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# SOME PROPERTIES OF THE GENERALIZED EIGENVALUE PROBLEM $\boldsymbol{M}\boldsymbol{x} = \lambda(\boldsymbol{\Gamma} - c\boldsymbol{I})\boldsymbol{x}$, WHERE $\boldsymbol{M}$ IS THE INFINITISEMAL GENERATOR OF A MARKOV PROCESS, AND $\boldsymbol{\Gamma}$ IS A REAL DIAGONAL MATRIX.

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# Some properties of the generalized eigenvalue problem $\boldsymbol{Mx} = \lambda(\boldsymbol{\Gamma} - c\boldsymbol{I})\boldsymbol{x}$, where $\boldsymbol{M}$ is the infinitisemal generator of a Markov process, and $\boldsymbol{\Gamma}$ is a real diagonal matrix.

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## 1 Introduction

In [1] a liquid model is described, in which an information buffer is represented by a "fluid buffer", filled up by several independent two-state sources, and drained by an output channel at a constant depletion rate $c$. The *on-off* transitions of the input sources are *Poisson* distributed.
In [2] *L. Kosten* makes a generalization in which the filling rate of the buffer depends on the momentary state of an $m$-state continuous time-parameter Markov proces. In this approach, the system is supposed to have $m$ "states" $S_1, S_2, \dots, S_m$. Each state $S_k$ has its own filling rate $\gamma_k$, and the transition between different states is described by the differential equation:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{M}\boldsymbol{p} \tag{1}$$

where the components $p_k(t)$ of the vector $\boldsymbol{p}(t)$ represent the probabilities of the system to be in the corresponding state $S_k$ at time $t$:

$$p_k(t) = \mathrm{P}\{S_k \text{ at time } t\}$$

The matrix $\boldsymbol{M}$, the infinitisemal generator of the Markov process, is assumed to be non-degenerate. It has the properties:

$$\begin{aligned} M_{k,k} &< 0 \\ M_{k,l} &\geq 0 \qquad \text{if } k \neq l \\ \sum_{k=1}^{m} M_{k,l} &= 0 \qquad l = 1, 2, \ldots, m \end{aligned} \tag{2}$$

In the state $S_k$, the buffer is filled at rate $\gamma_k$, and drained at rate $c$, hence the resulting filling rate is $\gamma_k - c$. This however doesn't hold if the buffer is empty and at the same time $\gamma_k < c$. In that situation, the incoming information is drained completely.
The question of interest is, of course, the probability of overflow, in relation to the "capacity" $c$ of the output channel. This question is dealt with by assuming a buffer of infinite capacity, and asking for the probability that some stochastic *level*-parameter $h$ will exceed some value $u$. If the system is in the state $S_k$, the level will change according to

$$\frac{dh}{dt} = \begin{cases} 0 & \text{if } (\gamma_k < c) \bigcap (h = 0) \\ \gamma_k - c & \text{in other cases} \end{cases}$$

The relevant quantity then is $\mathcal{P}\{\ h > u\ \}$ .
Kosten defines the stochastic vector $\boldsymbol{F}(u,t)$, of which the components $F_k(u,t)$ represent the probabilities that the level $h$ is *below* the value $u$, *and* the system is in the state $S_k$:

$$F_k(u,t) = \mathcal{P}\{\ S_k \wedge (h \leq u) \text{ at time } t\ \}$$

It can easily be derived, that $\boldsymbol{F}$ satisfies the following linear system of partial differential equations:

$$\frac{\partial \boldsymbol{F}}{\partial t} + (\boldsymbol{\Gamma} - c\boldsymbol{I})\frac{\partial \boldsymbol{F}}{\partial u} = \boldsymbol{M}\boldsymbol{F}(u,t)$$

If the system becomes stationary after some time, the differential equation turns over to an ordinary differential equation:

$$(\boldsymbol{\Gamma} - c\boldsymbol{I})\frac{d\boldsymbol{F}}{du} = \boldsymbol{M}\boldsymbol{F}(u) \tag{3}$$

where $\boldsymbol{\Gamma}$ is the diagonal $m \times m$ matrix of which the diagonal entries are the filling rates $\gamma_k$:

$$\boldsymbol{\Gamma} = \begin{pmatrix} \gamma_1 & 0 & 0 & \cdots & 0 \\ 0 & \gamma_2 & 0 & \cdots & 0 \\ 0 & 0 & \gamma_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_m \end{pmatrix} \tag{4}$$

Not all values of the depletion rate $c$ are of interest. If, for instance $c > \gamma_k$ for all states, the buffer simply doesn't fill at all, and no overflow can occur. On the other hand, if the *mean filling rate* exceeds the depletion rate, the level $h$ will increase permanently, so in every practical application the system certainly will arrive in a permanent state of overflow after a finite time. These considerations lead to the definition of an "interval of relevance" for the depletion rate.

**Definition 1 (Interval of relevance)** *Let $\boldsymbol{p}_\infty = \lim_{t\to\infty} \boldsymbol{p}(t)$, where $\boldsymbol{p}(t)$ is a solution of ( 1); let $\mathbf{1}$ denote the all-ones vector in $\mathbb{R}^m$, and let $\gamma_{\text{max}} = \max \gamma_k$. The interval of relevance for the depletion rate $c$ is the interval*

$$\mathcal{R} = \left(\mathbf{1}^T \boldsymbol{\Gamma} \boldsymbol{p}^\infty \ , \ \gamma_{\text{max}}\right) \tag{5}$$

*where the $\gamma_k$ are assumed to be in increasing order.*

The left point of $\mathcal{R}$ is the stability bound, the right point is the triviality bound.
If the stationary state equation is solved formally, the spectrum of the matrix $(\boldsymbol{\Gamma} - c\boldsymbol{I})^{-1}\boldsymbol{M}$ plays a key-role. The following questions are answered in this paper [1]

**1. Kosten's consistency conjecture.**

Let $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_m$ be the eigenvectors of $(\boldsymbol{\Gamma} - c\boldsymbol{I})^{-1}\boldsymbol{M}$, corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, and write the formal solution to ( 3 ) as

$$\boldsymbol{F}(u) = \sum_{k=1}^{m} a_k \boldsymbol{\phi}_k e^{\lambda_k u}$$

Then, since $\boldsymbol{F}(u)$ has to be bounded as $u \to \infty$, the coefficients $a_k$ must be zero if the corresponding eigenvalues $\lambda_k$ are in the strict right half of the

[1] These were brought under the authors attention by L. Kosten. The answers were conjectured by him, on the basis of experience with some classic buffer models with known analytic solution.

complex plane. On the other hand, the matrix $\boldsymbol{M}$, and therefore also the matrix $(\boldsymbol{\Gamma} - c\boldsymbol{I})^{-1}\boldsymbol{M}$, has an eigenvalue zero, corresponding to the eigenvector $\boldsymbol{p}^{\infty}$, i.e. the probability distribution of the states as the Markov process has become stationary. This information serves for determination of the corresponding coefficient $a_k$. Finally, if the value of some $\gamma_j$ exceeds $c$, the level *cannot be zero* (except of course for a single point in time), hence $F_j(0) \;=\; \mathcal{P}\{\; (h \le 0) \wedge S_j \;\} \;\;=\; 0$ in that case. In order to be able to determine the remaining coefficients $a_k$, i.e. the coefficients, corresponding to eigenvalues in the strict left half of the complex plane, we only can use these boundary conditions, therefore the number of these "left" eigenvalues *should equal the number of filling rates exceeding* $c$ .
This will be proved in this paper.

**2. Kosten's dominancy conjecture.**

Denoting by $F(u)$ the overall probability $\mathcal{P}\{\; h \le u \;\}$ (disregarding the state of the buffersystem), then

$$F(u) = \sum_{k=1}^{m} F_k(u) \;=\; \mathbf{1}^T \boldsymbol{F}(u)$$

The behaviour of $F(u)$ at large values of $u$ is mainly determined by the rightmost eigenvalues in the strict left half of the complex plane.
L.Kosten expected the existence of a "dominant" eigenvalue, i.e. one single eigenvalue in this halfplane, closest to the origin of the complex plane. The existence of this "left dominant eigenvalue" will be proved.

**3. Kosten's monotonicity conjecture.**

Finally, from experience with models as described in [1], it might be expected that at high depletion rate, the mean filling rate will be low. This should be expressed by the behaviour if the left dominant eigenvalue as function of $c$. Kosten conjectured that this eigenvalue *decreases monotonically as a function of* $c$.
Also this statement will be proved.

## 2 Barrier matrices

The key operations in the analysis of Kostens conjectures depend on the simplicity of some critical eigenvalue $\mu$, and the positivity of the corresponding eigenvector.

In the case of the simple eigenvalue problem $\boldsymbol{M}\boldsymbol{x} = \lambda\boldsymbol{x}$, where $\boldsymbol{M}$ is as defined in ( 2), the critical eigenvalue $\mu = 0$, and the corresponding eigenvector and left eigenvector are more or less given. In order to prove the existence of a 'left dominant eigenvalue', more information about the existence of positive eigenvectors is required. This information can be found in the *Perron-Frobenius* theory on irreducible non-negative matrices. We start with some definitions.

**Definition 2** *A matrix $\boldsymbol{A}$ is* **non-negative** *(resp.* **positive***) if the entries satisfy $a_{k,l} \geq 0$ (resp. $a_{k,l} > 0$). A non-negative (positive) matrix is denoted by $\boldsymbol{A} \geq \boldsymbol{O}$ (resp. $\boldsymbol{A} > \boldsymbol{O}$). Similarly $\boldsymbol{A} > \boldsymbol{B}$ means $\boldsymbol{A} - \boldsymbol{B} > \boldsymbol{O}$, etc.*

**Definition 3** *A square matrix $\boldsymbol{A}$ is* **reducible** *if a permutation matrix $\boldsymbol{P}$ exists such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ can be partitioned as follows*

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} \boldsymbol{B}_{LL} & \boldsymbol{B}_{LR} \\ \boldsymbol{O} & \boldsymbol{B}_{RR} \end{pmatrix}$$

*where $\boldsymbol{B}_{LL}$ and $\boldsymbol{B}_{RR}$ are square matrices.*
*An* **irreducible** *matrix is a matrix that is not reducible.*

An alternative definition of reducibility is connected with eigenvalue problems: A matrix is reducible if and only if it has a proper "Cartesian" invariant subspace, that is an invariant subspace, generated by a proper subset of (Cartesian) basis-vectors. Another alternative definition can be based on an important property, stated as a lemma:

**Lemma 1** *Let $\boldsymbol{A}$ be an irreducible $(m \times m)$ -matrix, then for any proper subset $K$ of $I_m = \{1, 2, \ldots, m\}$, there exist $k \in K$, and $l \in I_m \setminus K$, such that $a_{k,l} \neq 0$.*

**Proof:** Suppose $K \subset I_m$ exist such that $a_{k,l} = 0,\ \forall\ k \in\ K,\ \forall\ l \in\ I_m \setminus\ K$. Choosing any permutation matrix that places the members of $K$ before those of $I_m \setminus K$, a zero left-down off-diagonal block is obtained in the partioning of the matrix $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ □

From this "definition", it is seen directly that reducibility of a matrix only depends on its off-diagonal part.
The following theorem is the famous theorem of *Frobenius*, stated before in a weaker form by *Perron*:

**Theorem 1 (Perron-Frobenius)** *Let $\boldsymbol{A}$ be an irreducible non-negative square matrix, then*

- *$\boldsymbol{A}$ has a positive eigenvalue $\lambda^*$ equal to the spectral radius of $\boldsymbol{A}$.*
- *The corresponding eigenvector $\boldsymbol{x}^*$ is positive.*
- *The eigenvalue $\lambda^*$ increases when any entry of $\boldsymbol{A}$ increases.*
- *The eigenvalue $\lambda^*$ is simple*

In the present analysis, we have no special interest in the fact that the spectral radius of a matrix is actualized by a simple eigenvalue. We use the Perron Frobenius theorem because this eigenvalue is 'dominant' in another sense than the usual, namely the property that *all other eigenvalues have smaller real parts.*. Therefore we redefine the terminology in this paper.

**Definition 4 (Dominant eigenvalue.)** *An eigenvalue $\mu$ of a matrix $\boldsymbol{A}$ is* **dominant** *if*

1. *$\mu$ is real,*
2. *$\Re(\lambda) < \mu$ for all $\lambda \neq \mu$.*

In relation to this definition of 'dominant', we'll investigate an slight extension of the class of irreducible non-negative matrices:

**Definition 5 (Barrier matrix)** *A $B$-matrix is a square irreducible matrix, with non-negative off-diagonal entries:*

$$a_{k,l} \geq 0, \quad k \neq l$$

(In the name "$B$-matrix, *B* is for *Barrier*).
The following lemma is almost trivial:

**Lemma 2** *Let $\boldsymbol{A}$ be a $B$-matrix, then*

*(a)* *$\boldsymbol{A}$ has a unique, simple, dominant eigenvalue $\beta$, corresponding to a positive eigenvector $\boldsymbol{u}$.*

$$\boldsymbol{A}\boldsymbol{u} = \beta\boldsymbol{u}, \quad \boldsymbol{u} > \boldsymbol{0}$$

$$(\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}) \wedge (\lambda \neq \beta) \wedge (\boldsymbol{x} \neq \boldsymbol{0}) \quad \Longrightarrow \Re(\lambda) < \beta$$

*(b)* *$\boldsymbol{A}$ has no other positive eigenvectors:*

$$(\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}) \wedge (\boldsymbol{x} > \boldsymbol{0}) \quad \Longrightarrow (\lambda = \beta) \wedge (\boldsymbol{x} = \alpha\boldsymbol{u})$$

*We'll call the dominant eigenvalue $\beta$ the* **Barrier value** *of the matrix.*

**Proof:** To prove part (a), let $a_{k,k}$ be the minimal diagonal entry of $\boldsymbol{A}$: $a_{k,k} \le a_{l,l}$, $l = 1, 2, \ldots, n$, then $\boldsymbol{B} = \boldsymbol{A} - a_{k,k}\boldsymbol{I}$ is an irreducible non-negative matrix. According to the Perron-Frobenius theorem, $\boldsymbol{B}$ has a simple eigenvalue $\lambda^*$, satisfying: $\boldsymbol{B}\boldsymbol{u} = \lambda^*\boldsymbol{u}$, with $\boldsymbol{u} > \boldsymbol{0}$, and such that for all other eigenvalues $|\lambda| \le \lambda^*$. Let $\lambda$ be any other eigenvalue, then

$$\lambda = |\lambda|[\cos(\varphi) + i\sin(\varphi)]$$

If $|\lambda| = \lambda^*$, then $\varphi \neq 0 \bmod 2\pi$, since otherwise $\lambda^*$ weren't simple. Hence $\cos(\varphi) < 1$ in that case, and therefore $\Re(\lambda) = \lambda^* \cos(\varphi) < \lambda^*$.
If $|\lambda| < \lambda^*$, then of course $\Re(\lambda) = |\lambda|\cos(\varphi) \le |\lambda| < \lambda^*$. So for all $\lambda \neq \lambda^*$:

$$\Re(\lambda) < \lambda^*$$

Since the eigenvalues of $\boldsymbol{A}$ are obtained by shifting the eigenvalues of $\boldsymbol{B}$ by $a_{k,k}$ to the right, the matrix $\boldsymbol{A}$ has a dominant eigenvalue $\beta = \lambda^* + a_{k,k}$.
To prove part (b), let $\boldsymbol{x} > \boldsymbol{0}$ be some positive eigenvector: $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$. From the definition follows that $\boldsymbol{A}^T$ is a $B$-matrix as well, and hence, a positive vector $\boldsymbol{w}$ exists such that $\boldsymbol{w}^T\boldsymbol{A} = \beta\boldsymbol{w}^T$. Since $\boldsymbol{w}^T\boldsymbol{u} > 0$, it follows

$$\boldsymbol{w}^T\boldsymbol{A}\boldsymbol{x} = \beta\boldsymbol{w}^T\boldsymbol{x} = \lambda\boldsymbol{w}^T\boldsymbol{x} \implies \lambda = \beta \implies \boldsymbol{x} = \alpha\boldsymbol{u}$$

□

**Lemma 3 (Separation lemma)** *Let $\boldsymbol{A}$ be an $m \times m$ B-matrix, let $\boldsymbol{D}$ be a non-singular diagonal matrix, and let the numbers $\sigma_k$ be defined by*

$$\sigma_k = \frac{a_{kk}}{d_k}, \quad k = 1, 2, \ldots, m \tag{6}$$

*Assume the generalized eigenproblem $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{D}\boldsymbol{x}$ has a solution with strictly positive eigenvector $\boldsymbol{u}$, with corresponding eigenvalue $\mu$,
then $\mu$ is real, and satisfies the following inequalities:*

$$\sigma_k < \mu < \sigma_l \quad \textit{whenever } d_k > 0 \textit{ and } d_l < 0 \tag{7}$$

*Moreover, let the eigenvalues of the problem be ordered as*

$$\Re(\lambda_1) \le \Re(\lambda_2) \le \cdots \le \Re(\lambda_m) \tag{8}$$

*and let $s$ denote the number of $\sigma$'s to the left of $\mu$, then*

$$\Re(\lambda_s) \le \mu \le \Re(\lambda_{s+1}) \tag{9}$$

**Proof:** Write the equation $\boldsymbol{Au} = \mu\boldsymbol{Du}$ in components:

$$(\mu d_k - a_{kk})u_k = \sum_{l\neq k} a_{kl}u_l \quad k = 1, 2, \ldots, m$$

Since $\boldsymbol{A}$ is a $B$-matrix, its off-diagonal entries are non negative. Since $u_l > 0$ for all $l$, the righthand side is non negative. If it were zero for some $k$, then $a_{kl} = 0$, for all $l \neq k$, which would imply the matrix to be reducible. Hence $\mu d_k - a_{kk} > 0$ for all $k$, so the following inequalities must be satisfied:

$$\begin{array}{ll} \mu > \sigma_k, & d_k > 0 \\ \mu < \sigma_k, & d_k < 0 \end{array} \tag{10}$$

which is equivalent to ( 7).

For the proof of ( 9), we use Gershgorin's theorem on the matrices $\widetilde{\boldsymbol{A}}(\tau)$, with $\tau \in [0, 1]$, defined by

$$\widetilde{a}_{kk} = \sigma_k, \quad , \widetilde{a}_{kl} = \tau\frac{a_{kl}u_l}{d_k u_k}, \ \text{ for } l \neq k$$

The matrix $\widetilde{\boldsymbol{A}}(1)$ is similar to $\boldsymbol{D}^{-1}\boldsymbol{A}$, and therefore the eigenvalues of $\widetilde{\boldsymbol{A}}(1)$ are the same as the eigenvalues of the generalized eigenvalue problem $\boldsymbol{Ax} = \lambda\boldsymbol{Dx}$.

According to Gershgorins theorem, the eigenvalues of $\widetilde{\boldsymbol{A}}(\tau)$ are in the union of the circles $C_k$, with centre in $\sigma_k$, and radius $r_k = \sum_{l\neq k} |\widetilde{a}_{kl}(\tau)|$. So if $\lambda$ is an eigenvalue, then for some $k$

$$|\lambda - \sigma_k| \leq \sum_{l\neq k}\left|\tau\frac{a_{kl}u_l}{d_k u_k}\right| = \left|\tau\frac{\sum_{l\neq k} a_{kl}u_l}{d_k u_k}\right| = \left|\tau\frac{(\mu d_k - a_{kk})u_k}{d_k u_k}\right| = \tau|\mu - \sigma_k| \tag{11}$$

According to these inequalities, the Gershgorin circles are either completely to the right or completely to the left of the line $\Re(z) = \mu$ in the complex plane, at least as long as $\tau < 1$. If $\tau = 1$, all circles contain the point $z = \mu$.

Let $a < \min\sigma_k < \max\sigma_k < b$, then of course $a < \mu < b$. Now define the circular discs $\mathcal{C}_L$ and $\mathcal{C}_R$ by

$$C_L = \mathcal{C}(a, \mu - a), \quad C_R = \mathcal{C}(b, b - \mu)$$

Then $C_L$ and $C_R$ are respectively to the left and to the right of the line $\Re(z) = \mu$. They have the point $z = \mu$ in common. Now for all $\tau \in [0, 1)$, the circles $|z - \sigma_k| \leq |\mu - \sigma_k|$ are proper subsets of either $C_L$ or $C_R$.

The matrices $\widetilde{\boldsymbol{A}}(\tau)$ are continuous functions of $\tau$, therefore also the eigenvalues behave continuously [2]. For $\tau = 0$, the eigenvalues are in the centres of the Gershgorin circles. As long as $\tau < 1$, no eigenvalue can be on the boundaries of $C_L$ and $C_R$. hence the number of eigenvalues that are inside $C_L$ doesn't change. If $\tau$ tends to $1$,

[2] An elementary proof can be done with Rouché's theorem

interior eigenvalues in $C_L$ can possibly move to the boundary, but cannot move to the interior of the $C_R$, and vice versa.
$C_L$ contains the $\sigma_k$ values to the left of $\mu$, that are $s$ values. Hence for $\tau < 1$, there are precisely $s$ eigenvalues in $C_L$. If $\tau = 1$, one or more eigenvalues may be on the common boundary of the circles. Hence: $\Re(\lambda_k) \leq \mu$, $k = 1, 2, \ldots, s$.
Similarly $m - s$ eigenvalues are in $C_R$, and $\Re(\lambda_k) \geq \mu$, $k = s+1, s+2, \ldots, m$, from which ( 9) follows □

**Definition 6** *A real eigenvalue $\mu$ of the generalized eigenvalue problem $\boldsymbol{Ax} = \lambda \boldsymbol{Dx}$ is called a* **separator** *if the corresponding eigenvector is strictly positive.*

The existence of a *separator* value like $\mu$ in this lemma, is not a trivial question. It is necessary that for all pairs $[k, l]$ the following inequalities hold

$$\frac{a_{kk}}{d_k} < \frac{a_{ll}}{d_l}, \text{ whenever } d_k > 0 > d_l \tag{12}$$

This relates the ordering between the entries $d_k$ of the matrix $\boldsymbol{D}$ and the numbers $\sigma_k = a_{kk}/d_k$. For instance if $\boldsymbol{D}$ has positive as well as negative entries, whereas $\boldsymbol{A}$ has only positive diagonal entries, then ( 12) cannot be valid, and no positive eigenvector exists. If all diagonal entries of $\boldsymbol{A}$ are negative, which is the case if $\boldsymbol{A}$ is as described in ( 2), then the inequalities ( 12) hold automatically. In this case $\boldsymbol{u} = \boldsymbol{p}^\infty$ is a positive vector as used in the lemma. Since $\boldsymbol{Au} = \boldsymbol{0} = 0\boldsymbol{Du}$, the value $\mu = 0$ is a separator.
The existence, as well as the important properties of separating eigenvalues are analysed by tracking down the barrier values of a suitable family of barrier matrices.

**Definition 7 (Barrier function)** *Let $\boldsymbol{A}$ be an $m \times m$ B-matrix, and let $\boldsymbol{D}$ be a real diagonal matrix. All matrices $\boldsymbol{A}(\tau)$ defined by*

$$\boldsymbol{A}(\tau) = \boldsymbol{A} - \tau \boldsymbol{D}, \quad \tau \in \mathbb{R} \tag{13}$$

*are B-matrices, and therefore each $\boldsymbol{A}(\tau)$ has a barrier value $\beta(\tau)$.*
*The function $\beta$ is the* **Barrier function** *for the matrix pair $\{\boldsymbol{A}, \boldsymbol{D}\}$.*

**Lemma 4 (Convexity lemma)** *Let $\boldsymbol{A}$ be an $m \times m$ B-matrix, let $\boldsymbol{D}$ be a real diagonal matrix, and let $\beta$ be the barrier function corresponding to $\{\boldsymbol{A}, \boldsymbol{D}\}$. Then*

1. *The separators of $\boldsymbol{Ax} = \lambda \boldsymbol{Dx}$ are the zeros of $\beta$.*

2. *$\beta(\tau)$ is a convex function.*

**Proof:** Consider the family $\boldsymbol{A}(\tau)$ of $B$-matrices defined by ( 13). Let $\tau_0$ be a zero of the barrier function $\beta(\tau)$ for the pair $\{\boldsymbol{A}, \boldsymbol{D}\}$, let $\boldsymbol{u} > \boldsymbol{0}$ be a positive eigenvector, corresponding to $\beta(\tau_0)$, then

$$\boldsymbol{A}(\tau_0)\boldsymbol{u} = \beta(\tau_0)\boldsymbol{u} = \boldsymbol{0} \Longrightarrow \boldsymbol{A}\boldsymbol{u} = \tau_0 \boldsymbol{D}\boldsymbol{u}$$

So a zero of $\beta(\tau)$, is a separator for $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{D}\boldsymbol{x}$.
On the other hand, if $\mu$ is a separator, then $(\boldsymbol{A} - \mu\boldsymbol{D})\boldsymbol{u} = \boldsymbol{0}$ for some strictly positive $\boldsymbol{u}$, so apparently zero is the barrier value for $\boldsymbol{A} - \mu\boldsymbol{D}$.

*Therefore the separators are the zeros of $\beta(\tau)$.*

Next we'll prove the convexity of the function $\beta$ in 3 steps.

**1. Points of intersection with straight line.** We consider the larger family of matrices

$$\widetilde{\boldsymbol{A}}(\tau) = \boldsymbol{A}(\tau) - (p + q\tau)\boldsymbol{I} \tag{14}$$

For each real $p$, $q$, and $\tau$, this is a $B$-matrix, and since it differs from $\boldsymbol{A}(\tau)$ by a real multiple of the unit matrix, also the eigenvalues are shifted eigenvalues of $\boldsymbol{A}(\tau)$. Hence

$$\widetilde{\beta}(\tau) = \beta(\tau) - (p + q\tau) \tag{15}$$

where $\widetilde{\beta}$ can be interpreted as the barrier function for the matrix pair $\{\boldsymbol{A} - p\boldsymbol{I}, \boldsymbol{D} + q\boldsymbol{I}\}$. Hence zeros of $\widetilde{\beta}$ are separators for the problem $[\boldsymbol{A} - p\boldsymbol{I} - \lambda(\boldsymbol{D} + q\boldsymbol{I})]\boldsymbol{x} = \boldsymbol{0}$. If $\boldsymbol{D}$ and $q$ are such that $\boldsymbol{D} + q\boldsymbol{I}$ is not singular, then according to lemma 3 we have for each zero $\widetilde{\mu}$ of $\widetilde{\beta}$

$$\Re(\lambda_s) \le \widetilde{\mu} \le \Re(\lambda_{s+1})$$

where $s$ is the number of positive diagonal elements of $\boldsymbol{D} - q\boldsymbol{I}$.
Now $\widetilde{\mu}$ itself is an eigenvalue, so either $\widetilde{\mu} = \lambda_s$ or $\widetilde{\mu} = \lambda_{s+1}$. There can be no eigenvalue strictly between $\Re(\lambda_s)$ and $\Re(\lambda_{s+1})$. Therefore the number of zeros of $\widetilde{\beta}$ is maximal 2. It follows that the equation $\beta(\tau) = p + q\tau$ has at most two real solutions.
We may not apply lemma 3 in the case that the diagonal matrix $\boldsymbol{D} + q\boldsymbol{I}$ happens to be singular. This is not a real problem, as will be shown now. Let $q$ change continuously and let it pass a critical value $-d_k$. The matrix $\widetilde{\boldsymbol{A}}$ changes continuously too, and so do the barrier value, being a simple eigenvalue. Therefore also the number of zeros of the barrier function changes continuously, and the existence of more than 2 distinct zeros could occur only in an isolated situation. But a truly isolated situation cannot happen, since, by changing $p$, we could lift the graph of $\widetilde{\beta}$ in such way that the isolated third zero would change in a finite (nonzero) number of ordinary simple zeros [3].

[3] If $d_k \to \pm 0$, one of the eigenvalues of $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{D}\boldsymbol{x}$ will be close to $a_{kk}/d_k$, i.e. will run to $\pm\text{sign}(a_{kk}) * \infty$. This eigenvalue is far away from the separator values.

**2. Behaviour for $|\tau| \to \infty$.** Next consider the asymptotic behaviour of the function $\widetilde{\beta}$ for $\tau \to \pm\infty$. The eigenvalues of $\widetilde{\boldsymbol{A}}(\tau)/\tau$ tend to the numbers $-d_k - q$. Therefore, the eigenvalues of $\widetilde{\boldsymbol{A}}(\tau)$ behave asymptotically as $-\tau(d_k + q)$, for $k = 1, 2, \ldots, m$. The barrier value $\widetilde{\beta}(\tau)$ is then close to the rightmost of these numbers, so

$$\widetilde{\beta}(\tau) \approx \begin{cases} -\tau(d_{\text{max}} + q), & \tau \to -\infty \\ -\tau(d_{\text{min}} + q), & \tau \to +\infty \end{cases} \tag{16}$$

Let $\widetilde{\boldsymbol{u}}(\tau)$ and $\widetilde{\boldsymbol{w}}(\tau)$ be eigenvector, respectively left eigenvector of $\widetilde{\boldsymbol{A}}(\tau)$, corresponding to the barrier value $\widetilde{\beta}(\tau)$:

$$[\widetilde{\boldsymbol{A}}(\tau) - \widetilde{\beta}(\tau)\boldsymbol{I}]\widetilde{\boldsymbol{u}}(\tau) \equiv \boldsymbol{0}, \quad \widetilde{\boldsymbol{w}}^T(\tau)[\widetilde{\boldsymbol{A}}(\tau) - \widetilde{\beta}(\tau)\boldsymbol{I}] \equiv \boldsymbol{0}$$

Differentiation the expression $\widetilde{\boldsymbol{w}}^T(\widetilde{\boldsymbol{A}}(\tau) - \widetilde{\beta}(\tau)\boldsymbol{I}]\widetilde{\boldsymbol{u}}$ with respect to $\tau$ then yields

$$\widetilde{\boldsymbol{w}}^T[\widetilde{\boldsymbol{A}}'(\tau) - \widetilde{\beta}'(\tau)]\widetilde{\boldsymbol{u}} = 0 \Longrightarrow \widetilde{\beta}'(\tau) = -\frac{\widetilde{\boldsymbol{w}}^T(\boldsymbol{D} + q\boldsymbol{I})\widetilde{\boldsymbol{u}}}{\widetilde{\boldsymbol{w}}^T\widetilde{\boldsymbol{u}}}$$

This being a Rayleigh quotient for the diagonal matrix $\boldsymbol{D}$, we find for all $\tau$:

$$-d_{\text{max}} - q \le \widetilde{\beta}'(\tau) \le -d_{\text{min}} - q \tag{17}$$

According to ( 16), we have

$$\lim_{\tau \to \pm\infty} \widetilde{\beta}'(\tau) = \begin{cases} -d_{\text{min}} - q \\ -d_{\text{max}} - q \end{cases} \tag{18}$$

**3. Convexity relation.** Let $a < b < c$, and let $y(\tau) = p + q\tau$ be the first degree interpolation polynomial for $\beta(\tau)$: Then $\widetilde{\beta}(\tau) = \beta(\tau) - p - q\tau$ is the error of interpolation. Now $\widetilde{\beta}(a) = \widetilde{\beta}(b) = 0$, and according to Rolle's theorem, for some $\xi \in [a, b]$ we have $\widetilde{\beta}'(\xi) = 0$. Using this in the inequalities ( 17), we get

$$-d_{\text{max}} - q \le 0 \le -d_{\text{min}} - q$$

Suppose the equality sign holds in the right inequality, i.e. $-d_{\text{min}} - q = 0$, then ( 17) implies that for all $\tau$, $\widetilde{\beta}'(\tau) \le 0$, and consequently $\widetilde{\beta}$ is a monotonically nonincreasing function. From $\widetilde{\beta}(a) = \widetilde{\beta}(b) = 0$ then follows $\widetilde{\beta}(\tau) = 0, \ \forall \tau \in [a, b]$. We already know this isn't possible, hence $-d_{\text{min}} - q > 0$.
From ( 16) then follows $\widetilde{\beta}(\tau) > 0$ if $\tau \to \infty$. Since $a$ and $b$ are the only zeros of $\widetilde{\beta}$, this implies $\widetilde{\beta}(c) > 0$.
Now using the explicit formula for $\widetilde{\beta}(c)$, this can be expressed as:

$$\widetilde{\beta}(c) = \beta(c) - \beta(a) - \frac{c - a}{b - a}[\beta(b) - \beta(a)] > 0$$

which is equivalent to

$$\beta(b) < \frac{c-b}{c-a}\beta(a) + \frac{b-a}{c-a}\beta(c)$$

This holds for each triple with $a < b < c$, hence $\beta$ is a convex function. □

We next analyse the multiplicity of the separating eigenvalues.

**Lemma 5** *Let $\boldsymbol{A}$ be a B-matrix, let $\boldsymbol{D}$ be a diagonal matrix and $\mu$ be a separator for $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{D}\boldsymbol{x}$. Then $\mu$ is simple if and only if $\mu$ is a simple zero of the barrier function $\beta(\tau)$ for $\{\boldsymbol{A}, \boldsymbol{D}\}$*

**Proof:** Let $\boldsymbol{u}$ and $\boldsymbol{w}$ be respectively positive eigenvector and left eigenvector corresponding to $\mu$. Assume $\boldsymbol{x} \neq \boldsymbol{0}$ is another eigenvector, so

$$(\boldsymbol{A} - \mu\boldsymbol{D})\boldsymbol{x} = \boldsymbol{0}$$

Then $\boldsymbol{x}$ is also an (ordinary) eigenvector of $\boldsymbol{A} - \mu\boldsymbol{D}$, corresponding to the barrier value $\beta(\mu) = 0$ of this matrix. Since barrier eigenvalues are simple, this can only be true if $\boldsymbol{x} = \alpha\boldsymbol{u}$ for some $\alpha$.
So the geometric multiplicity of a separator is 1. However, the separator eigenvalue could be defective, causing the algebraic multiplicity to be higher.
Assume $\mu$ is defective, then $\boldsymbol{x} \neq \boldsymbol{0}$ exists such that $(\boldsymbol{A} - \mu\boldsymbol{D})\boldsymbol{x} = \boldsymbol{D}\boldsymbol{u}$, where $(\boldsymbol{A} - \mu\boldsymbol{D})\boldsymbol{u} = \boldsymbol{0}$. Necessary and sufficient for the eigenvalue $\mu$ to be defective is the condition that $\boldsymbol{D}\boldsymbol{u}$ is in the column space of $\boldsymbol{A} - \mu\boldsymbol{D}$. This is equivalent to the condition that $\boldsymbol{D}\boldsymbol{u}$ is perpendicular to the left nullspace of $\boldsymbol{A} - \mu\boldsymbol{D}$.

$$\boldsymbol{D}\boldsymbol{u} \perp \boldsymbol{x},\ \forall \boldsymbol{x} \in \mathit{Null}(\boldsymbol{A}^T - \mu\boldsymbol{D})$$

Now $\mathit{Null}(\boldsymbol{A}^T - \mu\boldsymbol{D})$ is spanned by left eigen vector $\boldsymbol{w}$, hence $\mu$ is defective if and only if $\boldsymbol{w}^T\boldsymbol{D}\boldsymbol{u} = 0$.
Differentiating the identity $\boldsymbol{w}^T(\tau)(\boldsymbol{A} - \tau\boldsymbol{D} - \beta(\tau)\boldsymbol{I})\boldsymbol{u}(\tau) = 0$ with respect to $\tau$, we get

$$\boldsymbol{w}^T(\boldsymbol{D} + \beta'(\tau)\boldsymbol{I})\boldsymbol{u} = 0 \Longrightarrow \beta'(\tau) = -\frac{\boldsymbol{w}^T\boldsymbol{D}\boldsymbol{u}}{\boldsymbol{w}^T\boldsymbol{u}}$$

Hence $\mu$ is defective if and only if $\beta'(\mu) = 0$. This proves the lemma □

# 3 Kosten's conjectures

First we analyse the relevance of irreducibility for an infinitisimal generators for a Markov proces.

**Lemma 6** *Let $\boldsymbol{M}$ be a a matrix as described in ( 2). Then the Markov process $\boldsymbol{p}'(t) = \boldsymbol{M}\boldsymbol{p}(t)$ has a unique positive stationary solution $\boldsymbol{p}^\infty$, satisfying $\mathbf{1}^T\boldsymbol{p}^\infty = 1$, if and only if the matrix $\boldsymbol{M}$ is irreducible.*

**Proof:** The "if part" is an immediate consequence of lemma 2, since $M$ is a $B$-matrix whenever it is irreducible. So any positive solution $\boldsymbol{x}$ must be a multiple of $\boldsymbol{p}^\infty$, at eigenvalue $\mu = 0$. The requirement $\mathbf{1}^T\boldsymbol{x} = 1$ then uniquely selects $\boldsymbol{p}^\infty$ itself. To prove the "only if part", suppose $\boldsymbol{M}$ were reducible, then for some permutation matrix $\boldsymbol{P}$

$$\boldsymbol{P}^{-1}\boldsymbol{M}\boldsymbol{P} = \begin{pmatrix} \boldsymbol{B}_{\mathrm{LL}} & \boldsymbol{B}_{\mathrm{LR}} \\ \boldsymbol{O} & \boldsymbol{B}_{\mathrm{RR}} \end{pmatrix}$$

Partitioning the stationary solution $\boldsymbol{p}^\infty$ and the vector $\mathbf{1}$ similarly:

$$\boldsymbol{P}^{-1}\boldsymbol{p}^\infty = \begin{pmatrix} \boldsymbol{p}_{\mathrm{L}} \\ \boldsymbol{p}_{\mathrm{R}} \end{pmatrix} \quad \boldsymbol{P}^{-1}\mathbf{1} = \begin{pmatrix} \mathbf{1}_{\mathrm{L}} \\ \mathbf{1}_{\mathrm{R}} \end{pmatrix}$$

The equations $\boldsymbol{M}\boldsymbol{p}^\infty = \mathbf{0}$ and $\mathbf{1}^T\boldsymbol{M} = \mathbf{0}^T$ then read as

$$\boldsymbol{B}_{\mathrm{LL}}\boldsymbol{p}_{\mathrm{L}} + \boldsymbol{B}_{\mathrm{LR}}\boldsymbol{p}_{\mathrm{R}} = \mathbf{0}_{\mathrm{L}}, \quad \boldsymbol{B}_{\mathrm{RR}}\boldsymbol{p}_{\mathrm{R}} = \mathbf{0}_{\mathrm{R}}$$

$$\mathbf{1}_{\mathrm{L}}^T\boldsymbol{B}_{\mathrm{LL}} = \mathbf{0}_{\mathrm{L}}^T, \quad \mathbf{1}_{\mathrm{L}}^T\boldsymbol{B}_{\mathrm{LR}} + \mathbf{1}_{\mathrm{R}}^T\boldsymbol{B}_{\mathrm{RR}}^T = \mathbf{0}_{\mathrm{R}}^T$$

Hence, both $\boldsymbol{B}_{\mathrm{LL}}$ and $\boldsymbol{B}_{\mathrm{RR}}$ are singular. So a nonzero vector $\boldsymbol{q}_{\mathrm{L}}$ exists such that $\boldsymbol{B}_{\mathrm{LL}}\boldsymbol{q}_{\mathrm{L}} = \mathbf{0}_{\mathrm{L}}$. Define $\boldsymbol{q}$ by

$$\boldsymbol{P}^{-1}\boldsymbol{q} = \begin{pmatrix} \boldsymbol{q}_{\mathrm{L}} \\ \mathbf{0}_{\mathrm{R}} \end{pmatrix}$$

then $\boldsymbol{q}$ is a different nonzero solution to $\boldsymbol{M}\boldsymbol{p} = \mathbf{0}$. Hence, we can choose nonzero numbers $\alpha$ and $\beta$ for which $\widetilde{\boldsymbol{p}} = \alpha\boldsymbol{p}^\infty + \beta\boldsymbol{q}$ is a different positive stationary solution for the Markov process, satisfying $\mathbf{1}^T\widetilde{\boldsymbol{p}} = 1$. This being a contradiction, we conclude $\boldsymbol{M}$ is irreducible. □

We now can prove the three conjectures stated in the introduction.

**Theorem 2** *Let $\boldsymbol{M}$, $\boldsymbol{\Gamma}$ be as described in ( 2) and ( 4), and such that $\boldsymbol{p}'(t) = \boldsymbol{M}\boldsymbol{p}(t)$ has a unique stationary solution $\boldsymbol{p}^\infty$ with $\mathbf{1}^T\boldsymbol{p}^\infty = 1$.*
*Let the interval of relevance be defined as in ( 5):*

$$\mathcal{R} = \left(\mathbf{1}^T\boldsymbol{\Gamma}\boldsymbol{p}^\infty \;,\; \gamma_{\max}\right)$$

*Then $\boldsymbol{M}$ is a $B$-matrix, and for the generalized eigenproblem $\boldsymbol{M}\boldsymbol{x} = \lambda(\boldsymbol{\Gamma} - c\boldsymbol{I})\boldsymbol{x}$ the following statements hold as long as $c$ in the interval of relevance:*

1. *The number of eigenvalues in the left halfplane equals the number of $\gamma_j > c$. (**Kosten's consistency conjecture**)*

2. *The rightmost eigenvalue in the left halfplane is simple, and there is a strictly positive corresponding eigenvector. (***Kosten's dominancy conjecture***)*

3. *The left dominant eigenvalue is a monotonically decreasing function of $c$. (***Kosten's monotonicity conjecture***).*

**Proof:**
The matrix $\boldsymbol{M}$ is irreducible according to lemma 6, and because $M_{k,l} \geq 0$, whenever $k \neq l$, $\boldsymbol{M}$ is a $B$-matrix. Let $\boldsymbol{D} = \boldsymbol{\Gamma} - c\boldsymbol{I}$, then $\mu = 0$ is a separator value corresponding to the positive eivenvector $\boldsymbol{u} = \boldsymbol{p}^{\infty}$. A strictly positive left eigenvector is $\boldsymbol{w} = \mathbf{1}$.
Let the eigenvalues of $\boldsymbol{D}^{-1}\boldsymbol{M}$ be 'ordered' according to increasing real parts, then the separation lemma 3, states

$$\Re(\lambda_s) \leq 0 \leq \Re(\lambda_{s+1})$$

where $s$ denotes the number of negative values for $M_{kk}/d_k$.
The diagonal elements $M_{kk}$ are stricly negative, otherwise $\boldsymbol{M}$ would have a zero row, contradicting its irreducibility. Therefore $s$ equals the number of *positive* elements of $\boldsymbol{D}$, that is *the number of filling rates exceeding the depletion rate*:

$$\widetilde{\gamma}_m \geq \widetilde{\gamma}_{m-1} \geq \cdots \geq \widetilde{\gamma}_{m-s+1} > c > \widetilde{\gamma}_{m-s} \geq \cdots \geq \widetilde{\gamma}_1$$

Where $\{\widetilde{\gamma}_k\}$ are the filling rates in increasing order.
We must determine wether the eigenvalue 0 belongs to the 'left' family or to the right. Now $c$ is in the interval of relevance, i.e. $\mathbf{1}^T\boldsymbol{\Gamma}\boldsymbol{p}^{\infty} < c < \gamma_m$, and hence $\mathbf{1}^T\boldsymbol{D}\boldsymbol{p}^{\infty} < 0$. Let $\beta(\tau)$ be the barrier value of $[\boldsymbol{M} - \tau\boldsymbol{D}]\boldsymbol{u} = \beta(\tau)\boldsymbol{u}$, then according to the convexity lemma 4, We have $\beta(0) = 0$, and

$$\beta'(0) = -\frac{\mathbf{1}^T\boldsymbol{D}\boldsymbol{p}^{\infty}}{\mathbf{1}^T\boldsymbol{p}^{\infty}} = -\mathbf{1}^T\boldsymbol{D}\boldsymbol{p}^{\infty} > 0$$

Therefore $\beta(\tau)$ is strictly increasing in a neighbourhood of $\tau = 0$, so $\beta(\tau) < 0$ for some $\tau < 0$.
On the other hand we have $d_{\text{max}} = d_m = \gamma_m - c > 0$, implying

$$\lim_{\tau \to -\infty} \beta(\tau) = +\infty$$

Therefore $\beta(\tau)$ must have another zero $\mu < 0$, and according to the separation lemma

$$\lambda_s = \mu < 0 = \lambda_{s+1}$$

This implies the consistency conjecture as well as the dominancy conjecture.

Finally the monotonicity conjecture. Since the separator values are simple for values of $c$ in the interval of relevance, they are continuously differentiable functions of $c$. Let $\boldsymbol{u}$ and $\boldsymbol{w}$ be respectively a positive eigenvector and a positive left eigenvector:

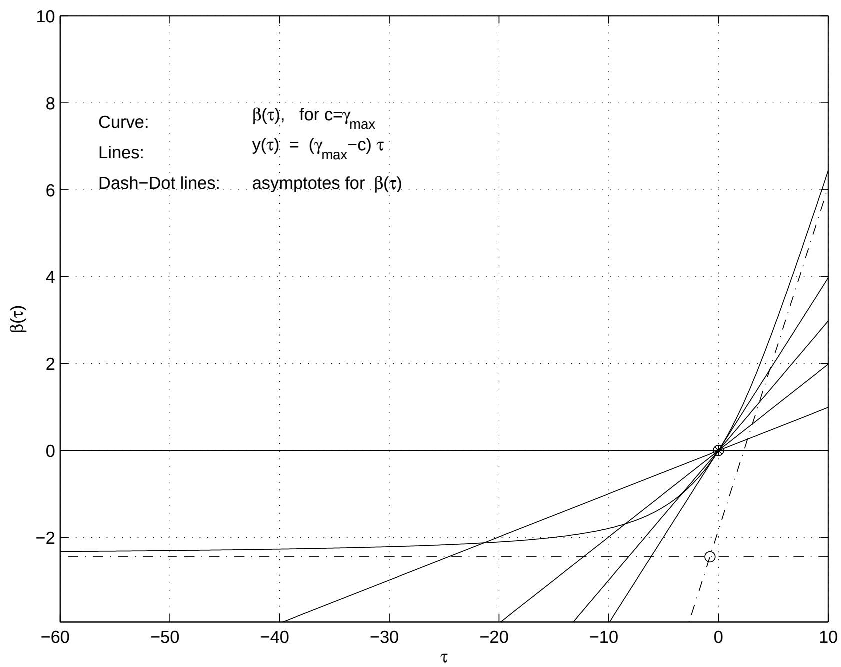
$$(\boldsymbol{M} - \mu \boldsymbol{D})\boldsymbol{u} = \boldsymbol{0}, \quad \boldsymbol{w}^T(\boldsymbol{M} - \mu \boldsymbol{D}) = \boldsymbol{0}^T$$

With $\boldsymbol{D} = \boldsymbol{\Gamma} - c\boldsymbol{I}$, the left separator is a function of $c$. Differentiate the identity $\boldsymbol{w}^T(\boldsymbol{M} - \mu(c)\boldsymbol{D}(c))\boldsymbol{u} = 0$ with respect to $c$, we get the identity

$$\boldsymbol{w}^T(-\mu'(c)\boldsymbol{D} + \mu(c)\boldsymbol{I})\boldsymbol{u} = 0 \Longrightarrow \mu'(c) = \mu(c)\frac{\boldsymbol{w}^T\boldsymbol{u}}{\boldsymbol{w}^T\boldsymbol{D}\boldsymbol{u}} = -\frac{\mu(c)}{\beta'(\mu)}$$

Since both $\beta'(\mu(c))$ and $\mu(c)$ are negative, it follows $\mu'(c) < 0$. This is the monotonicity statement. □

Figure 1: Graphical construction of separator values.




**Example.** In the above illustration, the barrier function $\widehat{\beta}$ is plotted for a random $5 \times 5$ matrix, satisfying ( 2), and a diagonal matrix $\boldsymbol{D} = \boldsymbol{\Gamma} - \gamma_{\max}\boldsymbol{I}$, where $\gamma_{\max}$ is the triviality bound of the interval of relevance.
The straight lines that are plotted, have slopes [4] that can be interpreted as $\gamma_{\max} - c$. The abcissa of the intersecttion points are the left dominant eigenvalues corresponding to the $c$-values.
The asymptotic behaviour of $\widehat{\beta}$ for $|\tau|$ large is plotted in a dash-dot pattern, according to the formula

$$\beta(\tau) = -\tau d_s + a_{ss}$$

where $d_s$ are the minimum and maximum values respectively for the entries of $\boldsymbol{D} = \boldsymbol{\Gamma} - \gamma_{\max}\boldsymbol{I}$
The calculations were carried out in Matlab, and it will not be difficult to extend the (small) program for more realistic purposes.

[4] Take the anisotropy of the picture into account!

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## References

[1] D. Anick, D. Mitra, M.M. Sondhi: *Stochastic theory of a data- handling system with multiple sources;* BSTJ **61,** 1871-1894, 1982

[2] L. Kosten: *Liquid Models for a Type of Information Buffer Problems;* Delft Progress Report **11,** 71-86 1986

[3] E.C. Titchmarsh: *The Theory of Functions;* Oxford University Press, London, 2nd edition, 1939-1960

[4] R.S. Varga: *Matrix Iterative Analysis;* Prentice-Hall, Eaglewood Cliffs, New Jersey, 1962