\bar{\tau})\right]^2 + V_X^2(X,\bar{\tau})}} + \right.$$
$$\left. V_X(X,\bar{\tau}) \right] + g\cos(\varphi) = 0,$$

where $\omega_o = \frac{T_o}{AE}$. The static or equilibrium state $\left(\hat{U}(X), \hat{V}(X)\right)$ follows from (2.5) by taking the time-derivatives equal to zero, yielding

$$\frac{E}{\rho}\frac{d}{dX}\left[\hat{U}_X(X) - \frac{1+\omega_o+\frac{\rho g X\sin(\varphi)}{E}+\hat{U}_X(X)}{\sqrt{\left[1+\omega_o+\frac{\rho g X\sin(\varphi)}{E}+\hat{U}_X(X)\right]^2+\hat{V}_X^2(X)}}\right] = 0,$$
$$\frac{E}{\rho}\frac{d}{dX}\left[\hat{V}_X(X) - \frac{\hat{V}_X(X)}{\sqrt{\left[1+\omega_o+\frac{\rho g X\sin(\varphi)}{E}+\hat{U}_X(X)\right]^2+\hat{V}_X^2(X)}}\right] = g\cos(\varphi), \tag{2.6}$$

with boundary conditions $\hat{U}(0) = \hat{V}(0) = \hat{U}(L) = \hat{V}(L) = 0$. In the next section the solution of the coupled system of second order ordinary differential equations (2.6) will be determined.

3 The static state

In applications the parameters $\epsilon = \frac{\rho g L}{E}$ and $\omega_o = \frac{T_o}{AE}$ are usually small parameters. So, actually two small parameters ϵ and ω_o are present in system (2.6). In this paper only the case $0 < \epsilon << \omega_o$ will be considered in detail, that is, it will be assumed that the tension in the string due to gravity is much smaller than the pretension T_o in the string. The system of ordinary differential equations (2.6) will now be solved exactly. First both equations in (2.6) are integrated once with respect to X, yielding

$$\hat{U}_X(X) - \frac{1+\omega_o+\frac{\rho g X\sin(\varphi)}{E}+\hat{U}_X(X)}{\sqrt{\left[1+\omega_o+\frac{\rho g X\sin(\varphi)}{E}+\hat{U}_X(X)\right]^2+\hat{V}_X^2(X)}} = k_1,$$
$$\hat{V}_X(X) - \frac{\hat{V}_X(X)}{\sqrt{\left[1+\omega_o+\frac{\rho g X\sin(\varphi)}{E}+\hat{U}_X(X)\right]^2+\hat{V}_X^2(X)}} = \frac{\rho g X\cos(\varphi)}{E} + k_2, \tag{3.1}$$

where k_1 and k_2 are constants of integration. Let $1+\omega_o+\frac{\rho g X\sin(\varphi)}{E}+\hat{U}_X(X) = R(X)\cos(\Psi(X))$ and $\hat{V}_X(X) = R(X)\sin(\Psi(X))$, then (3.1) becomes

$$(R-1)\cos(\Psi) = f_1,$$
$$(R-1)\sin(\Psi) = f_2, \tag{3.2}$$

where $f_1 = \frac{\rho g X\sin(\varphi)}{E} + k_1 + 1 + \omega_o$ and $f_2 = \frac{\rho g X\cos(\varphi)}{E} + k_2$. It follows from (3.2) that $R = 1 \pm \sqrt{f_1^2+f_2^2}$. Since $\hat{U}_X << 1$ it follows from $R\cos(\Psi) = 1+\omega_o+\frac{\rho g X\sin(\varphi)}{E}+$

$\hat{U}_X(X)$ that $R > 1$, and so $R = 1 + \sqrt{f_1^2 + f_2^2}$. Then, it follows from (3.1) and (3.2) that

$$\begin{aligned} \hat{U}_X(X) &= \frac{f_1}{\sqrt{f_1^2 + f_2^2}} + k_1, \\ \hat{V}_X(X) &= \frac{f_2^2}{\sqrt{f_1^2 + f_2^2}} + \frac{\rho g X \cos(\varphi)}{E} + k_2. \end{aligned} \tag{3.3}$$

By integrating the equations in (3.3) with respect to X the following expressions for $\hat{U}(X)$ and $\hat{V}(X)$ are obtained

$$\begin{aligned} \hat{U}(X) = \frac{E}{\rho g}\Bigg[&\sin(\varphi)\sqrt{\frac{\rho^2 g^2 X^2}{E^2} + \frac{2\rho g X}{E}a + a^2 + b^2} + (1 + \omega_o + k_1 - \\ &a\sin(\varphi))\ln\left(\frac{\rho g X}{E} + a + \sqrt{\frac{\rho^2 g^2 X^2}{E^2} + \frac{2\rho g X}{E}a + a^2 + b^2}\right)\Bigg] + k_1 X + k_3, \\ \hat{V}(X) = \frac{E}{\rho g}\Bigg[&\cos(\varphi)\sqrt{\frac{\rho^2 g^2 X^2}{E^2} + \frac{2\rho g X}{E}a + a^2 + b^2} + (k_2 - a\cos(\varphi))\ln\Bigg(\frac{\rho g X}{E} + \\ &a + \sqrt{\frac{\rho^2 g^2 X^2}{E^2} + \frac{2\rho g X}{E}a + a^2 + b^2}\Bigg) + \frac{\rho^2 g^2 X^2 \cos(\varphi)}{2E^2}\Bigg] + k_2 X + k_4, \end{aligned} \tag{3.4}$$

where k_3 and k_4 are constants of integration, and where

$$\begin{aligned} a &= (1 + \omega_o + k_1)\sin(\varphi) + k_2\cos(\varphi), \text{and} \\ b &= (1 + \omega_o + k_1)\cos(\varphi) - k_2\sin(\varphi). \end{aligned} \tag{3.5}$$

By using the boundary conditions for $\hat{U}$ and $\hat{V}$ at $X = 0$ and $X = L$ it follows that

$$\begin{aligned} k_3 &= -\frac{E}{\rho g}\Big[\sin(\varphi)\sqrt{a^2 + b^2} + (1 + \omega_o + k_1 - a\sin(\varphi))\ln(a + \sqrt{a^2 + b^2})\Big], \\ k_4 &= -\frac{E}{\rho g}\Big[\cos(\varphi)\sqrt{a^2 + b^2} + (k_2 - a\cos(\varphi)\ln(a + \sqrt{a^2 + b^2})\Big], \end{aligned} \tag{3.6}$$

and that k_1 and k_2 (or equivalently a and b) have to satisfy

$$\begin{aligned} &\sqrt{(a+\epsilon)^2 + b^2} - \sqrt{a^2 + b^2} + \epsilon[a - (1 + \omega_o)\sin(\varphi)] + \frac{1}{2}\epsilon^2\cos^2(\varphi) = 0, \\ &b\Bigg[\ln\left(a + \epsilon + \sqrt{(a+\epsilon)^2 + b^2}\right) - \ln\left(a + \sqrt{a^2 + b^2}\right)\Bigg] + \epsilon[b - (1 + \omega_o)\cos(\varphi)] \\ &\qquad - \frac{1}{4}\epsilon^2\sin(2\varphi) = 0, \end{aligned} \tag{3.7}$$

where a and b are given by (3.5). Having determined a and b from (3.5) (and so, k_1 and k_2) it follows that $\hat{U}(X)$ and $\hat{V}(X)$ are given by

$$\hat{U}(X) = \frac{E}{\rho g}\Bigg[\sin(\varphi)\Big(\sqrt{(\frac{\rho g X}{E} + a)^2 + b^2} - \sqrt{a^2 + b^2}\Big) + b\cos(\varphi)\Big(\ln\Big[a +$$

$$\frac{\rho g X}{E} + \sqrt{(\frac{\rho g X}{E} + a)^2 + b^2}\Big] - \ln(a + \sqrt{a^2 + b^2})\Big) + \frac{\rho g X}{E} k_1\Big],$$

$$\hat{V}(X) = \frac{E}{\rho g}\Big[\cos(\varphi)\Big(\sqrt{(\frac{\rho g X}{E} + a)^2 + b^2} - \sqrt{a^2 + b^2}\Big) - b\sin(\varphi)\Big(\ln\Big[a + \frac{\rho g X}{E} + \sqrt{(\frac{\rho g X}{E} + a)^2 + b^2}\Big] - \ln(a + \sqrt{a^2 + b^2})\Big) + \frac{\rho g X}{E} k_2 + \frac{\rho^2 g^2 X^2 \cos(\varphi)}{2E^2}\Big]. \tag{3.8}$$

Unfortunately it is not possible to solve (3.7) for a and b exactly. On the other hand, for small ϵ (3.7) can be solved in an approximate way by expanding a and b in power series in ϵ, that is,

$$a = \sum_{i=0}^{\infty} \epsilon^i a_i \quad \text{and} \quad b = \sum_{i=0}^{\infty} \epsilon^i b_i, \tag{3.9}$$

where a_i and b_i are of order $O(1)$. Substituting (3.9) into (3.7), and then by taking together terms of equal powers in ϵ, and by solving the so-obtained $O(\epsilon^n)$-problems for $n = 0, 1, 2, \ldots$, it follows that $a_0 = \omega_o \sin(\varphi)$, $b_0 = \omega_o \cos(\varphi)$, $a_1 = -\frac{1}{2}\cos^2(\varphi)$, $b_1 = \frac{1}{2}\sin(\varphi)\cos(\varphi)$, $a_2 = \frac{(1+3\omega_o)}{24\omega_o^2(1+\omega_o)}\sin(\varphi)\cos^2(\varphi)$, $b_2 = \frac{\cos(\varphi)}{24\omega_o^2(1+\omega_o)}\Big[(1+3\omega_o)\cos^2(\varphi) - 2\omega_o\Big]$, and so on. From (3.5) k_1 and k_2 can now be approximated, yielding

$$\begin{aligned} k_1 &= -1 + \epsilon^2 \frac{\cos^2(\varphi)}{24\omega_o^2} + O(\epsilon^3), \quad \text{and} \\ k_2 &= -\frac{1}{2}\epsilon\cos(\varphi) + \epsilon^2 \frac{\sin(\varphi)\cos(\varphi)}{12\omega_o(1+\omega_o)} + O(\epsilon^3). \end{aligned}$$

Finally, from (3.8) $\hat{U}(X)$ and $\hat{U}(X)$ can then be approximated, yielding

$$\begin{aligned} \hat{U}(X) &= -\frac{\rho^2 g^2 A^2 \cos^2(\varphi)}{12T_o^2} X(X-L)(2X-L) + \mathrm{O}(\tilde{\epsilon}^3), \\ \hat{V}(X) &= \frac{\rho g A \cos(\varphi)}{2T_o} X(X-L)\Big[1 + \omega_o - \frac{\rho g A \sin(\varphi)}{6T_o}(4X+L) + \mathrm{O}(\tilde{\epsilon}^2)\Big], \end{aligned} \tag{3.10}$$

where $\tilde{\epsilon} = \frac{\epsilon}{\omega_o}$, $\epsilon = \frac{\rho g L}{E}$, and $\omega_o = \frac{T_o}{AE}$. So, for small ϵ the displacement $\hat{V}(X)$ of the inclined string in the direction perpendicular to the string can be well approximated by a parabola.

4 Conclusions and remarks

In this paper the static state of an inclined string due to gravity has been considered. The string is assumed to be perfectly flexible and to be stretched uniformly between two fixed support which are situated at two different levels. It is assumed that the tension is sufficiently large such that the sag of the string due to gravity is small compared to the length of the string. By using a variational principle the equations describing the static state of the string in the direction along the string and in the direction perpendicular to the string are determined, and are given by a system of two nonlinearly coupled, second order ordinary differential equations. By solving these equations the static displacements due to gravity of the string in the direction along the string and in the direction perpendicular to the string are determined exactly.

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