On the in-plane response of an inclined stretched string due to a forcing at one of the boundaries

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abstract

The longitudinal and the transversal in-plane displacements of an inclined stretched string are studied. At one end of the string a parametrical and transversal excitation is applied and at the other end the string is kept fixed. By applying Kirchhoff's approach the coupled system of partial differential equations (PDEs) to describe the in-plane displacements of the string is reduced to a single PDE. The effect of gravity and of the external excitation on the in-plane displacements of the string are studied in detail. Complicated internal resonances can occur when the excitation-frequency is near an eigenfrequency of the linearized system. The existence and the stability of time-periodic solutions are investigated.

1 Introduction

In recent years the spans of the cable-stayed bridges become longer and longer, for instance the Akashi-Kaiko Bridge with three spans (960 m - 1991 m - 960 m) and the planned bridge in the Messina Straits with three spans (960 m - 3300 m - 810 m). By increasing the span lengths of the bridges the stay cables can become more prone to vibrations due to wind, rain, traffic, and so on. Hence, cable vibrations are of great interest to engineers. Pinto da Costa et.al. [1] studied the oscillations of the cable stays induced by periodic motions of the deck and/or towers. In [1] it has been shown that reasonably small anchoraged amplitudes may lead to important cable oscillations when conditions are met for lower-order classical or parametrical resonance of the cables. Jones and Scanlan [2] studied the wind effects on cable-supported bridges and provided an overview of the basic steps in the process of a typical aerodynamic analysis and design. Other papers related to this subject can

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for instance be found in [3, 4, 5].
The non-linear responses of suspended cables to primary resonance excitations have been studied by Arafat and Nayfeh [6]. Srinil et al. [7] analyzed numerically the large-amplitude free vibrations of a suspended cable having a small sag-to-span ratio. Many interesting results have been found. One of these results is that the cable vibrations describe qualitatively multi-harmonic responses due to geometric non-linearities in case the cable sag is significant. However, in both papers [6, 7], a parametric resonance is not involved. On the other hand, in long-span cable-stayed bridges the parametric excitation will be very probable due to the presence of many low frequencies in the cable stays as has been shown in [1, 8, 9].

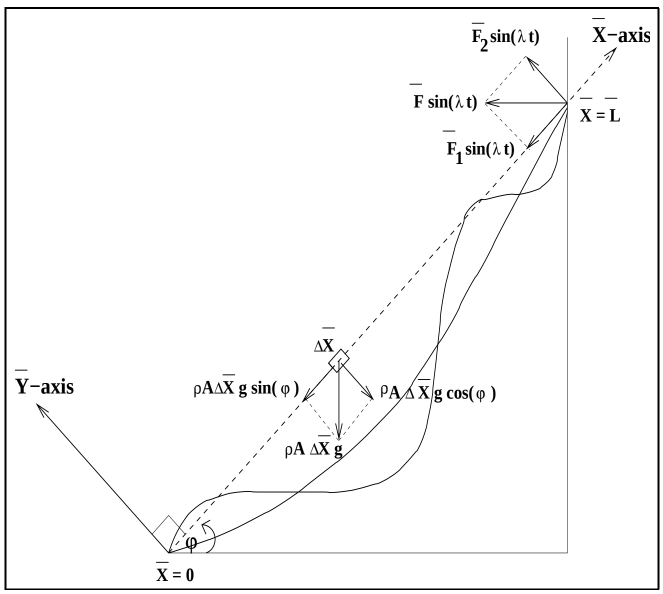


Figure 1: The inclined string in the dynamic state with a parametrical and transversal excitation at $\bar{X} = \bar{L}$.

In a recent paper by Caswita and van der Burgh [10], a perfectly flexible elastic stretched string which is fixed at one end and which is attached to a vibrating support at the other hand was considered as a model. Gravity effects were neglected. This model can be used as a starting point for the study of the dynamics of inclined cable-stays connecting the bridge deck and a pylon of a cable-stayed bridge as for example the Erasmus bridge in Rotterdam, The Netherlands. It follows from [1, 6, 11] that gravity influences the equilibrium state of the string and the dynamics of the string as well. Hence, the model that will be studied in this paper seems to be more applicable to study the dynamics of cables in cable-stayed-bridge than the model as given in [10]. The mathematical model for this system is given by (see [12]):

$$U_{\bar{\tau}\bar{\tau}}(X,\bar{\tau}) - \frac{E}{\rho}\frac{\partial}{\partial X}\Big[- \frac{1+\omega_o+\frac{\rho X g\sin(\varphi)}{E}+U_X(X,\bar{\tau})}{\sqrt{\Big[1+\omega_o+\frac{\rho X g\sin(\varphi)}{E}+U_X(X,\bar{\tau})\Big]^2+V_X^2(X,\bar{\tau})}} +$$

$$U_X(X,\bar{\tau})\Big] = -\bar{\alpha}_1 U_{\bar{\tau}},$$

$$V_{\bar{\tau}\bar{\tau}}(X,\bar{\tau}) - \frac{E}{\rho}\frac{\partial}{\partial X}\Big[- \frac{V_X(X,\bar{\tau})}{\sqrt{\Big[1+\omega_o+\frac{\rho X g\sin(\varphi)}{E}+U_X(X,\bar{\tau})\Big]^2+V_X^2(X,\bar{\tau})}} +$$

$$V_X(X,\bar{\tau})\Big] = -\bar{\alpha}_2 V_{\bar{\tau}} - g\cos(\varphi), \tag{1.1}$$

with boundary conditions:

$$U(0,\bar{\tau}) = V(0,\bar{\tau}) = 0,$$
$$U(L,\bar{\tau}) = \bar{F}_1\sin(\bar{\lambda}\bar{\tau}) \quad , \quad V(L,\bar{\tau}) = \bar{F}_2\sin(\bar{\lambda}\bar{\tau}), \tag{1.2}$$

where $\omega_o = \frac{T_o}{AE}$, U and V are the displacements in $\bar{X}$-direction and $\bar{Y}$-direction respectively (see also Fig. 1), U_X and V_X are the derivatives of $U(X,\bar{\tau})$ and $V(X,\bar{\tau})$ with respect to X, where X is the coordinate along the string describing the unstretched position (see also [12]), where $\bar{F}_1$ and $\bar{F}_2$ are the amplitudes of the external force applied at $\bar{X} = \bar{L}$, and $\bar{\lambda}$ is the excitation-frequency, where $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are positive damping coefficients (in $\frac{1}{second}$), and where the other symbols T_o, A, E, ρ, g, and φ are defined in [12].

This paper is organized as follows. In section 2 it is assumed that the dimensionless parameters $\epsilon = \frac{\rho g L}{E}$ and $\omega_o = \frac{T_o}{AE}$ satisfy the following relationship $0 < \epsilon < \omega_o$. By using this assumption and by applying Kirchhoff's approach the system of two PDEs (1.1) can be reduced to a single PDE describing the dynamic state of the string. In section 3 this single PDE is transformed into an infinite dimensional system of ordinary differential equations (ODEs). The averaging method is used to study this system of ODEs. It will turn out that complicated internal resonances can occur when the excitation frequency λ is near an eigenfrequency of the linearized system. In section 4 these resonance-cases will be studied in detail, and the existence of stable periodic solutions will be discussed. Finally, in section 5 some conclusions will be drawn and some remarks will be made.

2 Derivation of the dynamic state equation

In applications the oscillations of a string or cable are studied around its equilibrium (static) state. Now it will be assumed that $0 < \epsilon = \frac{\rho g L}{E} << \omega_o = \frac{T_o}{AE}$. This implies that the static state $(\hat{U}(X), \hat{V}(X))$ of (1.1) can be approximated up to $O(\tilde{\epsilon}^2)$ (see: [11, 12]) by:

$$\hat{U}_{ap}(X) = -\frac{\rho^2 g^2 A^2\cos^2(\varphi)}{12T_o^2}X(X-L)(2X-L),$$
$$\hat{V}_{ap}(X) = \frac{\rho g A\cos(\varphi)}{2T_o}X(X-L)\Big[1+\omega_o-\frac{\rho g A}{6T_o}(4X+L)\sin(\varphi)\Big], \tag{2.1}$$

where $\tilde{\epsilon} = \frac{\rho g A L}{T_o} = \frac{\epsilon}{\omega_o}$ is small compared to 1. It can readily be seen from (2.1) that

$$\frac{d\hat{U}_{ap}(X)}{dX} = -\frac{1}{2}(\frac{d\hat{V}_{ap}(X)}{dX})^2 + \frac{1}{2L}\int_0^L(\frac{d\hat{V}_{ap}(X)}{dX})^2dX + O(\tilde{\epsilon}^3). \tag{2.2}$$

Now introduce the new variables $\bar{U}(X,\bar{\tau})$ and $\bar{V}(X,\bar{\tau})$ by:

$$U(X,\bar{\tau}) = \hat{U}(X) + \bar{U}(X,\bar{\tau}),$$
$$V(X,\bar{\tau}) = \hat{V}(X) + \bar{V}(X,\bar{\tau}), \tag{2.3}$$

then $U_X = \hat{U}_X + \bar{U}_X$, $V_X = \hat{V}_X + \bar{V}_X$, $U_{XX} = \hat{U}_{XX} + \bar{U}_{XX}$, $V_{XX} = \hat{V}_{XX} + \bar{V}_{XX}$, $U_{\bar{\tau}} = \bar{U}_{\bar{\tau}}$, $V_{\bar{\tau}} = \bar{V}_{\bar{\tau}}$, $U_{\bar{\tau}\bar{\tau}} = \bar{U}_{\bar{\tau}\bar{\tau}}$, and $V_{\bar{\tau}\bar{\tau}} = \bar{U}_{\bar{\tau}\bar{\tau}}$. By substituting (2.3) into (1.1), and after rescaling by

$$x = \frac{(1+\omega_o)X}{\bar{L}}, \tau = \bar{\tau}\sqrt{\frac{E}{\rho\bar{L}^2}}, \hat{u}(x) = \frac{\hat{U}(X)}{\bar{L}}, \hat{v}(x) = \frac{\hat{V}(X)}{\bar{L}}, \bar{u}(x,\tau) =$$
$$\frac{\bar{U}(X,\bar{\tau})}{\bar{L}}, \text{ and } \bar{v}(x,\tau) = \frac{\bar{V}(X,\bar{\tau})}{\bar{L}}, \tag{2.4}$$

one obtains that (1.1) becomes:

$$\bar{u}_{\tau\tau}(x,\tau) - (1+\omega_o)\frac{\partial}{\partial x}\Big[- \frac{\omega(x) + \hat{u}_x + \bar{u}_x}{\sqrt{(\omega(x)+\hat{u}_x+\bar{u}_x)^2 + (\hat{v}_x+\bar{v}_x)^2}} + (1+\omega_o)(\hat{u}_x +$$
$$\bar{u}_x)\Big] = -\bar{\alpha}_1\sqrt{\frac{\rho\bar{L}^2}{E}}\bar{u}_\tau,$$
$$\bar{v}_{\tau\tau}(x,\tau) - (1+\omega_o)\frac{\partial}{\partial x}\Big[- \frac{\hat{v}_x + \bar{v}_x}{\sqrt{(\omega(x)+\hat{u}_x+\bar{u}_x)^2 + (\hat{v}_x+\bar{v}_x)^2}} + (1+\omega_o)(\hat{v}_x +$$
$$\bar{v}_x)\Big] = -\bar{\alpha}_2\sqrt{\frac{\rho\bar{L}^2}{E}}\bar{v}_\tau - \frac{\rho g\bar{L}\cos(\varphi)}{E}, \tag{2.5}$$

where $\omega(x) = 1 + \frac{\rho g L}{E(1+\omega_o)}x\sin(\varphi)$, and where the boundary conditions (1.2) become:

$$\bar{u}(0,\tau) = \bar{v}(0,\tau) = 0,$$
$$\bar{u}(1,\tau) = \frac{\bar{F}_1}{\bar{L}}\sin\Big(\bar{\lambda}\sqrt{\frac{\rho\bar{L}^2}{E}}\tau\Big) \quad ; \quad \bar{v}(1,\tau) = \frac{\bar{F}_2}{\bar{L}}\sin\Big(\bar{\lambda}\sqrt{\frac{\rho\bar{L}^2}{E}}\tau\Big). \tag{2.6}$$

It follows from (2.2) that

$$\hat{u}_x = -\frac{1}{2}(1+\omega_o)\hat{v}_x^2 + \frac{1}{2}(1+\omega_o)\int_0^1 \hat{v}_x^2 dx + O(\tilde{\epsilon}^3). \tag{2.7}$$

System (2.5) represents the equations of motion around the equilibrium state (after a rescaling of the variables).
It will now be assumed that $|u_x|$ and $|v_x|$ are small with respect to 1. System (2.5) may then be approximated by the following system:

$$\bar{u}_{\tau\tau}(x,\tau) - (1+\omega_o)^2\bar{u}_{xx}(x,\tau) = (1+\omega_o)\frac{\partial}{\partial x}\Big[\frac{1}{\omega^2(x)}\Big(\hat{v}_x\bar{v}_x + \frac{1}{2}\bar{v}_x^2\Big) -$$
$$\frac{1}{\omega^3(x)}\Big(\hat{u}_x(2\hat{v}_x\bar{v}_x + \bar{v}_x^2) + \bar{u}_x(\hat{v}_x+\bar{v}_x)^2\Big) - \frac{1}{\omega^4(x)}\Big(\frac{3}{8}(4\hat{v}_x^3\bar{v}_x + 6\hat{v}_x^2\bar{v}_x^2 + 4\hat{v}_x\bar{v}_x^3 +$$

$$\bar{v}_x^4) - \frac{3}{2}\hat{u}_x^2(2\hat{v}_x\bar{v}_x + \bar{v}_x^2) - \frac{3}{2}(2\hat{u}_x\bar{u}_x + \bar{u}_x^2)(\hat{v}_x + \bar{v}_x)^2\Big) + h.o.t.\Big] - \bar{\alpha}_1\sqrt{\frac{\rho\bar{L}^2}{E}}\bar{u}_\tau,$$

$$\bar{v}_{\tau\tau}(x,\tau) - (1+\omega_o)\frac{\partial}{\partial x}\Big[(1+\omega_o - \frac{1}{\omega(x)})\bar{v}_x\Big] = (1+\omega_o)\frac{\partial}{\partial x}\Big[\frac{1}{\omega^2(x)}\Big(\hat{u}_x\bar{v}_x + \bar{u}_x(\hat{v}_x + \bar{v}_x)\Big) + \frac{1}{\omega^3(x)}\Big(\frac{1}{2}(3\hat{v}_x^2\bar{v}_x + 3\hat{v}\bar{v}_x^2 + \bar{v}_x^3) - \hat{u}_x^2\bar{v}_x - (2\hat{u}_x\bar{u}_x + \bar{u}_x^2)(\hat{v}_x + \bar{v}_x)\Big) - \frac{1}{\omega^4(x)}\Big(\frac{3}{2}\hat{u}_x(3\hat{v}_x^2\bar{v}_x + 3\hat{v}_x\bar{v}_x^2 + \bar{v}_x^3) + \frac{3}{2}\bar{u}_x(\hat{v}_x + \bar{v}_x)^3 - \hat{u}_x^3\bar{v}_x - (3\hat{u}_x^2\bar{u}_x + 3\hat{u}_x\bar{u}_x^2 + \bar{u}_x^3)(\hat{v}_x + \bar{v}_x)\Big) + h.o.t.\Big] - \bar{\alpha}_2\sqrt{\frac{\rho\bar{L}^2}{E}}\bar{v}_\tau, \tag{2.8}$$

where $h.o.t.$ stands for the terms of order six or higher in $\hat{u}_x$, $\hat{v}_x$, $\bar{u}_x$, and $\bar{v}_x$. Let $c_1^2 = \frac{E}{\rho}$ and $c_2^2 = \frac{E\omega_o}{\rho}$, and introduce the time-rescaling $t = \frac{c_2\tau}{c_1}\sqrt{(1+\omega_o)}$. Then $c_2^2 << c_1^2$, and $\omega(x)$ tends to 1 as $\omega_o \to 0$. Hence, in a first order approximation the first equation in (2.8) reduces to:

$$-\bar{u}_{xx}(x,t) = \frac{\partial}{\partial x}\Big[\hat{v}_x\bar{v}_x + \frac{1}{2}\bar{v}_x^2 - \hat{u}_x(2\hat{v}_x\bar{v}_x + \bar{v}_x^2) - \bar{u}_x(\hat{v}_x + \bar{v}_x)^2 - \frac{3}{8}(4\hat{v}_x^3\bar{v}_x + \frac{3}{8}(4\hat{v}_x^3\bar{v}_x + 6\hat{v}_x^2\bar{v}_x^2 + 4\hat{v}_x\bar{v}_x^3 + \bar{v}_x^4) + \frac{3}{2}\hat{u}_x^2(2\hat{v}_x\bar{v}_x + \bar{v}_x^2) + \frac{3}{2}(2\hat{u}_x\bar{u}_x + \bar{u}_x^2)(\hat{v}_x + \bar{v}_x)^2\Big]. \tag{2.9}$$

If one considers additionally Kirchhoff's approach that $\bar{u} = O(\bar{v}^2)$ and $\bar{v}$, $\hat{v}$ are $O(\omega_o)$, then (2.9) can be reduced to (up to $O(\omega_o)$):

$$-\bar{u}_{xx} = \frac{\partial}{\partial x}(\hat{v}_x\bar{v}_x + \frac{1}{2}\bar{v}_x^2). \tag{2.10}$$

By integrating (2.10) with respect to x and then by using the boundary conditions for $\bar{u}(x,\tau)$ one obtains

$$\bar{u}_x(x,t) = -(\hat{v}_x\bar{v}_x + \frac{1}{2}\bar{v}_x^2) + \int_0^1(\hat{v}_x\bar{v}_x + \frac{1}{2}\bar{v}_x^2)dx + \frac{\bar{F}_1}{\bar{L}}\sin\Big(\frac{\bar{\lambda}L\sqrt{(1+\omega_o)}}{c_2}t\Big). \tag{2.11}$$

Up till now no assumptions have been made about the amplitudes $\bar{F}_1$ and $\bar{F}_2$ of the external force applied at $x = 1$. It will be now assumed that $\frac{\bar{F}_1}{\bar{L}} = O(\omega_o^2)$ and $\frac{\bar{F}_2}{\bar{L}} = O(\omega_o^2)$. Because $0 < \epsilon << \omega_o$, we can set $\omega_o = \bar{\epsilon}$ and $\epsilon = O(\bar{\epsilon}^2)$. Substituting $\hat{v}(x) = \bar{\epsilon}\check{v}(x)$, $\bar{v}(x,t) = \bar{\epsilon}\tilde{v}(x,t)$, $\frac{\bar{F}_1}{\bar{L}} = \bar{\epsilon}^2F_1$, $\frac{\bar{F}_2}{\bar{L}} = \bar{\epsilon}^2F_2$ into the second equation of (2.8), by using (2.7), (2.11), and the boundary conditions for $\bar{v}(x,t)$, one obtains (up to order $\bar{\epsilon}^2$):

$$\tilde{v}_{tt}(x,t) - \tilde{v}_{xx}(x,t) = \bar{\epsilon}\Big[(\check{v}_{xx} + \tilde{v}_{xx})\Big(\int_0^1(\check{v}_x\tilde{v}_x + \frac{1}{2}\tilde{v}_x^2)dx + F_1\sin(\lambda t)\Big) + \frac{\partial}{\partial x}\Big([\omega_1 x\sin(\varphi) + p]\tilde{v}_x\Big) - \alpha\tilde{v}_t\Big], \quad 0 < x < 1, \quad t > 0,$$

$$\text{BC's}: \quad \tilde{v}(0,t) = 0, \quad \tilde{v}(1,t) = \bar{\epsilon}F_2\sin(\lambda t), \tag{2.12}$$

where $\omega_1 = \frac{\rho g A^2 LE}{T_o^2}$, $\lambda = \frac{\bar{\lambda}L}{c_2}$, $p = \frac{1}{2}\int_0^1 \check{v}_x^2 dx = \frac{1}{24}\omega_1^2\cos^2(\varphi)$, and $\alpha = \frac{\bar{\alpha}_2 L c_1^2}{c_2^3}$. To obtain a problem for $\tilde{v}$ with homogeneous boundary conditions, the following transformation is introduced:

$$\tilde{v}(x,t) = \bar{\epsilon}F_2 x\sin(\lambda t) + \vartheta(x,t). \tag{2.13}$$

Substitution of (2.13) into (2.12) (and by introducing initial conditions) yields (up to $O(\bar{\epsilon}^2)$):

$$\begin{aligned}
&\vartheta_{tt} - \vartheta_{xx} = \bar{\epsilon}\Big[(\check{v}_{xx} + \vartheta_{xx})\Big(\int_0^1 (\check{v}_x\vartheta_x + \frac{1}{2}\vartheta_x^2)dx + F_1\sin(\lambda t)\Big) + \frac{\partial}{\partial x}\Big([p + \\
&\qquad \omega_1 x\sin(\varphi)]\vartheta_x\Big) + \lambda^2 F_2 x\sin(\lambda t) - \alpha\vartheta_t\Big], \quad 0 < x < 1, \quad t > 0, \\
&\text{BC's:} \quad \vartheta(0,t) = \vartheta(1,t) = 0, \qquad t > 0, \\
&\text{IC's:} \quad \vartheta(x,0) = f(x), \quad \vartheta_t(x,0) = g(x), \quad 0 \le x \le 1.
\end{aligned} \tag{2.14}$$

In this paper the functions $f(x)$ and $g(x)$ are assumed to be sufficiently smooth such that a two times continuously differentiable solution for the initial-boundary value problem (2.14) exists. The case $\frac{\rho g AL}{T_o} \equiv 0$ has been studied in [10], that is, the case without gravity. Now, the case $\frac{\rho g AL}{T_o} = O(\bar{\epsilon})$ will be analyzed by using the averaging method.

3 A Perturbation method

The eigenfunctions of the Sturm-Liouville problem related to homogeneous, unperturbed problem (2.14) (i.e. $\bar{\epsilon} = 0$) are $\vartheta_n(x) = \tilde{q}_n\sin(\mu_n x)$, $\mu_n = n\pi$, and $\tilde{q}_n$ (for $n = 1, 2, 3, \ldots$) are constants. Based on the boundary conditions the exact solution $\vartheta(x,t)$ of (2.14) can be written in the form of an eigenfunction-series, that is,

$$\vartheta(x,t) = \sum_{n=1}^{\infty} q_n(t)\sin(\mu_n x). \tag{3.1}$$

By substituting (3.1) into the first equation of (2.14), by multiplying the so-obtained equation with $2\sin(\mu_k x)$, where k is any positive but fixed integer, and then by integrating the outcome over $x \in [0,1]$, one obtains an infinite-dimensional system for $q_k(t)$, $k = 1, 2, 3, \ldots$:

$$\begin{aligned}
\ddot{q}_k(t) + \mu_k^2 q_k(t) \;&=\; -\bar{\epsilon}\Big\{\Big[\mu_k^2 q_k(t) - b_k\Big]\Big[\sum_{m=1}^{\infty}\Big(\frac{1}{4}\mu_m^2 q_m^2(t) + a_m q_m(t)\Big) + F_1\sin(\lambda t)\Big] \\
&\quad + \mu_k^2\Big(p + \frac{1}{2}\omega_1\sin(\varphi)\Big)q_k(t) + \sum_{n\neq k}^{\infty}[d_{(n,k)} - c_{(n,k)}]q_n(t) + \alpha\dot{q}_k(t) + \\
&\quad (-1)^k\frac{2\lambda^2 F_2\sin(\lambda t)}{\mu_k}\Big\},
\end{aligned} \tag{3.2}$$

where a_m, b_k, $c_{(n,k)}$, and $d_{(n,k)}$ are defined in Appendix A. The overdot in (3.2) indicates differentiation with respect to t. The series in the right-hand side of (3.2) will be convergent for sufficiently smooth initial conditions (see also [13]).
The non-linear terms in (3.2) can lead to internal resonances. By considering the excitation-frequency it can readily be seen that there are two sorts of values of λ that can cause resonances. In fact these values are (in an $O(\bar{\epsilon})$ neighbourhood of) μ_n and $2\mu_s$, where n and s are integers (with n odd). The case $\lambda = \mu_n + O(\bar{\epsilon})$ corresponds to a so-called transversal resonance, whereas $\lambda = 2\mu_s + O(\bar{\epsilon})$ corresponds to a so-called parametrical resonance. Therefore, to study the existence of periodic solutions both cases should be considered. The following transformation $(q_k(t), \dot{q}_k(t)) \rightarrow (A_k(t), B_k(t))$ is introduced:

$$\begin{aligned} q_k(t) &= A_k(t)\sin(\mu_k t) + B_k(t)\cos(\mu_k t), \\ \dot{q}_k(t) &= \mu_k[A_k(t)\cos(\mu_k t) - B_k(t)\sin(\mu_k t)]. \end{aligned} \tag{3.3}$$

By substituting (3.3) into (3.2) and then by solving the equations for $\dot{A}_k(t)$ and $\dot{B}_k(t)$, yields

$$\begin{aligned} \dot{A}_k(t) &= -\bar{\epsilon}G_k(\mathbf{A}, \mathbf{B}; \varphi, \omega_1, \lambda, t)\cos(\mu_k t), \\ \dot{B}_k(t) &= \bar{\epsilon}G_k(\mathbf{A}, \mathbf{B}; \varphi, \omega_1, \lambda, t)\sin(\mu_k t), \end{aligned} \tag{3.4}$$

where $\mathbf{A} = (A_1, A_2, \ldots, A_k, \ldots)$, $\mathbf{B} = (B_1, B_2, \ldots, B_k, \ldots)$, and

$$\begin{aligned} G_k \;=\; & \frac{1}{\mu_k}\Big\{\Big(\mu_k^2[A_k\sin(\mu_k t) + B_k\cos(\mu_k t)] - b_k\Big)\Big[\sum_{m=1}^{\infty}\Big(\frac{\mu_m^2}{4}[A_m\sin(\mu_m t) + \\ & B_m\cos(\mu_m t)]^2 + a_m[A_m\sin(\mu_m t) + B_m\cos(\mu_m t)]\Big) + F_1\sin(\lambda t)\Big] + \mu_k^2(p + \\ & \frac{1}{2}\omega_1\sin(\varphi))[A_k\sin(\mu_k t) + B_k\cos(\mu_k t)] + \sum_{n\neq k}^{\infty}[d_{(n,k)} - c_{(n,k)}][A_n\sin(\mu_n t) + \\ & B_n\cos(\mu_n t)] + (-1)^k\frac{2\lambda^2F_2\sin(\lambda t)}{\mu_k} + \alpha\mu_k[A_k\cos(\mu_k t) - B_k\sin(\mu_k t)]\Big\}. \end{aligned}$$

The terms in the right-hand side of (3.4) are periodic functions in t. Hence, the functions $A_k(t)$ and $B_k(t)$ can be approximated by using the averaging method (see for instance [14]). For $\lambda \neq \mu_n + O(\bar{\epsilon})$ and $\lambda \neq 2\mu_s + O(\bar{\epsilon})$ (the cases that cause no resonances up to $O(\bar{\epsilon})$), the averaged equations of (3.2) are:

$$\begin{aligned} \dot{\bar{A}}_k &= -\frac{1}{2}\bar{\epsilon}\Big[\alpha\bar{A}_k + \mu_k\bar{B}_k\Big(\sum_{m\neq k}^{\infty}\frac{1}{8}\mu_m^2(\bar{A}_m^2 + \bar{B}_m^2) + \frac{3}{16}\mu_k^2(\bar{A}_k^2 + \bar{B}_k^2) + p + \\ & \qquad \frac{1}{2}\omega_1\sin(\varphi) + \frac{2a_k^2}{\mu_k^2}\Big)\Big], \\ \dot{\bar{B}}_k &= -\frac{1}{2}\bar{\epsilon}\Big[\alpha\bar{B}_k - \mu_k\bar{A}_k\Big(\sum_{m\neq k}^{\infty}\frac{1}{8}\mu_m^2(\bar{A}_m^2 + \bar{B}_m^2) + \frac{3}{16}\mu_k^2(\bar{A}_k^2 + \bar{B}_k^2) + p + \\ & \qquad \frac{1}{2}\omega_1\sin(\varphi) + \frac{2a_k^2}{\mu_k^2}\Big)\Big], \end{aligned} \tag{3.5}$$

where $\bar{A}_k$ and $\bar{B}_k$ are the averaged approximations of A_k and B_k respectively. It follows from (3.5) that for $\alpha > 0$ $R_k(t) = \bar{A}_k^2(t) + \bar{B}_k^2(t) = R_k(0)e^{(-\bar{\epsilon}\alpha t)} \to 0$ as $t \to \infty$. Hence, if $\bar{A}_k(0) = 0$ and $\bar{B}_k(0) = 0$ then $\bar{A}_k(t) = 0$ and $\bar{B}_k(t) = 0$ for $t > 0$. So, if there is no initial energy in the n-th mode then there will be no energy present up to $O(\bar{\epsilon})$ on a time-scale of order $\bar{\epsilon}^{-1}$. This shows that the truncation method can be applied to those modes that have non-zero initial energy. When $\lambda = \mu_n + O(\bar{\epsilon})$ with n an odd fixed number, extra terms come in due to the internal resonances. In this case the averaged equations for $\bar{A}_n$ and $\bar{B}_n$ are:

$$
\begin{aligned}
\dot{\bar{A}}_n &= -\frac{1}{2}\bar{\epsilon}\Big[\alpha\bar{A}_n + \mu_n\bar{B}_n\Big(\sum_{m\neq n}^{\infty}\frac{1}{8}\mu_m^2(\bar{A}_m^2 + \bar{B}_m^2) + \frac{3}{16}\mu_n^2(\bar{A}_n^2 + \bar{B}_n^2) + p + \\
&\quad \frac{1}{2}\omega_1\sin(\varphi) + \frac{2a_n^2}{\mu_n^2}\Big)\Big], \\
\dot{\bar{B}}_n &= -\frac{1}{2}\bar{\epsilon}\Big[\alpha\bar{B}_n - \mu_n\bar{A}_n\Big(\sum_{m\neq n}^{\infty}\frac{1}{8}\mu_m^2(\bar{A}_m^2 + \bar{B}_m^2) + \frac{3}{16}\mu_n^2(\bar{A}_n^2 + \bar{B}_n^2) + p + \\
&\quad \frac{1}{2}\omega_1\sin(\varphi) + \frac{2a_n^2}{\mu_n^2}\Big) + \frac{b_nF_1}{\mu_n} + 2F_2\Big],
\end{aligned}
\tag{3.6}
$$

whereas the equations for $\bar{A}_k$ and $\bar{B}_k$, $k \neq n$ are given by (3.5). From (3.5) and (3.6) it can readily be seen that only the n-th mode exists (that is, will usually not tend to zero) for a long time, while the other modes tend to zero as $t \to \infty$. When $\lambda = \mu_n + O(\bar{\epsilon}) = 2\mu_s + O(\bar{\epsilon})$, extra terms turn up in the equations for $\bar{A}_s$, $\bar{B}_s$, and $\bar{B}_n$. The averaged equations then become:

$$
\begin{aligned}
\dot{\bar{A}}_n &= -\frac{1}{2}\bar{\epsilon}\Big[\alpha\bar{A}_n + \mu_n\bar{B}_n\Big(\sum_{m\neq n}^{\infty}\frac{1}{8}\mu_m^2(\bar{A}_m^2 + \bar{B}_m^2) + \frac{3}{16}\mu_n^2(\bar{A}_n^2 + \bar{B}_n^2) + p + \\
&\quad \frac{1}{2}\omega_1\sin(\varphi)\Big)\Big], \\
\dot{\bar{B}}_n &= -\frac{1}{2}\bar{\epsilon}\Big[\alpha\bar{B}_n - \mu_n\bar{A}_n\Big(\sum_{m\neq n}^{\infty}\frac{1}{8}\mu_m^2(\bar{A}_m^2 + \bar{B}_m^2) + \frac{3}{16}\mu_n^2(\bar{A}_n^2 + \bar{B}_n^2) + p + \\
&\quad \frac{1}{2}\omega_1\sin(\varphi)\Big) - 2F_2\Big], \\
\dot{\bar{A}}_s &= -\frac{1}{2}\bar{\epsilon}\Big[(\alpha + \frac{1}{2}\mu_sF_1)\bar{A}_s + \mu_s\bar{B}_s\Big(\sum_{m\neq s}^{\infty}\frac{1}{8}\mu_m^2(\bar{A}_m^2 + \bar{B}_m^2) + \frac{3}{16}\mu_s^2(\bar{A}_s^2 + \\
&\quad \bar{B}_s^2) + p + \frac{1}{2}\omega_1\sin(\varphi) + \frac{2a_s^2}{\mu_s^2}\Big)\Big], \\
\dot{\bar{B}}_s &= -\frac{1}{2}\bar{\epsilon}\Big[(\alpha - \frac{1}{2}F_1)\mu_s\bar{B}_s - \mu_s\bar{A}_s\Big(\sum_{m\neq s}^{\infty}\frac{1}{8}\mu_m^2(\bar{A}_m^2 + \bar{B}_m^2) + \frac{3}{16}\mu_s^2(\bar{A}_s^2 + \\
&\quad \bar{B}_s^2) + p + \frac{1}{2}\omega_1\sin(\varphi) + \frac{2a_s^2}{\mu_s^2}\Big)\Big],
\end{aligned}
\tag{3.7}
$$

whereas the equations for $\bar{A}_k$ and $\bar{B}_k$ with $k \neq n$ and $k \neq 2s$ are given by (3.5). It follows from (3.5) and (3.7) that the n-th and the s-th modes will not decay to zero for $t \to \infty$, while the other modes will tend to zero as $t \to \infty$. Therefore, in the next section only the modes that will exist for a long time will be studied.

4 The cases that cause resonances ($\lambda = \mu_n + O(\bar{\epsilon})$)

As has been shown in section 3 resonances will occur when $\lambda = \mu_n + O(\bar{\epsilon})$. In section 4.1 the case $\lambda = \mu_n + \bar{\epsilon}\eta$ with n an odd, fixed number will be studied in which η is a detuning parameter. In this case the transversal excitation only excites the n-th mode while the parametrical excitation has no influence. In section 4.2 the case $\lambda = \mu_n + \bar{\epsilon}\eta$ with n an even, fixed number will be studied. Again η is a detuning parameter. In this case both the transversal and the parametrical excitation influence the interaction between the n-th mode and the s-th mode (with $n = 2s$).

4.1 The case $\lambda = \mu_n + \bar{\epsilon}\eta$, where n is an odd, fixed number

As has been mentioned in section 3: to study the long-time behaviour it is sufficient to study only the n-th mode behaviour. By substituting $q_k(t) = 0$ for all $k \neq n$ into (3.2) and then by setting $\lambda t = \bar{t}$, $\lambda = \mu_n + \bar{\epsilon}\eta$, one finds the following second order ordinary differential equation:

$$q_n''(\bar{t}) + q_n(\bar{t}) = -\frac{\bar{\epsilon}}{\mu_n}\Big\{\alpha q_k'(\bar{t}) + \frac{1}{4}\mu_n^3 q_n^3(\bar{t})\frac{3}{2}a_n q_n^2(\bar{t}) + \Big[\mu_n(p + \frac{1}{2}\omega_1\sin(\varphi) + F_1\sin(\bar{t})) + \frac{2a_n^2}{\mu_n} - 2\eta\Big]q_n(\bar{t}) - (2F_2 + \frac{b_nF_1}{\mu_n})\sin(\bar{t})\Big\} + O(\bar{\epsilon}^2), \tag{4.1}$$

where the prime ($'$)denotes differentiation with respect to $\bar{t}$. By using a transformation similar to (3.3), the averaged equation of (4.1) up to order $\bar{\epsilon}$ is:

$$\begin{aligned}
\bar{A}_n'(\bar{t}) &= -\frac{\bar{\epsilon}}{2\mu_n}\Big[\alpha\bar{A}_n + \bar{B}_n\Big(\frac{3}{16}\mu_n^3(\bar{A}_n^2 + \bar{B}_n^2) + \mu_n(p + \frac{1}{2}\omega_1\sin(\varphi)) + \frac{2a_n^2}{\mu_n} - 2\eta\Big)\Big],\\
\bar{B}_n'(\bar{t}) &= -\frac{\bar{\epsilon}}{2\mu_n}\Big[\alpha\bar{B}_n - \bar{A}_n\Big(\frac{3}{16}\mu_n^3(\bar{A}_n^2 + \bar{B}_n^2) + \mu_n(p + \frac{1}{2}\omega_1\sin(\varphi)) + \frac{2a_n^2}{\mu_n} - 2\eta\Big) + (2F_2 + \frac{b_nF_1}{\mu_n})\Big].
\end{aligned} \tag{4.2}$$

After the rescalings $\bar{A}_n = \sqrt{\frac{16\tilde{\alpha}}{3\mu_n^3}}\tilde{A}_n$ and $\bar{B}_n = \sqrt{\frac{16\tilde{\alpha}}{3\mu_n^3}}\tilde{B}_n$ (with $\tilde{\alpha} = 1$ for $\alpha = 0$, and $\tilde{\alpha} = \alpha$ for $\alpha > 0$), the critical points of (4.2) satisfy:

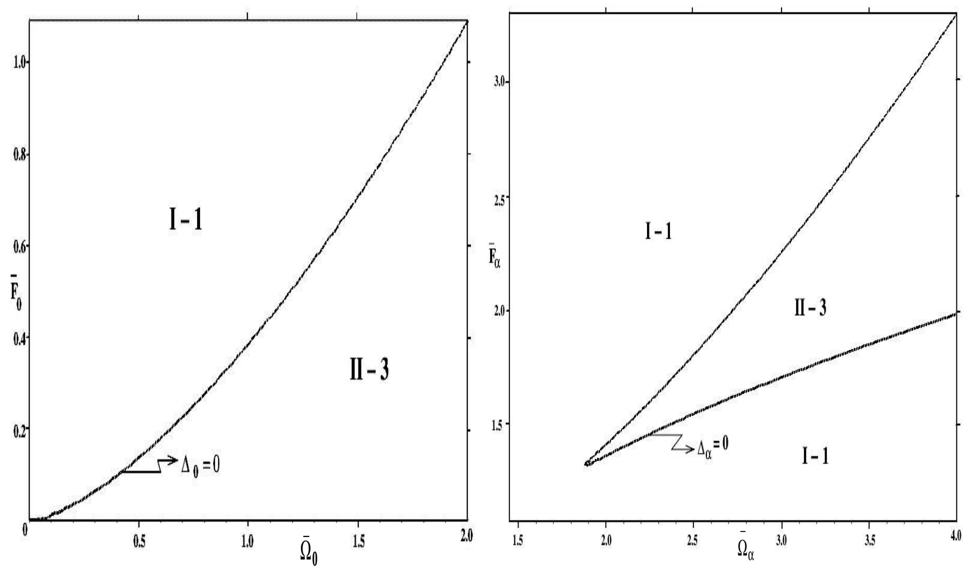
$$\begin{aligned}
&\frac{\alpha}{\tilde{\alpha}}\tilde{A}_n + \tilde{B}_n[\tilde{A}_n^2 + \tilde{B}_n^2 - \bar{\Omega}_\alpha] = 0,\\
&\frac{\alpha}{\tilde{\alpha}}\tilde{B}_n - \tilde{A}_n[\tilde{A}_n^2 + \tilde{B}_n^2 - \bar{\Omega}_\alpha] + \bar{F}_\alpha = 0,
\end{aligned} \tag{4.3}$$

where $\bar{\Omega}_\alpha = \frac{1}{\tilde{\alpha}}\Big[2\eta - \mu_n(p + \frac{1}{2}\omega_1\sin(\varphi)) - \frac{2a_n^2}{\mu_n}\Big]$ and $\bar{F}_\alpha = \sqrt{\frac{3\mu_n}{\tilde{\alpha}^3}}(\frac{1}{2}\mu_n F_2 + \frac{1}{4}b_n F_1)$. It should be observed that $\bar{\Omega}_\alpha$ and $\bar{F}_\alpha$ are arbitrary constants. However, it can be shown (later) that the case $\bar{F}_\alpha < 0$ gives the same results as the case $\bar{F}_\alpha > 0$. First, we consider $\alpha = 0$ (the case without damping). It follows from (4.3) with $\bar{F}_0 = 0$ that the critical points of (4.2) are the origin (0,0) and the points $(\sqrt{\frac{16}{3\mu_n^3}}\tilde{A}_n, \sqrt{\frac{16}{3\mu_n^3}}\tilde{B}_n)$, where $\tilde{A}_n$ and $\tilde{B}_n$ satisfy

$$\tilde{A}_n^2 + \tilde{B}_n^2 = \bar{\Omega}_0. \tag{4.4}$$

It can readily be seen from (4.4) that for $\bar{\Omega}_0 \leq 0$ the only critical point of (4.2) is the origin (0,0). If $F_0 \neq 0$ then the critical points of (4.2) are located in $(\sqrt{\frac{16}{3\mu_n^3}}\tilde{A}_n, 0)$, where $\tilde{A}_n$ satisfies:

$$\tilde{A}_n^3 - \bar{\Omega}_0\tilde{A}_n - \bar{F}_0 = 0. \tag{4.5}$$



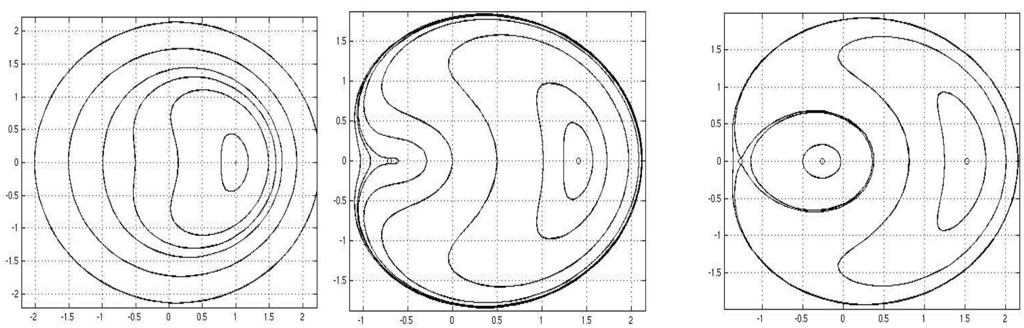
(i) $\alpha = 0$ (ii) $\alpha > 0$

Figure 2: The two domains of critical points of (4.2); $\Delta_0 = \frac{1}{4}\bar{F}_0^2 - \frac{1}{27}\bar{\Omega}_0^3$ and $\Delta_\alpha = \frac{1}{2916}(2\bar{\Omega}_\alpha^3 + 18\bar{\Omega}_\alpha - 27\bar{F}_\alpha^2)^2 + \frac{1}{27}(1 - \frac{1}{3}\bar{\Omega}_\alpha^2)^3$.

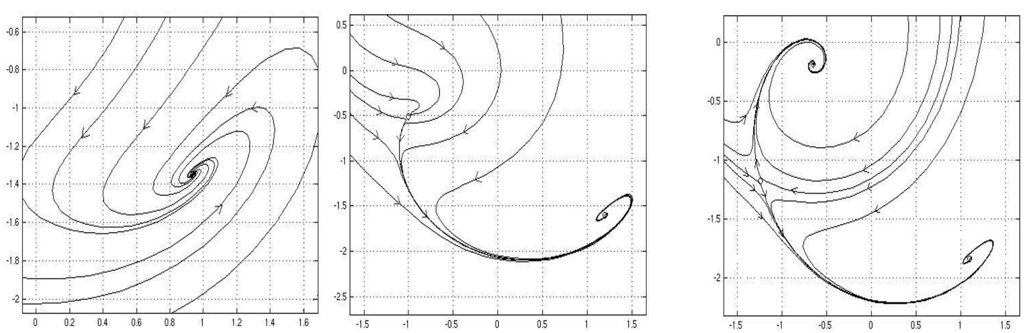
We now consider the case with positive damping, that is, $\alpha > 0$. If $\bar{F}_\alpha = 0$ then the only critical point of (4.2) is the origin (0,0) for all $\bar{\Omega}_\alpha$. Whereas for $\bar{F}_\alpha \neq 0$ the critical points of (4.2) are $(\sqrt{\frac{16\tilde{\alpha}}{3\mu_n^3}}\tilde{A}_n, \sqrt{\frac{16\tilde{\alpha}}{3\mu_n^3}}\tilde{B}_n)$, where $\tilde{A}_n = \frac{\bar{R}_n}{\bar{F}_\alpha}\Big(\bar{R}_n - \bar{\Omega}_\alpha\Big)$, $\tilde{B}_n = -\frac{\bar{R}_n}{\bar{F}_\alpha}$, and $\bar{R}_n$ satisfies (see Appendix B.1):

$$\tilde{R}_n^3 + (1 - \frac{1}{3}\bar{\Omega}_\alpha^2)\tilde{R}_n - \frac{1}{27}(27\bar{F}_\alpha^2 - 2\bar{\Omega}_\alpha^3 - 18\bar{\Omega}_\alpha) = 0,$$
$$\bar{R}_n = \tilde{R}_n + \frac{2}{3}\bar{\Omega}_\alpha, \tag{4.6}$$

where $\bar{R}_n = \tilde{A}_n^2 + \tilde{B}_n^2$. The number of critical points of (4.2) for the case $\alpha = 0$ and the case $\alpha > 0$ follow from the number of real solutions of (4.4), (4.5), and (4.6). The radicand Δ_0 of (4.5) and the radicand Δ_α of (4.6) are even functions with respect to $\bar{F}_0$ and $\bar{F}_\alpha$, respectively. This implies that the curves $\Delta_0 = 0$ and $\Delta_\alpha = 0$ are symmetric to the $\bar{\Omega}_0$-axis and the $\bar{\Omega}_\alpha$-axis, respectively. Hence, the domains of critical points of (4.2) for both cases $\alpha = 0$ and $\alpha > 0$ are only given for $\bar{F}_\alpha \geq 0$ (see Fig. 2). In Fig. 2 there are two domains: I-1 and II-3 with one or three critical points, respectively. On the curves $\Delta_0 = 0$ and $\Delta_\alpha = 0$ there are two critical points. On the non-positive $\bar{\Omega}_0$-axis there is one critical point (the origin), and on the positive $\bar{\Omega}_0$-axis there is an infinite number of critical points (see also (4.4)), whereas on the $\bar{\Omega}_\alpha$-axis there is only one critical point. Here, II-3 means that in this domain there exist three real critical points.



(i) $\bar{\Omega}_0 = 0.5$; $\bar{F}_0 = 0.5$. (ii) $\bar{\Omega}_0 = 1.5$; $\bar{F}_0 = \frac{\sqrt{2}}{2}$. (iii) $\bar{\Omega}_0 = 2$; $\bar{F}_0 = 0.5$.



(iv) $\bar{\Omega}_\alpha = 2$; $\bar{F}_\alpha = 2$. (v) $\bar{\Omega}_\alpha = 3.25$; $\bar{F}_\alpha = 2.5$. (vi) $\bar{\Omega}_\alpha = 4$; $\bar{F}_\alpha = 2.5$.

Figure 3: Phase portraits of system (4.2) for different values of $\bar{\Omega}_\alpha$ and $\bar{F}_\alpha$, where the horizontal and the vertical axis are the $\tilde{A}_n$-axis and the $\tilde{B}_n$-axis respectively. In (i)-(iii): $\alpha = 0$, and in (iv)-(vi): $\alpha > 0$.

The stability of the critical points will be analyzed by using the linearization method. Starting with the case $\alpha = 0$, it can readily be shown that for $\bar{F}_0 = 0$ the origin $(0, 0)$ is a stable point for $\bar{\Omega}_0 \neq 0$ (in the sense Lyapunov), and that it is a degenerate point for $\bar{\Omega}_0 = 0$. The points $(\sqrt{\frac{16}{3\mu_n^3}}\tilde{A}_n, \sqrt{\frac{16}{3\mu_n^3}}\tilde{B}_n)$ are degenerate points for $\bar{\Omega}_0 > 0$.

For $\bar{F}_0 \neq 0$ the eigenvalues of the coefficient matrix of the linearized system of (4.2) around the point $(\sqrt{\frac{16}{3\mu_n^3}}\tilde{A}_n, 0)$ are $\pm\frac{\bar{\epsilon}}{2\mu_n}\sqrt{-E(\tilde{A}_n)}$, where $E(\tilde{A}_n) = (\tilde{A}_n^2 - \bar{\Omega}_0)(3\tilde{A}_n^2 - \bar{\Omega}_0)$ and $\tilde{A}_n$ satisfies (4.5). This implies that the point $(\sqrt{\frac{16}{3\mu_n^3}}\tilde{A}_n, 0)$ will be unstable if $E(\tilde{A}_n) < 0$ and degenerate if $E(\tilde{A}_n) = 0$. Whereas for $E(\tilde{A}_n) > 0$ the stability of the point $(\sqrt{\frac{16}{3\mu_n^3}}\tilde{A}_n, 0)$ in the non-linear system (4.2) can not be determined from the linearized system. It should be observed that for $\alpha = 0$ (4.2) can be written as

$$\begin{aligned}\frac{d\bar{A}_n(\bar{t})}{d\bar{t}} &= -\frac{\bar{\epsilon}}{2\mu_n}\bar{B}_n(\bar{t})\Big(\frac{3}{16}\mu_n^3(\bar{A}_n^2(\bar{t}) + \bar{B}_n^2(\bar{t})) - \bar{\Omega}_0\Big),\\ \frac{d(\bar{A}_n^2(\bar{t}) + \bar{B}_n^2(\bar{t}))}{d\bar{t}} &= -\frac{2\bar{\epsilon}\bar{F}_0}{\mu_n^2\sqrt{3\mu_n}}\bar{B}_n(\bar{t}). \end{aligned} \tag{4.7}$$

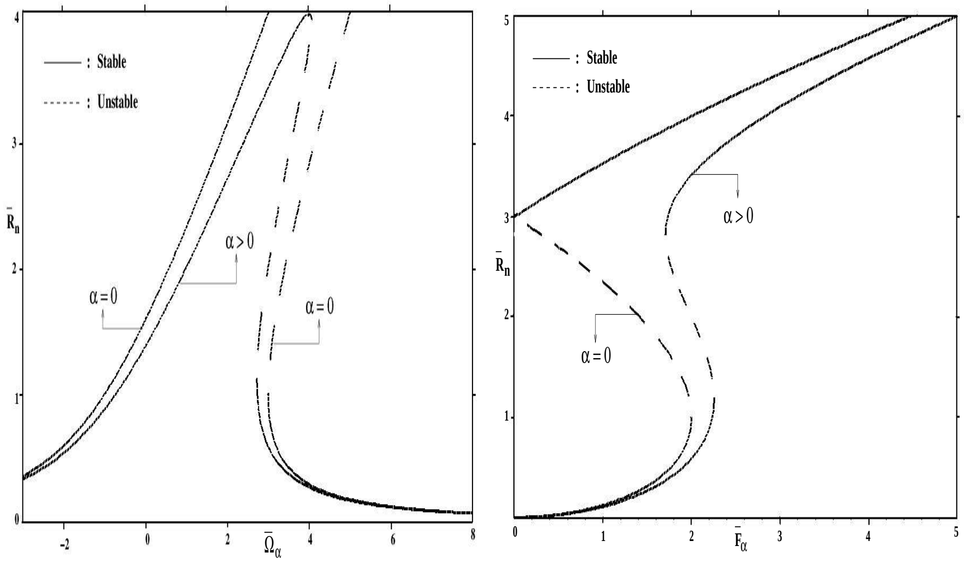
It follows from (4.7) that a Morse-function in a neighbourhood of the point $(\sqrt{\frac{16}{3\mu_n^3}}\tilde{A}_n, 0)$ (by using a first integral) can be defined by:

$$\begin{aligned}H(\bar{A}_n(\bar{t}), \bar{B}_n(\bar{t})) &= \frac{3}{32}\mu_n^3(\bar{A}_n^2(\bar{t}) + \bar{B}_n^2(\bar{t}))^2 - \bar{\Omega}_0(\bar{A}_n^2(\bar{t}) + \bar{B}_n^2(\bar{t})) -\\ &\quad \frac{8\bar{F}_0}{\mu_n\sqrt{3\mu_n}}\bar{A}_n(\bar{t}). \end{aligned} \tag{4.8}$$

The expansion of (4.8) in a neighbourhood of the point $(\sqrt{\frac{16}{3\mu_n^3}}\tilde{A}_n, 0)$ is given by

$$\begin{aligned}H(\bar{A}_n(\bar{t}), \bar{B}_n(\bar{t})) &= H(\sqrt{\frac{16}{3\mu_n^3}}\tilde{A}_n, 0) + (3\tilde{A}_n^2 - \bar{\Omega}_0)(\bar{A}_n - \sqrt{\frac{16}{3\mu_n^3}}\tilde{A}_n)^2 +\\ &\quad (\tilde{A}_n^2 - \bar{\Omega}_0)\bar{B}_n^2 + \text{h.o.t.}, \end{aligned} \tag{4.9}$$

where h.o.t. stands for the third or higher order terms in $(\bar{A}_n - \sqrt{\frac{16}{3\mu_n^3}}\tilde{A}_n)$ and $\bar{B}_n$. It can readily be observed from (4.9) that in the non-linear system the point $(\sqrt{\frac{16}{3\mu_n^3}}\tilde{A}_n, 0)$ is a centre (stable in the sense of Lyapunov) for $E(\tilde{A}_n) > 0$, and is a saddle point (unstable) for $E(\tilde{A}_n) < 0$. In domain I-1 in Fig. 2(i) (4.5) has one real solution: $x_{11} = (-\frac{1}{2}\bar{F}_0 + \sqrt{\Delta_0})^{\frac{1}{3}} - (\frac{1}{2}\bar{F}_0 + \sqrt{\Delta_0})^{\frac{1}{3}}$ and $E(x_{11}) > 0$. This implies that in domain I-1 in Fig. 2(i) system (4.4) has one stable point. On the curve $\Delta_0 = 0$ there are two solutions: $x_{21} = 2\sqrt{\frac{\bar{\Omega}_0}{3}}$ and $x_{22} = -\sqrt{\frac{\bar{\Omega}_0}{3}}$. Substituting these solutions into $E(\tilde{A}_n)$ yields $E(x_{21}) > 0$ and $E(x_{22}) = 0$. Thus, on this curve system (4.2) has a centre (stable) and a degenerate point (unstable). In domain II-3: $E(x_{31}) > 0$, $E(x_{33}) > 0$, while $E(x_{32}) < 0$, where $x_{3i} = 2\sqrt{\frac{\bar{\Omega}_0}{3}}\cos(\frac{\Psi_0 + 2(i-1)\pi}{3})$, for $i = 1, 2, 3$, and $\Psi_0 = \arccos(\frac{3\bar{F}_0}{2\sqrt{3\bar{\Omega}_0}})$. This implies that in domain II-3 system (4.2) has two stable points and an unstable one. For the case $\alpha > 0$ the results are similar to the case $\alpha = 0$. However, the stable points in this case are foci (asymptotically stable) and the degenerate point (0,0) is stable. As illustration the behaviour of the solutions is



(i) $\bar{F}_\alpha = 2$ (ii) $\bar{\Omega}_\alpha = 3$

Figure 4: The amplitude response-curves $\bar{R}_n = \tilde{A}_n^2 + \tilde{B}_n^2$ are given as function of the parameter $\bar{\Omega}_\alpha$ (with $\bar{F}_\alpha = 2$) and as function of $\bar{F}_\alpha$ (with $\bar{\Omega}_\alpha = 3$).

given in Fig. 3. This type of behaviour of solutions is well known (see also [15, 16]).

In Fig. 4 the amplitudes of the periodic solutions are plotted as function of $\bar{F}_\alpha$ and $\bar{\Omega}_\alpha$. In this figure it can be seen that the damping coefficient reduces the amplitudes of the periodic solutions. It should be noted that the parameter $\bar{\Omega}_\alpha$ depends on the detuning parameter of the excitation-frequency η, and on the gravitational acceleration g. Looking at Fig. 2 and at Fig. 4 it can readily be seen that gravity reduces the effect of the excitation-frequency. Another interesting result is that a jump up from a small amplitude solution to a large amplitude solution may occur for decreasing values of $\bar{\Omega}_\alpha$ or for increasing values of $\bar{F}_\alpha$.

4.2 The case $\alpha > 0$ and $\lambda = \mu_n + \bar{\epsilon}\eta$ with $n = 2s$, where s is a positive, fixed integer

In this case an interaction between the s-th mode and the n-th mode can be expected, while the other modes tend to zero for $\alpha > 0$. Hence, only the equations for $k = n$ and $k = s$ will be considered to study the behaviour for long times. By substituting $q_k(t) = 0$ for $k \neq n$ and $k \neq s$ into (3.2), and by introducing $\bar{t} = \frac{1}{2}\lambda t$, the following system of two coupled second order differential equations is obtained:

$$q_s''(\bar{t}) + q_s(\bar{t}) = -\frac{\bar{\epsilon}}{\mu_s}\Big\{\alpha q_s'(\bar{t}) + (\mu_s q_s(\bar{t}) - \frac{b_s}{\mu_s})[\mu_s^2 q_n^2(\bar{t}) + \frac{1}{4}\mu_s^2 q_s^2(\bar{t}) + F_1 \sin(2\bar{t})] +$$

$$a_s q_s(\bar{t}) + [\mu_s(p + \frac{1}{2}\omega_1 \sin(\varphi) - \eta]q_s(\bar{t}) + \frac{(d_{(n,s)} - c_{(n,s)})}{\mu_s} q_n(\bar{t}) +$$

$$(-1)^s 8F_2\sin(2\bar{t})\Big\},$$
$$q_n''(\bar{t})+4q_n(\bar{t}) = -\frac{\bar{\epsilon}}{\mu_s}\Big\{\alpha q_n'(\bar{t})+4\mu_s q_n(\bar{t})[\mu_s^2 q_n^2(\bar{t})+\frac{1}{4}\mu_s^2 q_s^2(\bar{t})+a_s q_s(\bar{t})+ F_1\sin(2\bar{t})+p+\frac{1}{2}\omega_1\sin(\varphi)-\frac{\eta}{\mu_s}]+\frac{(d_{(s,n)}-c_{(s,n)})}{\mu_s}q_s(\bar{t})+4F_2\sin(2\bar{t})\Big\}. \tag{4.10}$$

This system (4.10) actually describes a 1 : 2 resonance case. By using the transformation (3.3) and by applying the averaging method, it follows that the averaged system of (4.10) is given by:

$$\begin{aligned}
\bar{A}_n' &= -\frac{\bar{\epsilon}}{2\mu_s}\Big[\alpha\bar{A}_n+\bar{B}_n(\frac{3\mu_s^3}{2}(\bar{A}_n^2+\bar{B}_n^2)+\frac{\mu_s^3}{4}(\bar{A}_s^2+\bar{B}_s^2)+2\mu_s(p+\frac{\omega_1}{2}\sin(\varphi))- 2\eta)\Big],\\
\bar{B}_n' &= -\frac{\bar{\epsilon}}{2\mu_s}\Big[\alpha\bar{B}_n-\bar{A}_n(\frac{3\mu_s^3}{2}(\bar{A}_n^2+\bar{B}_n^2)+\frac{\mu_s^3}{4}(\bar{A}_s^2+\bar{B}_s^2)+2\mu_s(p+\frac{\omega_1}{2}\sin(\varphi))- 2\eta)-2F_2\Big],\\
\bar{A}_s' &= -\frac{\tilde{\epsilon}}{2\mu_s}\Big[(\alpha+\frac{1}{2}\mu_s F_1)\bar{A}_s+\bar{B}_s(\frac{\mu_s^3}{2}(\bar{A}_n^2+\bar{B}_n^2)+\frac{3\mu_s^3}{16}(\bar{A}_s^2+\bar{B}_s^2)+\mu_s(p+ \frac{\omega_1}{2}\sin(\varphi))+\frac{2a_s^2}{\mu_s}-\eta)\Big],\\
\bar{B}_s' &= -\frac{\bar{\epsilon}}{2\mu_s}\Big[(\alpha-\frac{1}{2}\mu_s F_1)\bar{B}_s-\bar{A}_s(\frac{\mu_s^3}{2}(\bar{A}_n^2+\bar{B}_n^2)+\frac{3\mu_s^3}{16}(\bar{A}_s^2+\bar{B}_s^2)+\mu_s(p+ \frac{\omega_1}{2}\sin(\varphi))+\frac{2a_s^2}{\mu_s}-\eta)\Big],
\end{aligned} \tag{4.11}$$

where $\bar{A}_n$, $\bar{B}_n$, $\bar{A}_s$, and $\bar{B}_s$ are the averaged approximations of A_n, B_n, A_s, and B_s respectively. By introducing the rescaling $\bar{A}_n=\sqrt{\frac{\tilde{F}_1}{\mu_s^2}}\tilde{A}_n$, $\bar{B}_n=\sqrt{\frac{\tilde{F}_1}{\mu_s^2}}\tilde{B}_n$, $\bar{A}_s=\sqrt{\frac{8\tilde{F}_1}{\mu_s^2}}\tilde{A}_s$, and $\bar{B}_s=\sqrt{\frac{8\tilde{F}_1}{\mu_s^2}}\tilde{B}_s$ it follows that the critical points of (4.11) satisfy:

$$\begin{aligned}
&\bar{\alpha}\tilde{A}_n+\tilde{B}_n[3(\tilde{A}_n^2+\tilde{B}_n^2)+4(\tilde{A}_s^2+\tilde{B}_s^2)-2\bar{K}]=0,\\
&\bar{\alpha}\tilde{B}_n-\tilde{A}_n[3(\tilde{A}_n^2+\tilde{B}_n^2)+4(\tilde{A}_s^2+\tilde{B}_s^2)-2\bar{K}]-\bar{M}=0,\\
&(\bar{\alpha}+\sigma)\tilde{A}_s+\tilde{B}_s[\tilde{A}_n^2+\tilde{B}_n^2+3(\tilde{A}_s^2+\tilde{B}_s^2)+\bar{\beta}_s-\bar{K}]=0,\\
&(\bar{\alpha}-\sigma)\tilde{B}_s-\tilde{A}_s[\tilde{A}_n^2+\tilde{B}_n^2+3(\tilde{A}_s^2+\tilde{B}_s^2)+\bar{\beta}_s-\bar{K}]=0,
\end{aligned} \tag{4.12}$$

where $\bar{\alpha}=\frac{2\alpha}{\mu_s\tilde{F}_1}$, $\bar{\beta}_s=\frac{4a_s^2}{\mu_s^2\tilde{F}_1}$, $\bar{K}=\frac{2}{\mu_s\tilde{F}_1}(\eta-\mu_s(p+\frac{1}{2}\omega_1\sin(\varphi))$, $\bar{M}=\frac{4F_2}{\tilde{F}_1\sqrt{\tilde{F}_1}}$, and $\sigma=\frac{F_1}{\tilde{F}_1}$ (with $\tilde{F}_1=1$ for $F_1=0$, and $\tilde{F}_1=|F_1|$ for $F_1\neq 0$). By considering the values of σ it should be observed that the solutions of (4.12) for $\sigma=-1$ can be determined from those for $\sigma=1$. This can be seen as follows: let $\{\tilde{A}_n^+,\tilde{B}_n^+,\tilde{A}_s^+,\tilde{B}_s^+\}$ and $\{\tilde{A}_n^-,\tilde{B}_n^-,\tilde{A}_s^-,\tilde{B}_s^-\}$ be solutions of (4.12) for $\sigma=1$ and $\sigma=-1$, respectively. Due to the symmetry of (4.12) these solutions satisfy the relations $\tilde{A}_n^- = \tilde{A}_n^+$, $\tilde{B}_n^- = \tilde{B}_n^+$, $\tilde{A}_s^- = -\tilde{B}_s^+$, and $\tilde{B}_s^- = \tilde{A}_s^+$. Moreover, the stability of those points is exactly the same. For $\sigma = 0$ it follows from the third and the fourth equation of (4.12) with $\bar{\alpha} \neq 0$ that the only solutions for $\tilde{A}_s$ and $\tilde{B}_s$ are $\tilde{A}_s = \tilde{B}_s = 0$. This implies that for $\sigma = 0$ the only periodic solution consists of one mode as studied in the previous subsection. Hence, only the case $\sigma = 1$ has to be considered. System (4.12) then contains four parameters: $\bar{\alpha}$, $\bar{\beta}_s$, $\bar{K}$, and $\bar{M}$. The dependence of the critical points on these parameters will be investigated. Clearly, the number of critical points of (4.11) is equal to the number of real solutions of (4.12). The number and the stability of the critical points of system (4.11) for a fixed, positive value of $\bar{\alpha}$ will now be studied. For $\bar{\alpha} > 1$ it follows from the two last equations in (4.12) that there are no non-trivial solutions for $\tilde{A}_s$ and $\tilde{B}_s$. This also implies (because of the damping) that $\bar{A}_s(\bar{t})$ and $\bar{B}_s(\bar{t})$ tend to zero as $\bar{t} \to \infty$. As a consequence the interaction between the n-th mode and the s-th mode does not occur. This implies that for $\tilde{\alpha} > 1$ the behaviour of the solutions of (4.11) can be described as has been described in section 4.1. Hence, in the further analysis only the case $0 < \bar{\alpha} \leq 1$ will be considered. It follows from the first two equations in (4.12) that for $\bar{M} = 0$ the only solutions $\tilde{A}_n$ and $\tilde{B}_n$ of (4.12) are $\tilde{A}_n = \tilde{B}_n = 0$. This implies that the periodic solution of (4.11) only consists of one (parametrically excited) mode. It can also readily be deduced from the first two equations in (4.12) that if for $\bar{M} \neq 0$ a solution of (4.12) exists then $\tilde{B}_n \neq 0$. In this case, that is, for $0 < \bar{\alpha} \leq 1$ and $\bar{M} \neq 0$ one can expect a periodic solution consisting of two modes. It should be observed that $\tilde{A}_s = \tilde{B}_s = 0$ is a solution of (4.12). In order to have non-trivial solutions for $\tilde{A}_s$ and $\tilde{B}_s$ one of the following conditions should be satisfied:

$$
\begin{aligned}
&\text{(a)} \quad (\tilde{A}_n^2 + \tilde{B}_n^2) + 3(\tilde{A}_s^2 + \tilde{B}_s^2) + \bar{\beta}_s - \bar{K} = -\sqrt{1-\bar{\alpha}^2}, \\
&\text{(b)} \quad (\tilde{A}_n^2 + \tilde{B}_n^2) + 3(\tilde{A}_s^2 + \tilde{B}_s^2) + \bar{\beta}_s - \bar{K} = \sqrt{1-\bar{\alpha}^2}.
\end{aligned} \tag{4.13}
$$

These conditions follow directly from the last two equations in (4.12). By substituting (4.13)(a) into the third equation of (4.12) one obtains $\tilde{A}_s = \sqrt{\frac{1-\bar{\alpha}}{1+\bar{\alpha}}}\tilde{B}_s$, and similarly by substituting (4.13)(b) into the third equation of (4.12) gives $\tilde{A}_s = -\sqrt{\frac{1-\bar{\alpha}}{1+\bar{\alpha}}}\tilde{B}_s$. Therefore, for the critical points of (4.11) with $0 < \tilde{\alpha} \leq 1$ the following types can be distinguished:

$$
\begin{aligned}
\text{CP-type 1}: (\bar{A}_s, \bar{B}_s, \bar{A}_n, \bar{B}_n) &= (0, 0, \sqrt{\frac{F_1}{\mu_s^2}}\tilde{A}_n, \sqrt{\frac{F_1}{\mu_s^2}}\tilde{B}_n), \\
\text{CP-type 2}: (\bar{A}_s, \bar{B}_s, \bar{A}_n, \bar{B}_n) &= (\sqrt{\frac{8F_1(1-\bar{\alpha})}{\mu_s^2(1+\bar{\alpha})}}\tilde{B}_s, \sqrt{\frac{8F_1}{\mu_s^2}}\tilde{B}_s, \sqrt{\frac{F_1}{\mu_s^2}}\tilde{A}_n, \\
&\quad \sqrt{\frac{F_1}{\mu_s^2}}\tilde{B}_n), \\
\text{CP-type 3}: (\bar{A}_s, \bar{B}_s, \bar{A}_n, \bar{B}_n) &= (-\sqrt{\frac{8F_1(1-\bar{\alpha})}{\mu_s^2(1+\bar{\alpha})}}\tilde{B}_s, \sqrt{\frac{8F_1}{\mu_s^2}}\tilde{B}_s, \sqrt{\frac{F_1}{\mu_s^2}}\tilde{A}_n,
\end{aligned}
$$

$$\sqrt{\frac{F_1}{\mu_s^2}}\tilde{B}_n). \tag{4.14}$$

When $\bar{\alpha} = 1$ it is obvious from (4.14) that the $\bar{A}_s$ components of the CP-type 2 and the CP-type 3 are zero. In what follows the dependence of the critical points on the parameters will be analyzed. Starting with the CP-type 1, the solutions of (4.11) with $\tilde{A}_s = \tilde{B}_s = 0$ are $\tilde{A}_n = \tilde{B}_n = 0$ for $\bar{M} = 0$, while for $\bar{M} \neq 0$ $\tilde{A}_n = -\frac{\bar{R}_n(3\bar{R}_n - 2\bar{K})}{\bar{M}}$ and $\tilde{B}_n = \frac{\bar{\alpha}\bar{R}_n}{\bar{M}}$, with $\bar{R}_n = X + \frac{4}{9}\bar{K}$, where X satisfies (see Appendix B.2):

$$\begin{aligned}
&X^3 + \kappa_0 X + \delta_0 = 0, \\
&\kappa_0 = -\frac{1}{27}(4\bar{K}^2 - 3\bar{\alpha}^2), \\
&\delta_0 = -\frac{1}{27}(3\bar{M}^2 - \frac{4\bar{K}}{27}[4\bar{K}^2 + 9\bar{\alpha}^2]).
\end{aligned} \tag{4.15}$$

In a similar way, the critical points of type 2 and of type 3 can be found by substituting (4.13)(a) and (4.13)(b) into the first two equations in (4.12) respectively. After some calculations (see Appendix B.2), one obtains $\tilde{A}_n = \tilde{B}_n = 0$ and $\tilde{B}_s^2 = \frac{1+\bar{\alpha}}{6}(\bar{K} - \bar{\beta}_s + (-1)^j\sqrt{1-\bar{\alpha}^2})$ for $\bar{M} = 0$, while for $\bar{M} \neq 0$ $\tilde{A}_n = -\frac{\bar{R}_n(5\bar{R}_n - 2\bar{K}_j)}{3\bar{M}}$ and $\tilde{B}_n = \frac{\bar{\alpha}\bar{R}_n}{\bar{M}}$, with $\tilde{R}_n = Y_j + \frac{4}{15}\bar{K}_j$ where Y_j, $j = 1, 2$, satisfies:

$$\begin{aligned}
&Y_j^3 + \kappa_j Y_j + \delta_j = 0, \\
&\mathrm{Cond}_{2+j}(Y_j) = Y_j + \frac{23}{15}\bar{\beta}_s - 11\bar{K} - (-1)^j\frac{23}{15}\sqrt{(1-\bar{\alpha}^2)} < 0, \\
&\tilde{R}_s = \tilde{A}_s^2 + \tilde{B}_s^2 = -\frac{1}{3}\mathrm{Cond}_{2+j}(Y_j),
\end{aligned} \tag{4.16}$$

with $\kappa_j = -\frac{1}{75}(4\bar{K}_j^2 - 27\bar{\alpha}^2)$, $\delta_j = -\frac{1}{75}\Big(27\bar{M}^2 - \frac{4\bar{K}_j}{45}[4\bar{K}_j^2 + 81\bar{\alpha}^2]\Big)$, and $\bar{K}_j = \bar{K} + 2\bar{\beta}_s + (-1)^{(1+j)}2\sqrt{(1-\bar{\alpha}^2)}$. In (4.16) the parameter $j = 1$ and the parameter $j = 2$ correspond to the CP-type 2 and the CP-type 3, respectively. An overview of the number of critical points is given in Table 1 and can be restricted to the case $\bar{M} \geq 0$ due to the symmetry in (4.12). By using (4.15) and (4.16) the different domains in the $(\bar{K}, \bar{M})$-plane are given for fixed values of $\bar{\beta}_s$ and $\bar{\alpha}$ in Figs. 5-10. By increasing the damping the size of the domains (in which critical points of type 2 and 3 exist) decreases. When one increases the damping coefficient from $\bar{\alpha} = 0.25$ to $\bar{\alpha} = 0.90$ then the motion indeed will change for fixed $\bar{K}$, $\bar{M}$, and $\bar{\beta}_s$ values as can be seen by comparing Figs. 5-10. The number of critical points of the types 1 to 3 in the domains as given in the Figs. 5-10 are presented in Table 1.

In Fig. 11 the stability response-curves of the transversal and of the parametrical modes are presented as function of $\bar{K}$. Looking at this figure one sees that the critical points of type 2 are always unstable. This result is the same as in the case without gravity (see also [10]). From Fig. 11 it can be deduced that stable motion can consist of transversally and parametrically excited modes. A remarkable result is the presence of jump phenomena in Fig. 11. These phenomena are due to the nonlinearities and the excitations. To explain this one starts with $\bar{\alpha} = 0.90$, $\bar{\beta}_s = 0$, $\bar{M} =$

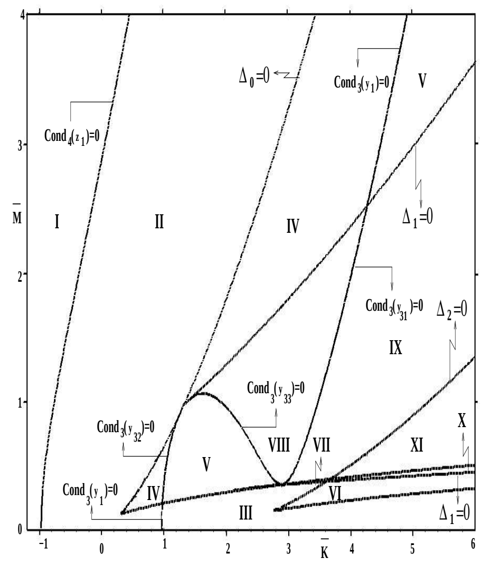


Figure 5: The bifurcation diagram of the critical points of system (4.11) for $\bar{\alpha} = 0.25$ and $\bar{\beta}_s = 0$ (y and z are the real solutions of (4.16) for $j = 1$ and $j = 2$, respectively).

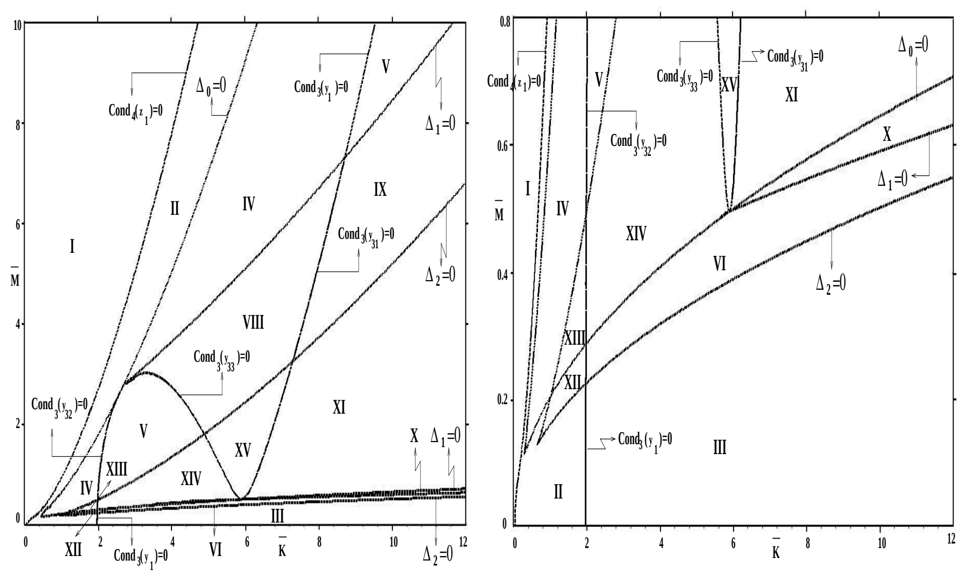


Figure 6: The bifurcation diagram of the critical points of system (4.11) for $\bar{\alpha} = 0.25$ and $\bar{\beta}_s = 1$ (y and z are the real solutions of (4.16) for $j = 1$ and $j = 2$, respectively).

2, and a small value for $\bar{K}$ such that the starting point is in domain I in Fig. 8. At

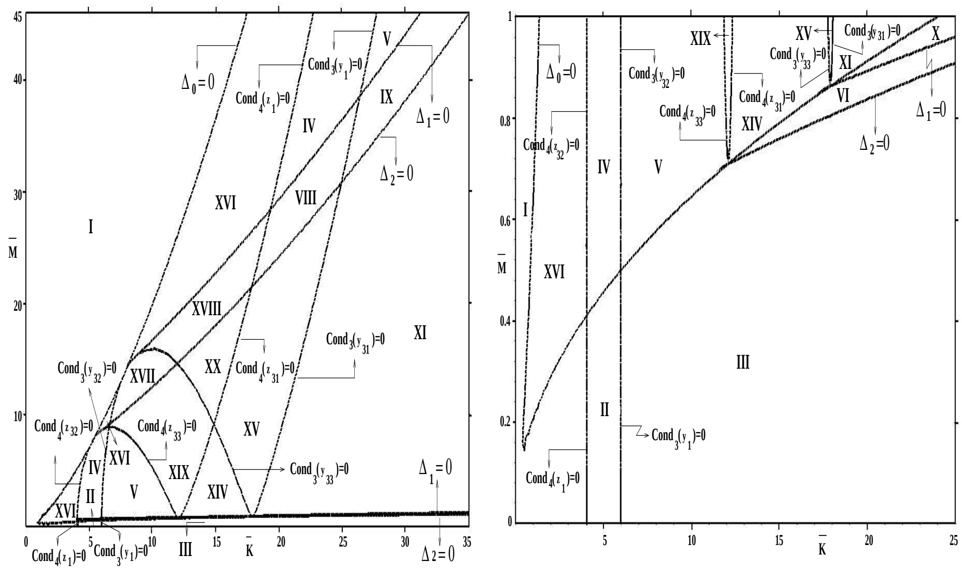
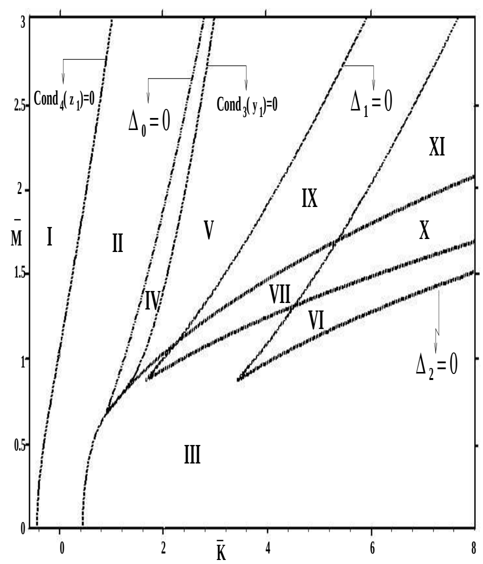


Figure 7: The bifurcation diagram of the critical points of system (4.11) for $\bar{\alpha} = 0.25$ and $\bar{\beta}_s = 5$ (y and z are the real solutions of (4.16) for $j = 1$ and $j = 2$, respectively).



(i) $\bar{\beta}_s = 0$

Figure 8: The bifurcation diagram of the critical points of system (4.11) for $\bar{\alpha} = 0.90$ and $\bar{\beta}_s = 0$ (y and z are the real solutions of (4.16) for $j = 1$ and $j = 2$, respectively).

the point Q_1 in Fig. 11(i) one smoothly leaves (for increasing $\bar{K}$ values) the CP-type

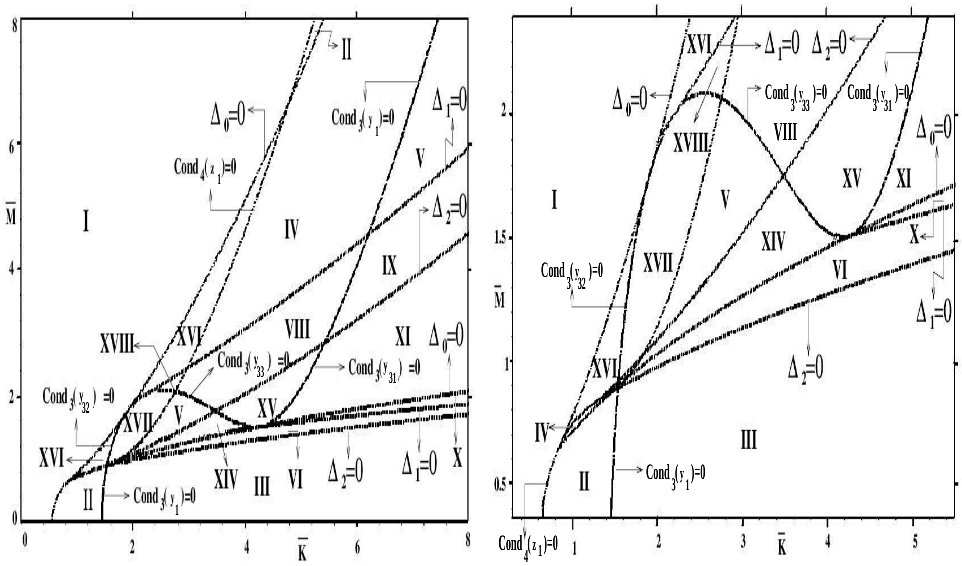


Figure 9: The bifurcation diagram of the critical points of system (4.11) for $\bar{\alpha} = 0.90$ and $\bar{\beta}_s = 1$ (y and z are the real solutions of (4.16) for $j = 1$ and $j = 2$, respectively).

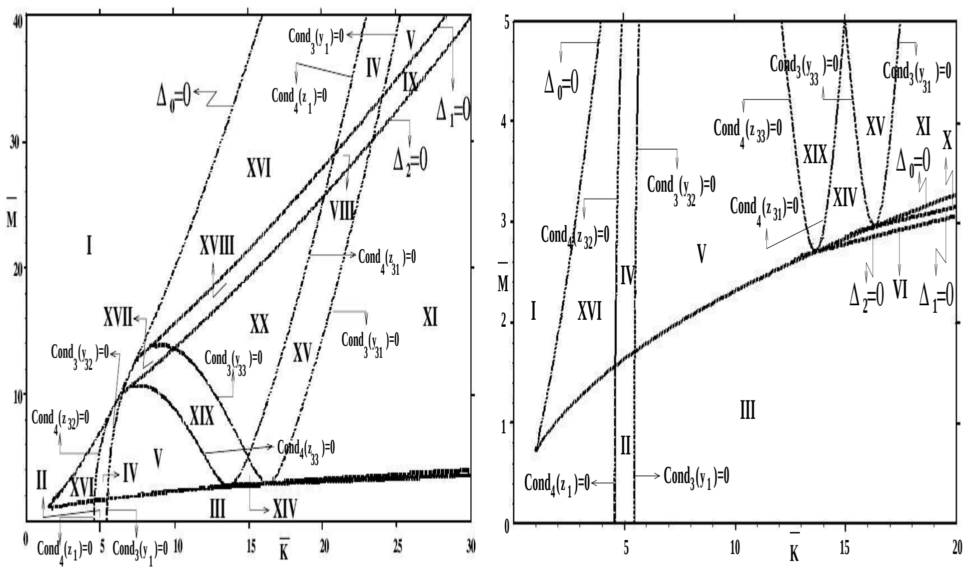


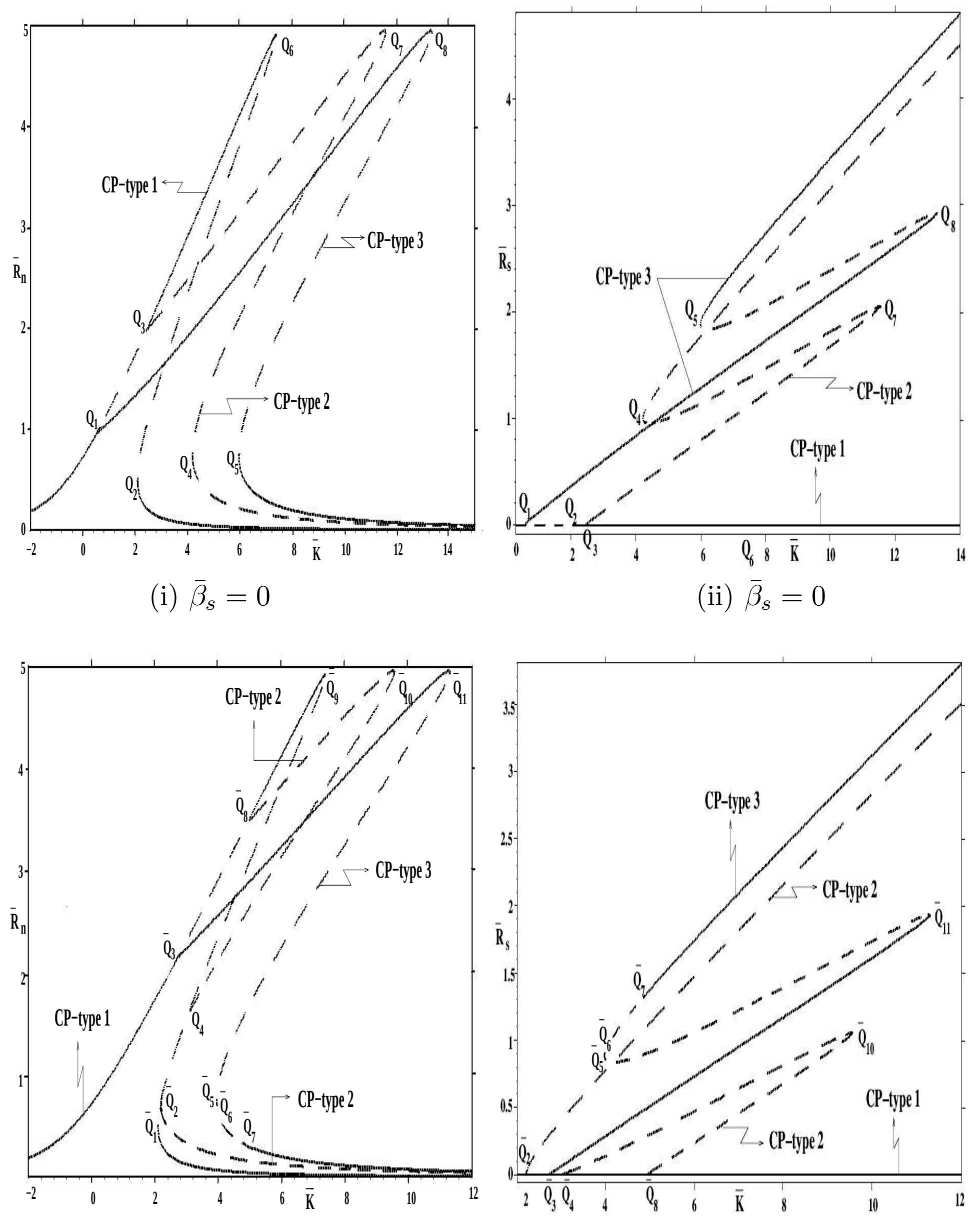
Figure 10: The bifurcation diagram of the critical points of system (4.11) for $\bar{\alpha} = 0.90$ and $\bar{\beta}_s = 5$ (y and z are the real solutions of (4.16) for $j = 1$ and $j = 2$, respectively).

1 to the CP-type 3 because the parametric excitation comes in as has been shown

Table 1: The number of critical points of (4.11) as described in the Figs. 5-10.

Domain	**The number of critical points**			**Total**
	CP-type 1	CP-type 2	CP-type 3	
I	1	0	0	1
II	1	0	1	2
III	1	1	1	3
IV	3	0	1	4
V	3	1	1	5
VI	1	1	3	5
VII	1	3	1	5
VIII	3	2	1	6
IX	3	3	1	7
X	1	3	3	7
XI	3	3	3	9
XII	1	0	3	4
XIII	3	0	3	6
XIV	3	1	3	7
XV	3	2	3	8
XVI	3	0	0	3
XVII	3	1	0	4
XVIII	3	2	0	5
XIX	3	1	2	6
XX	3	2	2	7

in Fig. 11(ii). By increasing $\bar{K}$ one arrives at Q_8 and a jump from the CP-type 3 to itself occurs followed by a sudden decrease in amplitude of the transversally excited mode ($\bar{R}_n$) and by an increase in the amplitude of the parametrically excited mode ($\bar{R}_s$). By reversing the procedure, again a jump phenomenon occurs at Q_5, where now the amplitude of the transversally excited mode increases, while simultaneously the amplitude of the parametrically excited mode decreases. An other possible phenomenon occurs at Q_2, where for decreasing values of $\bar{K}$ a jump occurs from the CP-type 1 to the CP-type 3. This phenomenon is followed by an increase in the amplitudes of the transversal and of the parametrical modes. It is also of interest to know the effect of $\bar{\beta}_s$ (which depends on the static state due to gravity and the mode interactions). By taking $\bar{\beta}_s = 1$ the stability response-curve is different from the case $\bar{\beta}_s = 0$ ($\bar{\beta}_s = 0$ corresponds to $g = 0$ or s is an even number). This can be seen by comparing the upper and the lower figures in Fig. 11. For instance, part of the curve $\bar{Q}_6\bar{Q}_7$ of the lower CP-type 3 curve as indicated in Fig. 11(iii) becomes unstable. The critical point of type 3 for $\bar{\beta}_s = 0$ describes the motion consisting of two even modes, while the critical point of type 3 for $\bar{\beta}_s \neq 0$ describes the motion consisting of an odd and an even mode. Hence, by considering $\bar{\beta}_s$ as function of s the



(i) $\bar{\beta}_s = 0$ (ii) $\bar{\beta}_s = 0$

(iii) $\bar{\beta}_s = 1$ (iv) $\bar{\beta}_s = 1$

Figure 11: The stability response-curves $\bar{R}_n = \tilde{A}_n^2 + \tilde{B}_n^2$ and $\bar{R}_s = \tilde{A}_s^2 + \tilde{B}_s^2$ of system (4.11) for $\tilde{\alpha} = 0.90$ and $\bar{M} = 2$. The dashed line represents an unstable solution and the solid line represents a stable solution.

instability of part of the $\bar{Q}_6\bar{Q}_7$ curve in Fig. 11(iii) is due to an interaction between an odd and an even mode.
The influence of gravity can be studied by varying the values of $\bar{\beta}_s$ or $\bar{K}$ with $\bar{\eta}$ kept fixed. It can readily be seen from the Figures 5 to 11 that gravity reduces the domain

of existence of string-motion consisting of two (parametric and transversal) modes. This implies that the existence of parametrically excited mode is reduced due to the presence of gravity. Moreover, in case of internal resonances occurring between an odd parametrically excited mode (s an odd number) and an even transversally excited mode the stability of this non-planar motion with small amplitudes becomes unstable due to gravity.

5 Conclusions and remarks

In this paper a mathematical model to describe the in-plane vibrations of an inclined stretched string has been derived. This model consists of a system of two coupled partial differential equations (PDEs) representing the longitudinal and the transversal displacement of the string. By using Kirchhoff's approach the model equations can be reduced to a single PDE representing the transversal displacement. In [10] a model has been studied by neglecting the effect of gravity. This is possible whenever the force due to gravity is very small compared to the pretension in the string. In this paper it is assumed that the effects due to gravity are small, but can not be neglected.

In this paper the influence of the parameters on the existence (and stability) of time-periodic solutions has been investigated. There are two values of the excitation-frequency that cause resonances, i.e. $\lambda = \mu_{(2s-1)} + O(\bar{\epsilon})$ and $\lambda = \mu_{2s} + O(\bar{\epsilon})$, where $\mu_{(2s-1)}$ and μ_{2s} are eigenfrequencies of the linearized model-problem. When $\lambda = \mu_{(2s-1)} + O(\bar{\epsilon})$ the periodic solution consists of only one mode which is the transversally excited mode. While when $\lambda = \mu_{2s} + O(\bar{\epsilon})$ the periodic solutions can consist of transversally and parametrically excited modes. The domain of existence of all types of periodic solutions and their stability are given. Moreover, a number of jump-phenomena are found with saddle-node bifurcations as underlying mechanism. These phenomena occur not only from semi-trivial to non-trivial solutions, but also from non-trivial solutions to itself.

The gravitational force influences not only the static state, but also the dynamic state. When gravity is ignored (see also [10]) the ultimate periodic string-motion can not consist of an odd mode (s an odd number) and an even mode. However, in contrast to the case without gravity the interaction between an odd mode and an even mode may occur when gravity is not neglected.

Appendix

A. Modal expansion

By substituting (3.1) into (2.14), and then by multiplying the so-obtained equation with $2\sin(\mu_k x)$, where k is a fixed, arbitrary integer, one obtains:

$$\sum_{n=1}^{\infty}(\ddot{q}_n + \mu_n^2 q_n) 2\sin(\mu_n x)\sin(\mu_k x) = \bar{\epsilon}\Big\{\Big[\sum_{m=1}^{\infty}(\frac{1}{4}\mu_m^2 q_m^2 + a_m q_m) +$$

$$F_1 \sin(\lambda t)\Big]\Big[\check{v}_{xx}(x) - \sum_{n=1}^{\infty} \mu_n^2 q_n \sin(\mu_n x)\Big] 2\sin(\mu_k x) - \sum_{n=1}^{\infty} 2\mu_n^2 q_n [p +$$
$$\omega_1 x \sin(\varphi)] \sin(\mu_n x)\sin(\mu_k x) + \sum_{n=1}^{\infty} 2\omega_1 \sin(\varphi)\mu_n q_n(t)\cos(\mu_n x)\sin(\mu_k x)$$
$$+2\lambda^2 F_2 \sin(\lambda t) x \sin(\mu_k x) - \sum_{n=1}^{\infty} 2\alpha \dot{q}_n(t)\sin(\mu_n x)\sin(\mu_k x)\Big\}, \tag{A.1}$$

where:

$$a_m = \int_0^1 \mu_m \check{v}_x(x)\cos(\mu_m x)dx = \begin{cases} O(\bar{\epsilon}) & , \text{ for m is even,} \\ -\frac{2\omega_1\cos(\varphi)}{\mu_m} + O(\bar{\epsilon}) & , \text{ for m is odd.} \end{cases}$$

By integrating (A.1) over $x \in [0,1]$ the following infinite-dimensional system of non-linear ordinary differential equations for $q_k(t)$ is found:

$$\begin{aligned} \ddot{q}_k + \mu_k^2 q_k \;=\; & -\bar{\epsilon}\Big\{\Big[\sum_{m=1}^{\infty}\Big(\frac{1}{4}\mu_m^2 q_m^2 + a_m q_m\Big) + F_1\sin(\lambda t)\Big]\Big[\mu_k^2 q_k - b_k\Big] + \\ & \sum_{n=1}^{\infty}(d_{(n,k)} - c_{(n,k)})q_n + (-1)^k \frac{2\lambda^2 F_2 \sin(\lambda t)}{\mu_k} + \alpha\dot{q}_k\Big\}, \end{aligned} \tag{A.2}$$

where:

$$b_k = \int_0^1 2\check{v}_{xx}\sin(\mu_k x)dx = \begin{cases} O(\bar{\epsilon}) & , \text{ for k is even,} \\ \frac{4\omega_1\cos(\varphi)}{\mu_k} + O(\bar{\epsilon}) & , \text{ for k is odd,} \end{cases}$$

$$\begin{aligned} c_{(n,k)} &= \int_0^1 2\omega_1\sin(\varphi)\mu_n\cos(\mu_n x)\sin(\mu_k x)dx \\ &= \begin{cases} 0 & , \text{ for } (n+k) \text{ is even,} \\ -\frac{4\omega_1\mu_n\mu_k\sin(\varphi)}{(\mu_n^2-\mu_k^2)} & , \text{ for } (n+k) \text{ is odd,} \end{cases} \end{aligned}$$

$$\begin{aligned} d_{(n,k)} &= \int_0^1 2\mu_n^2(p+\omega_1 x\sin(\varphi))\sin(\mu_n x)\sin(\mu_k x)dx \\ &= \begin{cases} \mu_k^2(p+\frac{1}{2}\omega_1\sin(\varphi)) & , \text{ for } n = k, \\ 0 & , \text{ for } (n+k),\, n \neq k, \text{ is even,} \\ -\frac{8\omega_1\mu_n^3\mu_k\sin(\varphi)}{(\mu_n+\mu_k)^2(\mu_n-\mu_k)^2} & , \text{ for } (n+k) \text{ is odd.} \end{cases} \end{aligned}$$

Note that in case $(n+k)$ is odd, $d_{(n,k)} - c_{(n,k)} = -\frac{4\omega_1\mu_k\mu_n(\mu_n^2+\mu_k^2)\sin(\varphi)}{(\mu_n^2-\mu_k^2)^2}$.

B. Critical points

B.1. The case $\lambda = \mu_n + \bar{\epsilon}\eta$, where n is an odd, fixed number

For $\alpha = 0$ it can readily be seen that the solutions of (4.3) satisfy (4.4) for $\bar{F}_0 = 0$ and (4.5) for $\bar{F}_0 \neq 0$. Now the case $\alpha > 0$ is considered. For $\bar{F}_\alpha = 0$ it can easily be shown that the only solution of (4.4) is $\tilde{A}_n = \tilde{B}_n = 0$. For $\bar{F}_\alpha \neq 0$ it should be observed that $\tilde{B}_n$ can not be zero. This implies that (4.3) can be written as:

$$\begin{aligned} \bar{F}_\alpha \tilde{B}_n &= -(\tilde{A}_n^2 + \tilde{B}_n^2), \\ \bar{F}_\alpha \tilde{A}_n &= (\tilde{A}_n^2 + \tilde{B}_n^2)(\tilde{A}_n^2 + \tilde{B}_n^2 - \bar{\Omega}_\alpha). \end{aligned} \tag{B.1}$$

By squaring both sides of (B.1), and then by adding the so-obtained equations, one obtains

$$\bar{R}_n^3 - 2\bar{\Omega}_\alpha \bar{R}_n^2 + (1 + \bar{\Omega}_\alpha^2)\bar{R}_n - \bar{F}_\alpha^2 = 0, \tag{B.2}$$

where $\bar{R}_n = \tilde{A}_n^2 + \tilde{B}_n^2$. By introducing a new variable $\bar{R}_n = \tilde{R}_n + \frac{2}{3}\bar{\Omega}_\alpha$ the first equation in (4.6) follows from (B.2).

B.2. The case $\lambda = \mu_n + \bar{\epsilon}\eta$ with $n = 2s$, where s is a positive, fixed integer

In this section the case with $\sigma = 1$, $0 < \bar{\alpha} \leq 1$, and $\bar{M} \neq 0$ is only considered. Because $\bar{M} \neq 0$ it follows from the first two equations in (4.12) that $\tilde{B}_n$ is not equal to zero. It should be observed that $\tilde{A}_s = \tilde{B}_s = 0$ is a solution of (4.12). By substituting $\tilde{A}_s = \tilde{B}_s = 0$ into the first two equations in (4.12) yields (after a simple reformulation of (4.12)):

$$\begin{aligned} \bar{M}\tilde{B}_n &= \bar{\alpha}(\tilde{A}_n^2 + \tilde{B}_n^2), \\ \bar{M}\tilde{A}_n &= -(\tilde{A}_n^2 + \tilde{B}_n^2)(3(\tilde{A}_n^2 + \tilde{B}_n^2) - 2\bar{K}). \end{aligned} \tag{B.3}$$

In a similar way as to obtain (B.2) but now by using (B.3) and the variable $\bar{R}_n = X + \frac{4}{9}\bar{K}$ one obtains (2.13). The non-trivial solutions for $\tilde{A}_s$ and $\tilde{B}_s$ satisfy $\tilde{A}_s = (-1)^{(1+j)}\sqrt{\frac{1-\bar{\alpha}}{1-\bar{\alpha}}}\tilde{B}_s$, $j = 1, 2$, under the conditions (4.13). By substituting these solutions into (4.12) yields

$$\begin{aligned} \bar{M}\tilde{B}_n &= \bar{\alpha}(\tilde{A}_n^2 + \tilde{B}_n^2), \\ \bar{M}\tilde{A}_n &= -\frac{1}{3}(\tilde{A}_n^2 + \tilde{B}_n^2)(5(\tilde{A}_n^2 + \tilde{B}_n^2) - 2\bar{K}_j), \end{aligned} \tag{B.4}$$

where $\bar{K}_j = \bar{K} + 2\bar{\beta}_s + (-1)^{(1+j)}2\sqrt{1-\bar{\alpha}^2}$. By introducing the variable $\bar{R}_n = Y_j + \frac{4}{15}\bar{K}_j$ system (4.16) can readily be found from (B.4) and (4.13).

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