

On the planar and whirling motion of a stretched string due to a parametric harmonic excitation

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Abstract

In this paper a model of the dynamics of a stretched string is derived. The sag of the string due to gravity is neglected. The string is suspended between a fixed support and a vibrating support. Due to the vibrating support the oscillation of the string in vertical direction is influenced by a parametrical excitation. The parametric term originates from a longitudinal vibration caused by an elastic elongation and then influences the transversal vibrations in- and out-of-plane. The study will be focused on the existence and stability of time-periodic solutions in transversal direction. The stability is analyzed by using a linearisation method. In addition the different types of periodic motions of the string will be determined.

1 Introduction

The vibrations of stretched strings have been investigated by many researchers, because a variety of physical systems can be described by stretched strings, e.g., the stay cable of a cable-stayed-bridge, an overhead power transmission line, and so on. Most of the studies of oscillations of stretched strings has mainly focused on the transverse displacements. The earliest experiment was successfully done by Melde in [1]. He observed that the string can oscillate transversally with an amplitude of about 4% of the length of string, although the excitation force is purely longitudinal. A number of papers corresponding to this subject has been published, for instance in [2, 3, 4].

The oscillations of the strings can be caused by many factors [5, 6]. Lilien and Pinto da Costa [7] studied the vibrations caused by a purely parametrical excitation of inclined cables of a cable-stayed-bridge. Pinto da Costa et al. [8] also studied the steady-state response of inclined cables when the ratio between the excitation frequency and the first natural frequency of the cables is close to two. The dominant phenomenon in that case is a parametric excitation. Corresponding to this case it has been shown in [9] that the mode generated by a parametric excitation may easily have amplitudes ten times larger than the amplitudes of the transversally excited modes.

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In [9, 8, 7] the vibrations of the strings are only studied in the plane. On the other hand if the frequency of excitation falls in a certain resonance range, the string movement in the plane becomes unstable, and leads to out of plane vibrations (see also [10, 12, 11, 13]). An experiment to show this phenomenon has been done by Matsumoto et al. [14].

In a recent paper by Zhao et al. [15] the in- and out-of-plane excitation of an inclined elastic cable is investigated (without considering the primary parametric excitation in longitudinal direction). Lee and Renshaw [16] studied the stability of parametrically excited systems using a spectral collocation method. In this paper, the model of an stretched string motion will be derived by neglecting the sag of the string due to gravity. The periodic solutions will be studied by using the averaging method, whereas their stability will be studied by linearizing the averaged equations.

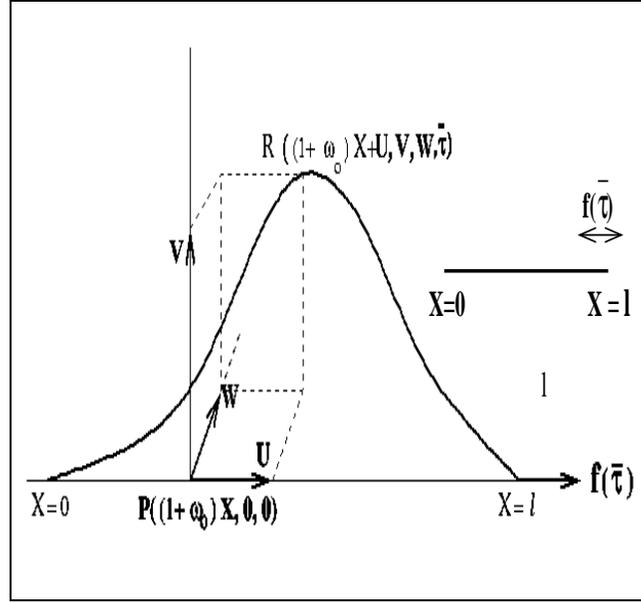


Figure 1: The dynamic state of the string suspended between a fixed support at $x = 0$ and a vibrating support at $x = l$.

2 The derivation of the model equation

We consider a perfectly flexible, elastic, unstretched string with length $l < 1$. Let $(X,0,0)$ be the coordinate of each material point P of the string with $X \in [0, l]$. The string is stretched uniformly so that the stretched length is 1. In the stretched state the point P will have the coordinates $((1 + \omega_0)X, 0, 0)$, where $\omega_0 = \frac{1}{l} - 1$ is the initial strain. Denote the dynamic displacement of the point P by $U(X, \bar{\tau})\mathbf{i}$, $V(X, \bar{\tau})\mathbf{j}$ and $W(X, \bar{\tau})\mathbf{k}$, where \mathbf{i} , \mathbf{j} and \mathbf{k} are the unit vectors along the axes of the Cartesian coordinate system and $\bar{\tau}$ is time. U and V are the displacements in horizontal and vertical direction, respectively, whereas W represents the displacement perpendicular to the picture as indicated in Fig. 1. So the vector position $\mathbf{R}(X, \bar{\tau})$ of the point P in the dynamic state can be written as:

$$\mathbf{R}(X, \bar{\tau}) = [(1 + \omega_0)X + U(X, \bar{\tau})]\mathbf{i} + V(X, \bar{\tau})\mathbf{j} + W(X, \bar{\tau})\mathbf{k}. \quad (2.1)$$

The relative strain per unit length of the stretched string is:

$$\left| \frac{\partial}{\partial X} \mathbf{R}(\mathbf{X}, \bar{\tau}) \right| - 1 = \sqrt{[(1 + \omega_o) + U_X]^2 + V_X^2 + W_X^2} - 1, \quad (2.2)$$

where U_X , V_X , and W_X represent the derivative of $U(X, \bar{\tau})$, $V(X, \bar{\tau})$, and $W(X, \bar{\tau})$ with respect to X , respectively. Introduce the new coordinate x by:

$$x = (1 + \omega_o)X, \quad (2.3)$$

implying that $x \in [0, 1]$ and set $U(X, \bar{\tau}) = U\left(\frac{x}{(1 + \omega_o)}, \bar{\tau}\right) = \tilde{u}(x, \bar{\tau})$, $V(X, \bar{\tau}) = V\left(\frac{x}{(1 + \omega_o)}, \bar{\tau}\right) = \tilde{v}(x, \bar{\tau})$, and $W(X, \bar{\tau}) = W\left(\frac{x}{(1 + \omega_o)}, \bar{\tau}\right) = \tilde{w}(x, \bar{\tau})$, then the relative strain $r(x, \bar{\tau})$ per unit length becomes:

$$r(x, \bar{\tau}) = (1 + \omega_o) \sqrt{1 + (2\tilde{u}_x + \tilde{u}_x^2 + \tilde{v}_x^2 + \tilde{w}_x^2)} - 1. \quad (2.4)$$

The kinetic and the potential energy densities of the system are given by:

$$\mathbb{K} = \frac{1}{2} \rho (\tilde{u}_{\bar{\tau}}^2 + \tilde{v}_{\bar{\tau}}^2 + \tilde{w}_{\bar{\tau}}^2) \quad \text{and} \quad \mathbb{P} = \frac{1}{2} E r^2(x, \bar{\tau}), \quad (2.5)$$

respectively, where ρ is the mass of the string per unit length and E is Young's modulus. By assuming that $|\tilde{u}_x|$, $|\tilde{v}_x|$, and $|\tilde{w}_x|$ are small with respect to 1, the potential energy may be approximated by its Taylor expansion \mathbb{P}_6 up to terms of the sixth degree:

$$\begin{aligned} \mathbb{P}_6 = & \frac{1}{2} E \left[\omega_o^2 + 2\omega_o(1 + \omega_o)\tilde{u}_x + (1 + \omega_o)^2 \tilde{u}_x^2 + \omega_o(1 + \omega_o)(\tilde{v}_x^2 + \tilde{w}_x^2) + \tilde{u}_x(1 + \omega_o)(\tilde{v}_x^2 + \right. \\ & \tilde{w}_x^2) - (1 + \omega_o)\tilde{u}_x^2(\tilde{v}_x^2 + \tilde{w}_x^2) + \frac{1}{4}(1 + \omega_o)(\tilde{v}_x^2 + \tilde{w}_x^2)^2 - \frac{3}{4}\tilde{u}_x(1 + \omega_o)(\tilde{v}_x^2 + \tilde{w}_x^2)^2 + \\ & (1 + \omega_o)\tilde{u}_x^3(\tilde{v}_x^2 + \tilde{w}_x^2) + \frac{3}{2}(1 + \omega_o)\tilde{u}_x^2(\tilde{v}_x^2 + \tilde{w}_x^2)^2 - (1 + \omega_o)\tilde{u}_x^4(\tilde{v}_x^2 + \tilde{w}_x^2) - \frac{1}{8}(1 + \\ & \left. \omega_o)(\tilde{v}_x^2 + \tilde{w}_x^2)^3 \right]. \quad (2.6) \end{aligned}$$

The Lagrangian density $\mathbb{D} = \mathbb{K} - \mathbb{P}_6$ is used in a variational principle [17] to obtain the following equations of motion:

$$\begin{aligned} \rho \tilde{u}_{\bar{\tau}\bar{\tau}} - E(1 + \omega_o)^2 \tilde{u}_{xx} &= \frac{1}{2} E(1 + \omega_o) \frac{\partial}{\partial x} \left[(\tilde{v}_x^2 + \tilde{w}_x^2) - 2\tilde{u}_x(\tilde{v}_x^2 + \tilde{w}_x^2) - \frac{3}{4}(\tilde{v}_x^2 + \tilde{w}_x^2)^2 + \right. \\ & \left. 3\tilde{u}_x^2(\tilde{v}_x^2 + \tilde{w}_x^2) + 3\tilde{u}_x(\tilde{v}_x^2 + \tilde{w}_x^2)^2 - 4\tilde{u}_x^3(\tilde{v}_x^2 + \tilde{w}_x^2) \right], \\ \rho \tilde{v}_{\bar{\tau}\bar{\tau}} - E\omega_o(1 + \omega_o)\tilde{v}_{xx} &= \frac{1}{2} E(1 + \omega_o) \frac{\partial}{\partial x} \left[2\tilde{u}_x\tilde{v}_x - 2\tilde{u}_x^2\tilde{v}_x + \tilde{v}_x(\tilde{v}_x^2 + \tilde{w}_x^2) - 3\tilde{u}_x\tilde{v}_x(\tilde{v}_x^2 + \right. \\ & \left. \tilde{w}_x^2) + 2\tilde{u}_x^3\tilde{v}_x + 6\tilde{u}_x^2\tilde{v}_x(\tilde{v}_x^2 + \tilde{w}_x^2) - 2\tilde{u}_x^4\tilde{v}_x - \frac{3}{4}\tilde{v}_x(\tilde{v}_x^2 + \tilde{w}_x^2)^2 \right], \\ \rho \tilde{w}_{\bar{\tau}\bar{\tau}} - E\omega_o(1 + \omega_o)\tilde{w}_{xx} &= \frac{1}{2} E(1 + \omega_o) \frac{\partial}{\partial x} \left[2\tilde{u}_x\tilde{w}_x - 2\tilde{u}_x^2\tilde{w}_x + \tilde{w}_x(\tilde{v}_x^2 + \tilde{w}_x^2) - 3(\tilde{v}_x^2 + \right. \\ & \left. \tilde{w}_x^2)\tilde{u}_x\tilde{w}_x + 2\tilde{u}_x^3\tilde{w}_x + 6\tilde{u}_x^2\tilde{w}_x(\tilde{v}_x^2 + \tilde{w}_x^2) - 2\tilde{u}_x^4\tilde{w}_x - \frac{3}{4}\tilde{w}_x(\tilde{v}_x^2 + \tilde{w}_x^2)^2 \right]. \quad (2.7) \end{aligned}$$

Replace now $\bar{\tau}$ by $\frac{\tau}{\sqrt{1 + \omega_o}}$ and then set $\frac{E}{\rho} = c_1^2$ and $\frac{E\omega_o}{\rho} = c_2^2$. By taking the damping forces (proportional to the velocity) into account, the model problem (2.7) becomes:

$$\tilde{u}_{\tau\tau} - c_1^2(1 + \omega_o)\tilde{u}_{xx} = -\tilde{\alpha}\tilde{u}_{\tau} + \frac{1}{2}c_1^2 \frac{\partial}{\partial x} \left[(\tilde{v}_x^2 + \tilde{w}_x^2) - 2\tilde{u}_x(\tilde{v}_x^2 + \tilde{w}_x^2) - \frac{3}{4}(\tilde{v}_x^2 + \tilde{w}_x^2)^2 + \right.$$

$$\begin{aligned}
& 3\tilde{u}_x^2(\tilde{v}_x^2 + \tilde{w}_x^2) + 3\tilde{u}_x(\tilde{v}_x^2 + \tilde{w}_x^2)^2 - 4\tilde{u}_x^3(\tilde{v}_x^2 + \tilde{w}_x^2) \Big], \\
\tilde{v}_{\tau\tau} - c_2^2 \tilde{v}_{xx} &= -\tilde{\alpha}_1 \tilde{v}_\tau + c_1^2 \frac{\partial}{\partial x} \left[\tilde{u}_x \tilde{v}_x - \tilde{u}_x^2 \tilde{v}_x + \frac{1}{2} \tilde{v}_x (\tilde{v}_x^2 + \tilde{w}_x^2) - \frac{3}{2} \tilde{u}_x \tilde{v}_x (\tilde{v}_x^2 + \tilde{w}_x^2) + \tilde{u}_x^3 \tilde{v}_x + \right. \\
& \left. 3\tilde{u}_x^2 \tilde{v}_x (\tilde{v}_x^2 + \tilde{w}_x^2) - \tilde{u}_x^4 \tilde{v}_x - \frac{3}{8} \tilde{v}_x (\tilde{v}_x^2 + \tilde{w}_x^2)^2 \right], \\
\tilde{w}_{\tau\tau} - c_2^2 \tilde{w}_{xx} &= -\tilde{\alpha}_2 \tilde{w}_\tau + c_1^2 \frac{\partial}{\partial x} \left[\tilde{u}_x \tilde{w}_x - \tilde{u}_x^2 \tilde{w}_x + \frac{1}{2} \tilde{w}_x (\tilde{v}_x^2 + \tilde{w}_x^2) + \tilde{u}_x^3 \tilde{w}_x - \frac{3}{2} \tilde{u}_x \tilde{w}_x (\tilde{v}_x^2 + \right. \\
& \left. \tilde{w}_x^2) + 3\tilde{u}_x^2 \tilde{w}_x (\tilde{v}_x^2 + \tilde{w}_x^2) - \tilde{u}_x^4 \tilde{w}_x - \frac{3}{8} \tilde{w}_x (\tilde{v}_x^2 + \tilde{w}_x^2)^2 \right], \tag{2.8}
\end{aligned}$$

where $0 < x < 1$, $\tau > 0$, $\tilde{\alpha}$, $\tilde{\alpha}_1$, and $\tilde{\alpha}_2$ are nonnegative damping parameters. The following boundary conditions are considered:

$$\begin{aligned}
\tilde{u}(0, \tau) &= \tilde{v}(0, \tau) = \tilde{w}(0, \tau) = 0, \\
\tilde{u}(1, \tau) &= \mathbf{f}(\tau), \quad \text{and} \quad \tilde{v}(1, \tau) = \tilde{w}(1, \tau) = 0, \tag{2.9}
\end{aligned}$$

where $\mathbf{f}(\tau)$ is a periodic function. Note that the string is fixed at $x = 0$ and that at $x = 1$ there is a time varying forcing in the x-direction implying a parametrical (or longitudinal) forcing. By assuming $c_2^2 \ll c_1^2$ and by setting $\omega_0 = \frac{c_2^2}{c_1^2} = \varepsilon$ (small) and $c_2 \tau = \bar{t}$, system (2.8), after dividing by c_1^2 , becomes:

$$\begin{aligned}
\varepsilon \tilde{u}_{\bar{t}\bar{t}} - (1 + \varepsilon) \tilde{u}_{xx} &= -\varepsilon \frac{\tilde{\alpha}}{c_2} \tilde{u}_{\bar{t}} + \frac{1}{2} \frac{\partial}{\partial x} \left[(\tilde{v}_x^2 + \tilde{w}_x^2) - 2\tilde{u}_x (\tilde{v}_x^2 + \tilde{w}_x^2) - \frac{3}{4} (\tilde{v}_x^2 + \tilde{w}_x^2)^2 + 3(\tilde{v}_x^2 + \right. \\
& \left. \tilde{w}_x^2) \tilde{u}_x^2 + 3\tilde{u}_x (\tilde{v}_x^2 + \tilde{w}_x^2)^2 - 4\tilde{u}_x^3 (\tilde{v}_x^2 + \tilde{w}_x^2) \right], \\
\varepsilon \tilde{v}_{\bar{t}\bar{t}} - \varepsilon \tilde{v}_{xx} &= -\varepsilon \frac{\tilde{\alpha}_1}{c_2} \tilde{v}_{\bar{t}} + \frac{\partial}{\partial x} \left[\tilde{u}_x \tilde{v}_x - \tilde{u}_x^2 \tilde{v}_x + \frac{1}{2} \tilde{v}_x (\tilde{v}_x^2 + \tilde{w}_x^2) - \frac{3}{2} \tilde{u}_x \tilde{v}_x (\tilde{v}_x^2 + \tilde{w}_x^2) + \tilde{u}_x^3 \tilde{v}_x + \right. \\
& \left. 3\tilde{u}_x^2 \tilde{v}_x (\tilde{v}_x^2 + \tilde{w}_x^2) - \tilde{u}_x^4 \tilde{v}_x - \frac{3}{8} \tilde{v}_x (\tilde{v}_x^2 + \tilde{w}_x^2)^2 \right], \\
\varepsilon \tilde{w}_{\bar{t}\bar{t}} - \varepsilon \tilde{w}_{xx} &= -\varepsilon \frac{\tilde{\alpha}_2}{c_2} \tilde{w}_{\bar{t}} + \frac{\partial}{\partial x} \left[\tilde{u}_x \tilde{w}_x - \tilde{u}_x^2 \tilde{w}_x + \frac{1}{2} \tilde{w}_x (\tilde{v}_x^2 + \tilde{w}_x^2) - \frac{3}{2} \tilde{u}_x \tilde{w}_x (\tilde{v}_x^2 + \tilde{w}_x^2) + \tilde{u}_x^3 \tilde{w}_x \right. \\
& \left. + 3\tilde{u}_x^2 \tilde{w}_x (\tilde{v}_x^2 + \tilde{w}_x^2) - \tilde{u}_x^4 \tilde{w}_x - \frac{3}{8} \tilde{w}_x (\tilde{v}_x^2 + \tilde{w}_x^2)^2 \right]. \tag{2.10}
\end{aligned}$$

In a first order approximation, that is, for $\varepsilon \rightarrow 0$ the first equation of (2.10) reduces to:

$$\begin{aligned}
-\tilde{u}_{xx} &= \frac{1}{2} \frac{\partial}{\partial x} \left[(\tilde{v}_x^2 + \tilde{w}_x^2) - 2\tilde{u}_x (\tilde{v}_x^2 + \tilde{w}_x^2) - \frac{3}{4} (\tilde{v}_x^2 + \tilde{w}_x^2)^2 + 3\tilde{u}_x^2 (\tilde{v}_x^2 + \tilde{w}_x^2) + \right. \\
& \left. 3\tilde{u}_x (\tilde{v}_x^2 + \tilde{w}_x^2)^2 - 4\tilde{u}_x^3 (\tilde{v}_x^2 + \tilde{w}_x^2) \right]. \tag{2.11}
\end{aligned}$$

If one assumes additionally that the transversal vibrations \tilde{v} and \tilde{w} are of $O(\varepsilon)$ it follows from (2.11) that $\tilde{u}(x, \bar{t}) = O(\varepsilon^2)$. This implies that one only keeps the term $\frac{1}{2} \frac{\partial}{\partial x} (\tilde{v}_x^2 + \tilde{w}_x^2)$ in the right hand side of (2.11) and the others can be neglected in a first order approximation:

$$-\tilde{u}_{xx} = \frac{1}{2} \frac{\partial}{\partial x} (\tilde{v}_x^2 + \tilde{w}_x^2). \tag{2.12}$$

After integration of (2.12) with respect to x and by using the boundary conditions (2.9) for \tilde{u} the following expression for \tilde{u}_x can be derived:

$$\tilde{u}_x = -\frac{1}{2} (\tilde{v}_x^2 + \tilde{w}_x^2) + \mathbf{f}(\bar{t}) + \frac{1}{2} \int_0^1 (\tilde{v}_x^2 + \tilde{w}_x^2) dx. \tag{2.13}$$

By using the new time variable \bar{t} and by substituting (2.13) into the second and third equations of (2.10) (and by introducing initial conditions), the following model equations are obtained (up to $O(\varepsilon)$):

$$\begin{aligned} v_{\bar{t}\bar{t}}(x, \bar{t}) - v_{xx}(x, \bar{t}) &= \varepsilon v_{xx}(x, \bar{t}) \left[\frac{1}{2} \int_0^1 (v_x^2(x, \bar{t}) + w_x^2(x, \bar{t})) dx + f(\bar{t}) \right] - \varepsilon \alpha_1 v_{\bar{t}}(x, \bar{t}), \\ &0 < x < 1; \bar{t} > 0, \\ w_{\bar{t}\bar{t}}(x, \bar{t}) - w_{xx}(x, \bar{t}) &= \varepsilon w_{xx}(x, \bar{t}) \left[\frac{1}{2} \int_0^1 (v_x^2(x, \bar{t}) + w_x^2(x, \bar{t})) dx + f(\bar{t}) \right] - \varepsilon \alpha_2 w_{\bar{t}}(x, \bar{t}), \\ &0 < x < 1; \bar{t} > 0, \\ \text{IC's: } v(x, 0) &= \bar{v}_0(x), w(x, 0) = \bar{w}_0(x), v_{\bar{t}}(x, 0) = \bar{v}_1(x), w_{\bar{t}}(x, 0) = \bar{w}_1(x), \\ \text{BC's: } v(0, \bar{t}) &= w(0, \bar{t}) = 0, \quad v(1, \bar{t}) = w(1, \bar{t}) = 0, \end{aligned} \quad (2.14)$$

where v , w , f , α_1 , and α_2 are respectively defined by : $\bar{v} = \varepsilon v$, $\bar{w} = \varepsilon w$, $\mathbf{f} = \varepsilon^2 f$, $\frac{\alpha_1}{c_2} = \varepsilon \alpha_1$, $\frac{\alpha_2}{c_2} = \varepsilon \alpha_2$. The function $f(\bar{t})$ and the parameters α_1 and α_2 are ε -independent. Additionally it will be assumed that $f(\bar{t}) = F \sin(\lambda \bar{t})$, in which without loss of generality λ and F are taken positive constants.

3 Discretization of the model equation

By considering the boundary conditions (2.14), the solutions $v(x, \bar{t})$ and $w(x, \bar{t})$ of (2.14) can be expanded in eigenfunction-series:

$$v(x, \bar{t}) = \sum_{n=1}^{\infty} v_n(\bar{t}) \sin(\mu_n x) \quad \text{and} \quad w(x, \bar{t}) = \sum_{n=1}^{\infty} w_n(\bar{t}) \sin(\mu_n x), \quad (3.1)$$

where $\mu_n = n\pi$, $n = 1, 2, 3, \dots$. By substituting (3.1) into (2.14) and by using the orthogonality properties of the eigenfunctions, one obtains the following infinite dimensional system for $v_n(\bar{t})$ and $w_n(\bar{t})$:

$$\begin{aligned} \ddot{v}_n(\bar{t}) + \mu_n^2 v_n(\bar{t}) &= -\varepsilon \left[\alpha_1 \dot{v}_n(\bar{t}) + \mu_n^2 v_n(\bar{t}) \left(\frac{1}{4} \sum_{k=1}^{\infty} \mu_k^2 (v_k^2(\bar{t}) + w_k^2(\bar{t})) + F \sin(\lambda \bar{t}) \right) \right], \\ \ddot{w}_n(\bar{t}) + \mu_n^2 w_n(\bar{t}) &= -\varepsilon \left[\alpha_2 \dot{w}_n(\bar{t}) + \mu_n^2 w_n(\bar{t}) \left(\frac{1}{4} \sum_{k=1}^{\infty} \mu_k^2 (v_k^2(\bar{t}) + w_k^2(\bar{t})) + F \sin(\lambda \bar{t}) \right) \right], \\ v_n(0) &= 2 \int_0^1 \bar{v}_0(x) \sin(n\pi x) dx, \quad \dot{v}_n(0) = 2 \int_0^1 \bar{v}_1(x) \sin(n\pi x) dx, \\ w_n(0) &= 2 \int_0^1 \bar{w}_0(x) \sin(n\pi x) dx, \quad \dot{w}_n(0) = 2 \int_0^1 \bar{w}_1(x) \sin(n\pi x) dx, \end{aligned} \quad (3.2)$$

where $n = 1, 2, 3, \dots$ and the dot represents differentiation with respect to t . Considering the values of λ , there are two possibilities:

- (i) $\lambda \neq 2\mu_s + O(\varepsilon)$ for all s ; then the parametric forcing term does not influence the $O(1)$ approximations of the solutions on a time-scale of order ε^{-1} , and no $O(1)$ time varying motion will occur due to parametric excitation.
- (ii) $\lambda = 2\mu_s + O(\varepsilon)$ for an integer s ; then the parametric term is important in the equation for the s -th mode. This possibility implies that an $O(1)$ periodic solution can be expected due to parametric excitation.

In what follows the case λ near $2\mu_s$ will be studied. This value implies that the external force will excite the s -th mode (in- and out-of-plane) but not the other modes. The infinite dimensional system (3.2) is now truncated to only the s -th modes. This can be justified in the following sense. It can be shown rigorously (see [9]) that all other modes initially present will vanish exponentially up to $O(\varepsilon)$. This means that except for the s -th modes (in-plane and out-of-plane), the modes which are not present initially will not become larger than $O(\varepsilon)$. The interaction of the s -th modes in-plane and out-of-plane is described by:

$$\begin{aligned}\dot{v}_s + \mu_s^2 v_s &= -\varepsilon \left[\alpha_1 \dot{v}_s + \mu_s^2 v_s \left(\frac{1}{4} \mu_s^2 (v_s^2 + w_s^2) + F \sin(\lambda \bar{t}) \right) \right], \\ \dot{w}_s + \mu_s^2 w_s &= -\varepsilon \left[\alpha_2 \dot{w}_s + \mu_s^2 w_s \left(\frac{1}{4} \mu_s^2 (v_s^2 + w_s^2) + F \sin(\lambda \bar{t}) \right) \right].\end{aligned}\quad (3.3)$$

By setting $\lambda \bar{t} = 2t$, where $\lambda = 2(s\pi + \varepsilon\eta)$ and $\eta = O(1)$, system (3.3) becomes (up to order $O(\varepsilon)$):

$$\begin{aligned}w_s''(t) + w_s(t) &= -\frac{\varepsilon}{\mu_s} \left[\alpha_2 w_s'(t) + w_s(t) (\gamma_s (v_s^2(t) + w_s^2(t)) + 2\beta_s \sin(2t) - 2\eta) \right], \\ v_s''(t) + v_s(t) &= -\frac{\varepsilon}{\mu_s} \left[\alpha_1 v_s'(t) + v_s(t) (\gamma_s (v_s^2(t) + w_s^2(t)) + 2\beta_s \sin(2t) - 2\eta) \right],\end{aligned}\quad (3.4)$$

where $\gamma_s = \frac{\mu_s^3}{4}$, $\beta_s = \frac{\mu_s}{2} F$, and a prime represents differentiation with respect to t .

4 On the of periodic solutions of system (3.4)

In system (3.4) the excitation term $2\beta_s \sin(2t)$ describes a parametric resonance and will lead to an $O(1)$ amplitude response in the system. Moreover, an $O(1)$ interaction between the in-plane and out-of-plane modes will occur. From a practical point of view system (3.4) may represent the motion of a stretched string (stay cable) due to a pure parametric excitation.

Introduce the transformations $(v_s(t), \dot{v}_s(t)) \rightarrow (A_s(t), B_s(t))$ and $(w_s(t), \dot{w}_s(t)) \rightarrow (C_s(t), D_s(t))$:

$$\begin{aligned}v_s(t) &= A_s(t) \sin(t) + B_s(t) \cos(t), & w_s(t) &= C_s(t) \sin(t) + D_s(t) \cos(t), \\ \dot{v}_s(t) &= A_s(t) \cos(t) - B_s(t) \sin(t), & \dot{w}_s(t) &= C_s(t) \cos(t) - D_s(t) \sin(t).\end{aligned}\quad (4.1)$$

System (3.4) then becomes after averaging:

$$\begin{aligned}\dot{\bar{A}}_s(t) &= -\bar{\varepsilon} \left((\bar{\alpha}_1 + \bar{\beta}) \bar{A}_s + \bar{B}_s \left[\frac{1}{4} (3(\bar{A}_s^2 + \bar{B}_s^2) + (\bar{C}_s^2 + \bar{D}_s^2)) - 2\bar{\eta} \right] + \frac{1}{2} (\bar{A}_s \bar{C}_s + \bar{B}_s \bar{D}_s) \bar{D}_s \right), \\ \dot{\bar{B}}_s(t) &= -\bar{\varepsilon} \left((\bar{\alpha}_1 - \bar{\beta}) \bar{B}_s - \bar{A}_s \left[\frac{1}{4} (3(\bar{A}_s^2 + \bar{B}_s^2) + (\bar{C}_s^2 + \bar{D}_s^2)) - 2\bar{\eta} \right] - \frac{1}{2} (\bar{A}_s \bar{C}_s + \bar{B}_s \bar{D}_s) \bar{C}_s \right), \\ \dot{\bar{C}}_s(t) &= -\bar{\varepsilon} \left((\bar{\alpha}_2 + \bar{\beta}) \bar{C}_s + \bar{D}_s \left[\frac{1}{4} ((\bar{A}_s^2 + \bar{B}_s^2) + 3(\bar{C}_s^2 + \bar{D}_s^2)) - 2\bar{\eta} \right] + \frac{1}{2} (\bar{A}_s \bar{C}_s + \bar{B}_s \bar{D}_s) \bar{B}_s \right), \\ \dot{\bar{D}}_s(t) &= -\bar{\varepsilon} \left((\bar{\alpha}_2 - \bar{\beta}) \bar{D}_s - \bar{C}_s \left[\frac{1}{4} ((\bar{A}_s^2 + \bar{B}_s^2) + 3(\bar{C}_s^2 + \bar{D}_s^2)) - 2\bar{\eta} \right] - \frac{1}{2} (\bar{A}_s \bar{C}_s + \bar{B}_s \bar{D}_s) \bar{A}_s \right),\end{aligned}\quad (4.2)$$

where $\bar{\varepsilon} = \frac{\gamma_s}{2\mu_s} \varepsilon$, $\bar{\beta} = \frac{\beta_s}{\gamma_s}$, $\bar{\alpha}_1 = \frac{\alpha_1}{\gamma_s}$, $\bar{\alpha}_2 = \frac{\alpha_2}{\gamma_s}$, and $\bar{\eta} = \frac{\eta}{\gamma_s}$, and where $\bar{A}_s(t)$, $\bar{B}_s(t)$, $\bar{C}_s(t)$, and $\bar{D}_s(t)$ are the averaged approximations of $A_s(t)$, $B_s(t)$, $C_s(t)$, and $D_s(t)$, respectively. It should be observed that for $\bar{\beta} < 0$ a simple transformation ($\bar{A}_s := \bar{B}_s$, $\bar{B}_s := -\bar{A}_s$, $\bar{C}_s := \bar{D}_s$,

and $\bar{D}_s := -\bar{C}_s$) leads again to system (4.2) for $\bar{\beta} \geq 0$. Hence, the analysis can be restricted to the case $\bar{\beta} \geq 0$. From system (4.2) it can readily be deduced that

$$\frac{d}{dt}(\bar{A}_s \bar{D}_s(t) - \bar{B}_s \bar{C}_s(t)) = -\bar{\epsilon}(\bar{\alpha}_1 + \bar{\alpha}_2)(\bar{A}_s \bar{D}_s(t) - \bar{B}_s \bar{C}_s(t)), \quad (4.3)$$

implying that a first integral of system (4.2) is given by:

$$\mathbf{G}(\bar{A}_s, \bar{B}_s, \bar{C}_s, \bar{D}_s) = (\bar{A}_s \bar{D}_s(t) - \bar{B}_s \bar{C}_s(t)) e^{\bar{\epsilon}(\bar{\alpha}_1 + \bar{\alpha}_2)t}. \quad (4.4)$$

By using the transformation (4.1) it can easily be shown that (4.4) is a first integral of the original system (3.3). In what follows the critical points of system (4.2) and their dependence on the parameters $\bar{\alpha}_1$, $\bar{\alpha}_2$, $\bar{\eta}$, and $\bar{\beta}$ will be investigated. These critical points correspond to periodic solutions of system (3.4). Depending on the values of $\bar{\alpha}_1$ and $\bar{\alpha}_2$ the following two cases will be studied: $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$ (no damping) and $\bar{\alpha}_1, \bar{\alpha}_2 > 0$ (positive damping).

4.1 The case without damping: $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$

The critical points of (4.2) with $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$ satisfy the following algebraic system:

$$\begin{aligned} \bar{\beta} \bar{A}_s + \bar{B}_s \left[\frac{1}{4}(3(\bar{A}_s^2 + \bar{B}_s^2) + (\bar{C}_s^2 + \bar{D}_s^2)) - 2\bar{\eta} \right] + \frac{1}{2}(\bar{A}_s \bar{C}_s + \bar{B}_s \bar{D}_s) \bar{D}_s &= 0, \\ \bar{\beta} \bar{B}_s + \bar{A}_s \left[\frac{1}{4}(3(\bar{A}_s^2 + \bar{B}_s^2) + (\bar{C}_s^2 + \bar{D}_s^2)) - 2\bar{\eta} \right] + \frac{1}{2}(\bar{A}_s \bar{C}_s + \bar{B}_s \bar{D}_s) \bar{C}_s &= 0, \\ \bar{\beta} \bar{C}_s + \bar{D}_s \left[\frac{1}{4}((\bar{A}_s^2 + \bar{B}_s^2) + 3(\bar{C}_s^2 + \bar{D}_s^2)) - 2\bar{\eta} \right] + \frac{1}{2}(\bar{A}_s \bar{C}_s + \bar{B}_s \bar{D}_s) \bar{B}_s &= 0, \\ \bar{\beta} \bar{D}_s + \bar{C}_s \left[\frac{1}{4}((\bar{A}_s^2 + \bar{B}_s^2) + 3(\bar{C}_s^2 + \bar{D}_s^2)) - 2\bar{\eta} \right] + \frac{1}{2}(\bar{A}_s \bar{C}_s + \bar{B}_s \bar{D}_s) \bar{A}_s &= 0. \end{aligned} \quad (4.5)$$

Obviously, $\bar{A}_s = \bar{B}_s = \bar{C}_s = \bar{D}_s = 0$ is a solution of (4.5). The non-zero solutions of (4.5) can be classified as semi-trivial (that is, $\bar{A}_s = \bar{B}_s = 0$ or $\bar{C}_s = \bar{D}_s = 0$) or as non-trivial (that is, $\bar{A}_s \neq 0$ or $\bar{B}_s \neq 0$ and $\bar{C}_s \neq 0$ or $\bar{D}_s \neq 0$). After some calculations (see Appendix 5.1) it will turn out that four types of critical points of (4.2) with $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$ can be distinguished:

$$\begin{aligned} \text{CP-type 1: } (\bar{A}_s, \bar{B}_s, \bar{C}_s, \bar{D}_s) &= (0, 0, 0, 0), \\ \text{CP-type 2: } (\bar{A}_s, \bar{B}_s, \bar{C}_s, \bar{D}_s) &= (-\tilde{B}_s, \tilde{B}_s, -\tilde{D}_s, \tilde{D}_s), \\ \text{CP-type 3: } (\bar{A}_s, \bar{B}_s, \bar{C}_s, \bar{D}_s) &= (\tilde{B}_s, \tilde{B}_s, \tilde{D}_s, \tilde{D}_s), \\ \text{CP-type 4: } (\bar{A}_s, \bar{B}_s, \bar{C}_s, \bar{D}_s) &= (\tilde{A}_s, \tilde{B}_s, \tilde{C}_s, \tilde{D}_s), \quad \text{where } (\tilde{A}_s \tilde{B}_s + \tilde{C}_s \tilde{D}_s) < 0 \\ &\quad \text{and } \tilde{A}_s \tilde{D}_s - \tilde{B}_s \tilde{C}_s \neq 0. \end{aligned} \quad (4.6)$$

The second and third type of critical points describe semi-trivial ($\tilde{B}_s = 0$ or $\tilde{D}_s = 0$) and non-trivial ($\tilde{B}_s \neq 0$ and $\tilde{D}_s \neq 0$) solutions, respectively, and these solutions describe a planar motion of the string. Whereas the fourth type describes non-trivial solutions. These solutions describe a non-planar (whirling) motion of the string.

The critical point of type 1 exists for all values of the parameters, but its stability depends on the parameter values. In what follows a $(\bar{\eta}, \bar{\beta})$ -diagram will be constructed which gives an overview of all possible critical points. Starting with $\bar{A}_s = -\bar{B}_s$ and $\bar{C}_s = -\bar{D}_s$ it follows after some calculations (see also Appendix 5.1) that a critical point of type 2 satisfies:

$$\begin{aligned} \tilde{B}_s^2 + \tilde{D}_s^2 &= \frac{2}{3}(2\bar{\eta} + \bar{\beta}), \\ \text{Cond}_1 &= (2\bar{\eta} + \bar{\beta}) > 0. \end{aligned} \quad (4.7)$$

The curve defined by $\text{Cond}_1 = (2\bar{\eta} + \bar{\beta}) = 0$ divides the $(\bar{\eta}, \bar{\beta})$ -plane into two domains in which the CP-type 2 exists or not. When one looks separately at the case $\bar{A}_s = \bar{B}_s$ and $\bar{C}_s = \bar{D}_s$ a critical point of type 3 satisfies (see Appendix 5.1):

$$\begin{aligned}\tilde{B}_s^2 + \tilde{D}_s^2 &= \frac{2}{3}(2\bar{\eta} - \bar{\beta}), \\ \text{Cond}_2 &= (2\bar{\eta} - \bar{\beta}) > 0.\end{aligned}\quad (4.8)$$

By setting $\text{Cond}_2 = 0$ one obtains a curve on which critical points of type 3 can be found.

It follows from (A.7)-(A.11) (see Appendix 5.1) that a critical point of type 4 satisfies:

$$\begin{aligned}\tilde{A}_s^2 + \tilde{C}_s^2 &= \tilde{B}_s^2 + \tilde{D}_s^2 = 2\bar{\eta}, \\ \tilde{A}_s\tilde{B}_s + \tilde{C}_s\tilde{D}_s &= -2\bar{\beta}, \\ \text{Cond}_3 &= \bar{\eta} - \bar{\beta} > 0.\end{aligned}\quad (4.9)$$

The resulting boundary curves and domains on or in which the different types of critical points can be found, are presented in Fig. 2. The types of critical points and their stability in different domains and on different boundary curves are given in Table 1.

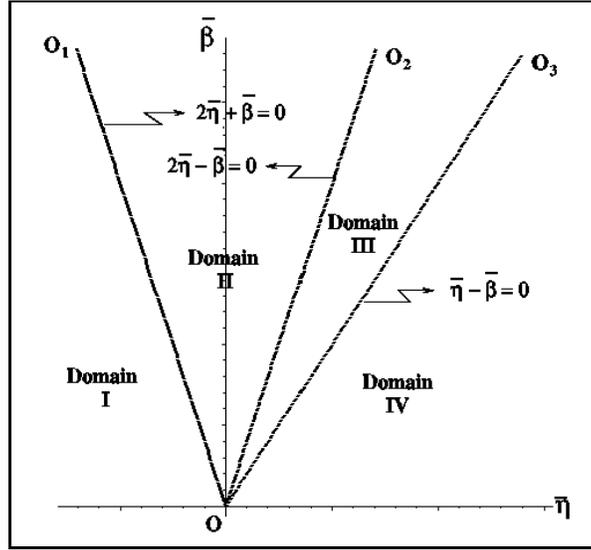


Figure 2: Four domains with real solutions of type 1 - 4 of system (4.5) in the $(\bar{\eta}, \bar{\beta})$ -plane for $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$.

Looking at (4.7), (4.8), or (4.9) one sees that for fixed parameter values the number of critical points of type 2 - 4 (when existing) is infinite, and these critical points can be considered as curves in a four dimensional space, namely the curves **S**, **U**, and **L**, respectively. The characteristics of the curves **S** and **U** are similar, while the characteristics of the curve **L** are different from the others. The characteristics of the curve **S** are: the projection of this curve on the (\bar{A}_s, \bar{B}_s) ((\bar{C}_s, \bar{D}_s)) -plane is a straight line passing through the point $(0,0)$ with gradient -1, while the projections on the (\bar{A}_s, \bar{C}_s) ((\bar{B}_s, \bar{D}_s)) and (\bar{A}_s, \bar{D}_s) ((\bar{B}_s, \bar{C}_s))-plane are circles with radius ρ_o^+ , where $\rho_o^+ = \sqrt{\frac{2}{3}(2\bar{\eta} + \bar{\beta})}$. The characteristics of the curve **L** are: the projection of this curve on the (\bar{A}_s, \bar{B}_s) ((\bar{C}_s, \bar{D}_s)) -plane is an ellipse, projection on the (\bar{A}_s, \bar{C}_s) ((\bar{B}_s, \bar{D}_s)) -plane is a circle, and projection on the (\bar{A}_s, \bar{D}_s) ((\bar{B}_s, \bar{C}_s)) -plane leads to two ellipses. This is illustrated in Fig. 3.

Table 1: The critical points of system (4.2) and their stability for $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$. The stability is determined by using the linearisation method.

Domains/ curves	number of critical points	location of the critical points	behaviour in system (4.2)
I and $\overline{OO_1}$	1	(0,0,0,0)	stable ("4d-center")
II and $\overline{OO_2}$	2	(0,0,0,0) $(-\tilde{B}_s, \tilde{B}_s, -\tilde{D}_s, \tilde{B}_s)$	unstable ("4d-saddle") stable (degenerate)
III and $\overline{OO_3}$	3	(0,0,0,0) $(-\tilde{B}_s, \tilde{B}_s, -\tilde{D}_s, \tilde{D}_s)$ $(\tilde{B}_s, \tilde{B}_s, \tilde{D}_s, \tilde{D}_s)$	stable ("4d-center") stable (degenerate) unstable (degenerate)
IV	4	(0,0,0,0) $(-\tilde{B}_s, \tilde{B}_s, -\tilde{D}_s, \tilde{D}_s)$ $(\tilde{B}_s, \tilde{B}_s, \tilde{D}_s, \tilde{D}_s)$ $(\tilde{A}_s, \tilde{B}_s, \tilde{C}_s, \tilde{D}_s)$	stable ("4d-center") unstable (degenerate) unstable (degenerate) stable (degenerate)

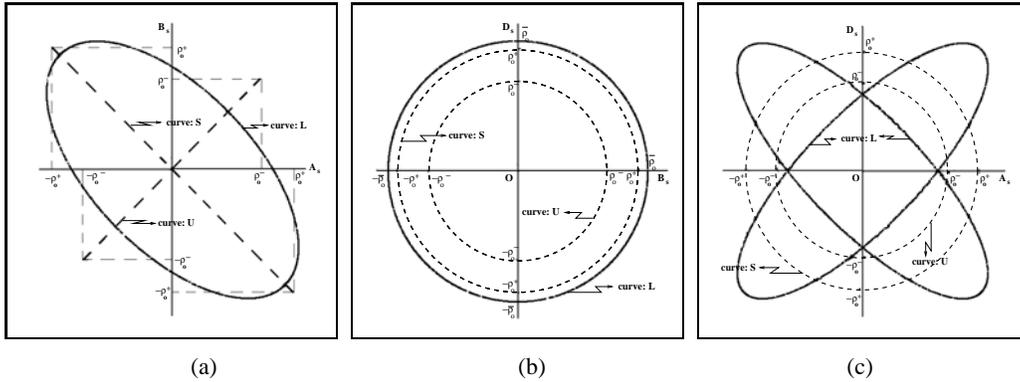


Figure 3: Projection of the curves **S**, **U**, and **L** for $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$ and $\bar{\eta} > \bar{\beta}$ on: (a) the (\bar{A}_s, \bar{B}_s) $((\bar{C}_s, \bar{D}_s))$ -plane; (b) the (\bar{A}_s, \bar{C}_s) $((\bar{B}_s, \bar{D}_s))$ -plane; (c) the (\bar{A}_s, \bar{D}_s) $((\bar{B}_s, \bar{C}_s))$ -plane (with $\rho_o^- = \sqrt{\frac{2}{3}(2\bar{\eta} - \bar{\beta})}$ and $\bar{\rho}_o = \sqrt{2\bar{\eta}}$). The solid and dashed line represent stable and unstable solutions, respectively.

Now we define the hyperplane \mathbf{G}_E by using the first integral (4.4), where $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$, as follows:

$$\mathbf{G}_E = \{(\bar{A}_s, \bar{B}_s, \bar{C}_s, \bar{D}_s) : |\mathbf{G}(\bar{A}_s, \bar{B}_s, \bar{C}_s, \bar{D}_s)| = E, E \geq 0\}. \quad (4.10)$$

It follows from (4.4) that the solutions of (4.2) satisfy $\bar{A}_s(t)\bar{D}_s(t) - \bar{B}_s(t)\bar{C}_s(t) = \bar{A}_s(0)\bar{D}_s(0) - \bar{B}_s(0)\bar{C}_s(0)$. This shows that the hyperplane \mathbf{G}_E is an invariant for (4.2) with $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$, for all E . The critical points of type 1 belong to the hyperplane \mathbf{G}_0 . It also follows from

(4.6) - (4.9) that the critical points of type 2 and 3 are in the hyperplane \mathbf{G}_0 , while the critical points of type 4 are in the hyperplane $\mathbf{G}_{\bar{E}}$ with $\bar{E} = 2\sqrt{\bar{\eta}^2 - \bar{\beta}^2}$. Hence, the other hyperplanes \mathbf{G}_E with $E \neq 0$ and $E \neq \bar{E}$ have no singular points when $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$. However, in these hyperplanes the solutions of (4.2) are bounded, because the hyperplanes \mathbf{G}_E with $E \neq 0$ and $E \neq \bar{E}$ are invariant for (4.2) with $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$.

In the $(\bar{A}_s, \bar{B}_s, \bar{C}_s, \bar{D}_s)$ -space the critical point of type 3 is unstable for $\bar{\eta} > \frac{1}{2}\bar{\beta}$ and the critical point of type 4 is stable (in the sense of Lyapunov) for $\bar{\eta} > \bar{\beta}$, whereas the critical point of type 2 is stable for $-\frac{1}{2}\bar{\beta} < \bar{\eta} \leq \bar{\beta}$ and unstable for $\bar{\eta} > \bar{\beta}$. It shows that for $\bar{\eta} > \bar{\beta}$ there is a solution of (4.2) with initial conditions in a neighbourhood of the critical point of type 2 which goes away from that point. In the hyperplane \mathbf{G}_0 the critical points of type 2 and 3 are degenerate points. Therefore, to study the behaviour of these points and the origin, we introduce the hyperplane \mathbf{H}_k :

$$\mathbf{H}_k = \begin{cases} \{(\bar{A}_s, \bar{B}_s, \bar{C}_s, \bar{D}_s) \in \mathbf{G}_0 \mid \bar{A}_s = k\bar{C}_s \text{ and } \bar{B}_s = k\bar{D}_s\} & \text{with } k \in \mathbb{R}, \\ \{(\bar{A}_s, \bar{B}_s, \bar{C}_s, \bar{D}_s) \in \mathbf{G}_0 \mid \bar{C}_s = 0 \text{ and } \bar{D}_s = 0\} & \text{for } k = \pm\infty. \end{cases} \quad (4.11)$$

It is clear that the hyperplane \mathbf{H}_k is an invariant of (4.2) for $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$. Also, one sees that the union and the intersection of \mathbf{H}_k are

$$\bigcup_k \mathbf{H}_k = \mathbf{G}_0 \quad \text{and} \quad \bigcap_k \mathbf{H}_k = O(0,0,0,0),$$

respectively. The intersection points of the hyperplane \mathbf{H}_k with the curve \mathbf{S} are the points $S_k^\pm(\pm k\rho_k^+, \mp k\rho_k^+, \pm\rho_k^+, \mp\rho_k^+)$, where:

$$\rho_k^+ = \frac{\rho_o^+}{1+k^2} \sqrt{(1+k^2)}. \quad (4.12)$$

Whereas the intersection points of the hyperplane \mathbf{H}_k with the curve \mathbf{U} are the points $U_k^\pm(\pm k\rho_k^-, \mp k\rho_k^-, \pm\rho_k^-, \mp\rho_k^-)$, where:

$$\rho_k^- = \frac{\rho_o^-}{1+k^2} \sqrt{(1+k^2)}. \quad (4.13)$$

In the hyperplane \mathbf{H}_k the critical point $O(0,0,0,0)$ is unstable in domain II in Fig. 2 and stable in the other domains. The points S_k^\pm are stable (centers in the hyperplane \mathbf{H}_k), while the points U_k^\pm are unstable (saddle points in the hyperplane \mathbf{H}_k).

The values of $\bar{\eta}$ and $\bar{\beta}$ as function of s (the mode number determined by the excitation-frequency) are strictly monotone decreasing. It follows from the first equation of (4.7) that large amplitudes of a periodic solution of type 2 is most likely for a low frequency of the excitation. Moreover, it follows from (4.9) that also large whirling motion can only occur in the low frequency case. Therefore the effect of parametric excitation will be most significant to the system for low frequencies as also mentioned in [7]. As illustration some values for the parameters $\bar{\eta}$ and $\bar{\beta}$ will be taken.

In Fig. 4 the response curves $r_1 = \sqrt{\bar{A}_s^2 + \bar{B}_s^2}$ are plotted as function of $\bar{\eta}$ for a fixed $\bar{\beta}$ -value. In this figure the response curves for the critical points of type 2 and of type 3 are given for $k = 1$ (that is, \mathbf{H}_1), so that a similar figure for the response curves $r_2 = \sqrt{\bar{C}_s^2 + \bar{D}_s^2}$ is found. It should be remarked that for other values of k similar results are obtained. In the H_1 -plane each point on the curve type 2 represents two centers, whereas on the curve type

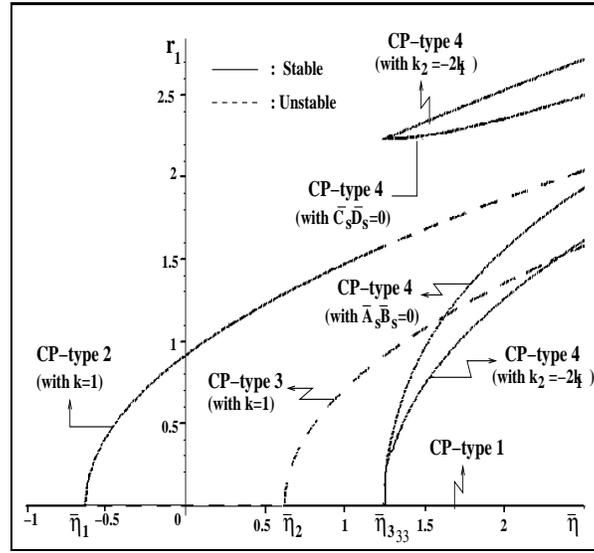


Figure 4: The stability response-curves $r_1 = \sqrt{\bar{A}_s^2 + \bar{B}_s^2}$ with respect to $\bar{\eta}$ for system (4.2) with $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$ and $\bar{\beta} = 1.25$: The curves for CP-type 2 and 3 represent response-curves in \mathbf{H}_k for $k = 1$ and the curves for CP-type 4 with $k_2 = -2k_1$, where $k_1 = \frac{\bar{A}_s}{\bar{C}_s}$ and $k_2 = \frac{\bar{B}_s}{\bar{D}_s}$, represent response-curves of periodic solutions with all of the components non-zero.

3 each point represents two saddle points. The bifurcations (for increasing values of $\bar{\eta}$) of the critical points of (4.2) in the hyperplane \mathbf{H}_1 will now be considered. For $\bar{\eta} \leq \bar{\eta}_1$ only the trivial critical point exists, and it is stable (center). When $\bar{\eta}$ passes $\bar{\eta}_1$ this solution bifurcates into two stable critical points of type 2 and an unstable trivial critical point. Finally when $\bar{\eta}$ passes $\bar{\eta}_2$ the trivial critical point again bifurcates into two unstable critical points of type 3 and a stable critical point. As illustration these results and the behaviour of the solutions are given in Fig. 5.

In Fig. 6 the response curves r_1 are given as function of $\bar{\beta}$ for a fixed value of $\bar{\eta}$. This figure clearly indicates that the whirling motion can only occur for relatively small excitation amplitudes (that is, $\bar{\beta} < \bar{\eta}$). Hence, for larger amplitudes the string is always moving in the plane. One can also see that for increasing excitation amplitudes the amplitudes of the periodic solutions become larger.

In Fig. 7 the types of motions of a stretched string as modeled by system (4.2) are described. These motions can be planar or non-planar (whirling) motions. In this figure the analytic solutions are compared with the numerical results as obtained by using a Runge-Kutta method. One can see that both results are very close to each other. Fig. 7(b) and Fig. 7(c) describe the planar motions corresponding to the critical points of type 2 and 3 in the hyperplane \mathbf{H}_1 and \mathbf{H}_{-1} , respectively. If the motions correspond to the critical points of type 3 then these motions are always unstable. Whereas the planar motions corresponding to the critical points of type 2 are stable for $-\frac{1}{2}\bar{\beta} < \bar{\eta} \leq \bar{\beta}$ and unstable for $\bar{\eta} > \bar{\beta}$ (see Table 1). The instability of planar motion type 2 gives way to a whirling motion, that is, the string begins to whirl like a jump rope as presented in Fig. 7(d) and 7(g). These whirling motions are a consequence of the equal frequency of the motions in both planes, while the phase of them is different. Thus, the motion is composed of two modes which are strongly coupled. Fig. 7(d) and Fig. 7(e) describe the whirling motions corresponding to the type 4 in case $\bar{A}_s \bar{B}_s = 0$. The inclination of the motion curve depends on the phase difference $\phi = \phi_p - \phi_w$,

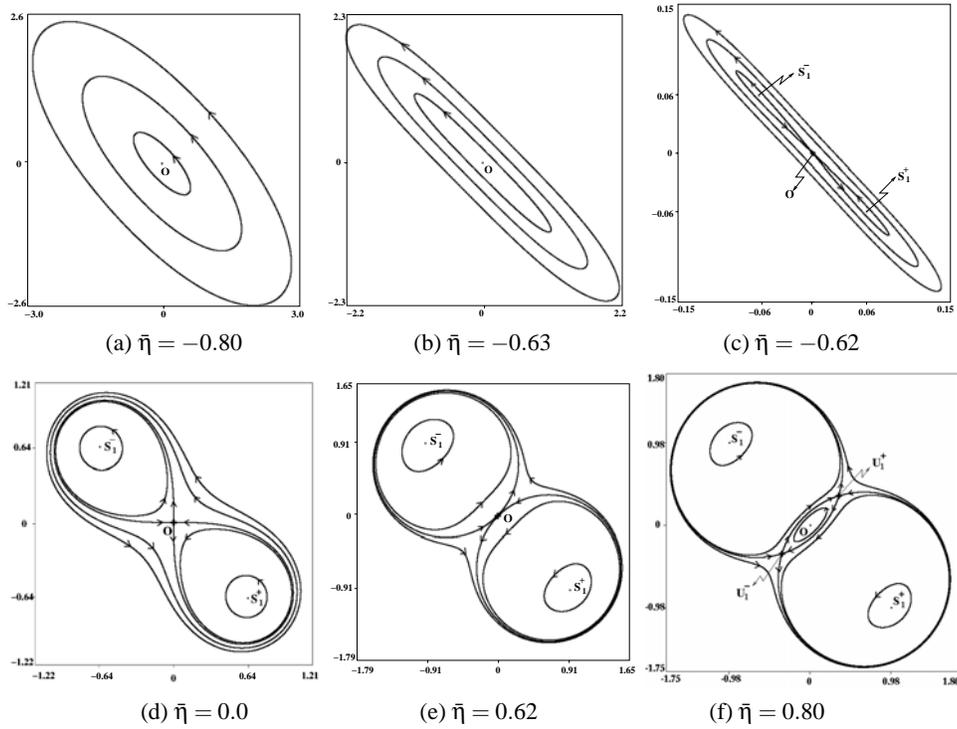


Figure 5: Behaviour of the critical points of system (4.2) in the hyperplane \mathbf{H}_k for $k = 1$, $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$, and $\bar{\beta} = 1.25$. The horizontal and the vertical axis are the \bar{A}_s (or the \bar{C}_s) -axis and the \bar{B}_s (or the \bar{D}_s) -axis, respectively.

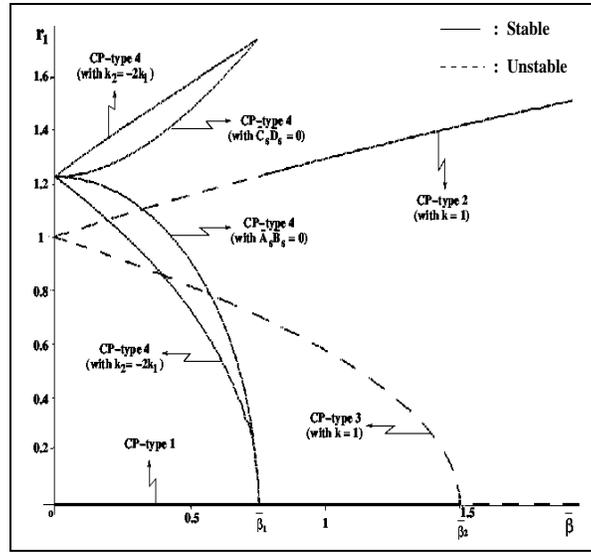


Figure 6: The stability response-curves $r_1 = \sqrt{\bar{A}_s^2 + \bar{B}_s^2}$ of system (4.2) with respect to $\bar{\beta}$ for $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$ and $\bar{\eta} = 0.75$: the curves of CP-type 2 and 3 represent response-curves in \mathbf{H}_1 , and the curves of CP-type 4 with $k_2 = -2k_1$, where $k_1 = \frac{\bar{A}_s}{\bar{C}_s}$ and $k_2 = \frac{\bar{B}_s}{\bar{D}_s}$, represent response curves of the periodic solutions in case all of the components are non-zero.

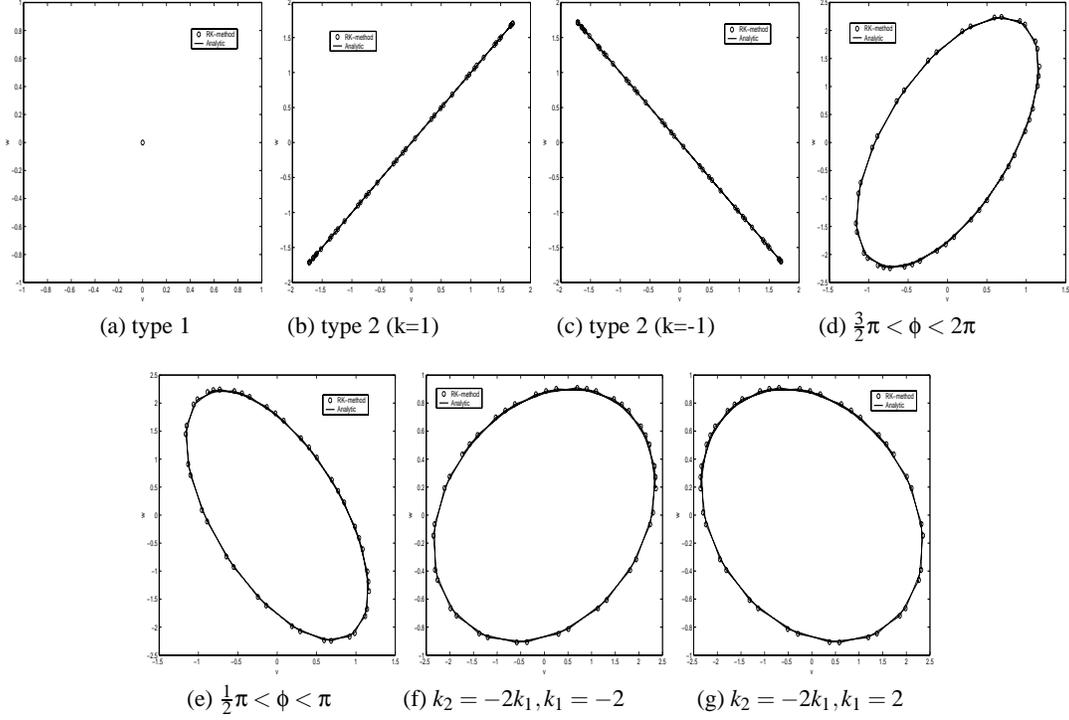


Figure 7: The types of stable motion of the string for $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$, $\bar{\eta} = 1.6014$, and $\bar{\beta} = 1.2159$. The first three figures describe planar motions, while the other figures describe whirling (non-planar) motions.

where ϕ_v and ϕ_w are the phase of the in-plane and out-of-plane motion, respectively. Similar results can be obtained for the other cases.

4.2 The case with (positive) damping: $\bar{\alpha}_1, \bar{\alpha}_2 > 0$

The critical points of system (4.2) with $\bar{\alpha}_1, \bar{\alpha}_2 > 0$ satisfy the following algebraic equations:

$$\begin{aligned}
 (\bar{\beta} + \bar{\alpha}_1)\bar{A}_s + \bar{B}_s \left[\frac{1}{4}(3(\bar{A}_s^2 + \bar{B}_s^2) + (\bar{C}_s^2 + \bar{D}_s^2)) - 2\bar{\eta} \right] + \frac{1}{2}(\bar{A}_s\bar{C}_s + \bar{B}_s\bar{D}_s)\bar{D}_s &= 0, \\
 (\bar{\beta} - \bar{\alpha}_1)\bar{B}_s + \bar{A}_s \left[\frac{1}{4}(3(\bar{A}_s^2 + \bar{B}_s^2) + (\bar{C}_s^2 + \bar{D}_s^2)) - 2\bar{\eta} \right] + \frac{1}{2}(\bar{A}_s\bar{C}_s + \bar{B}_s\bar{D}_s)\bar{C}_s &= 0, \\
 (\bar{\beta} + \bar{\alpha}_2)\bar{C}_s + \bar{D}_s \left[\frac{1}{4}((\bar{A}_s^2 + \bar{B}_s^2) + 3(\bar{C}_s^2 + \bar{D}_s^2)) - 2\bar{\eta} \right] + \frac{1}{2}(\bar{A}_s\bar{C}_s + \bar{B}_s\bar{D}_s)\bar{B}_s &= 0, \\
 (\bar{\beta} - \bar{\alpha}_2)\bar{D}_s + \bar{C}_s \left[\frac{1}{4}((\bar{A}_s^2 + \bar{B}_s^2) + 3(\bar{C}_s^2 + \bar{D}_s^2)) - 2\bar{\eta} \right] + \frac{1}{2}(\bar{A}_s\bar{C}_s + \bar{B}_s\bar{D}_s)\bar{A}_s &= 0.
 \end{aligned} \tag{4.14}$$

From system (4.14) one can derive the following equation:

$$(\bar{\alpha}_1 + \bar{\beta})\bar{A}_s^2 + (\bar{\alpha}_2 + \bar{\beta})\bar{C}_s^2 + (\bar{\alpha}_1 - \bar{\beta})\bar{B}_s^2 + (\bar{\alpha}_2 - \bar{\beta})\bar{D}_s^2 = 0. \tag{4.15}$$

Obviously, for $\bar{\alpha}_1, \bar{\alpha}_2 > \bar{\beta}$ system (4.14) only has the trivial solution as a solution, that is, the origin $O(0,0,0,0)$, and this is a stable solution. Moreover, for $\bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\beta}$ the components \bar{A}_s and \bar{C}_s of the semi-trivial and non-trivial solutions of (4.14) are zero. So, in that case the critical points of (4.2) are in the (\bar{B}_s, \bar{D}_s) -plane. These points exist for

$\bar{\eta} > 0$ and are unstable (degenerate) solutions, while the origin is a stable solution. In this subsection the study will be divided into two cases: $\bar{\alpha}_1 = \bar{\alpha}_2$ and $\bar{\alpha}_1 \neq \bar{\alpha}_2$.

4.2.1 The case $\bar{\alpha}_1 = \bar{\alpha}_2 = \alpha$

In this case it follows from (4.4) that the solutions of system (4.2) satisfy $(\bar{A}_s(t)\bar{D}_s(t) - \bar{B}_s(t)\bar{C}_s(t)) \rightarrow 0$ as $t \rightarrow \infty$. Therefore, for given initial conditions in the hyperplane \mathbf{G}_E , for arbitrary E , the solutions of (4.2) move into the hyperplane \mathbf{G}_0 as t tends to infinity. This means that the critical points of (4.2) can only be found in the hyperplane \mathbf{G}_0 . Moreover, the stability of the critical points in the $(\bar{A}_s, \bar{B}_s, \bar{C}_s, \bar{D}_s)$ -space is exactly the same as in the hyperplane \mathbf{G}_0 .

Clearly, the origin $O(0, 0, 0, 0)$ is always a critical point of (4.2). The critical points of (4.2) can be classified as follows (see Appendix 5.2 for the calculations):

$$\begin{aligned} \text{type 1 : } (\bar{A}_s, \bar{B}_s, \bar{C}_s, \bar{D}_s) &= (0, 0, 0, 0), \\ \text{type 2 : } (\bar{A}_s, \bar{B}_s, \bar{C}_s, \bar{D}_s) &= \left(-\sqrt{\frac{\bar{\beta}-\alpha}{\bar{\beta}+\alpha}}\bar{B}_s, \bar{B}_s, -\sqrt{\frac{\bar{\beta}-\alpha}{\bar{\beta}+\alpha}}\bar{D}_s, \bar{D}_s\right), \\ \text{type 3 : } (\bar{A}_s, \bar{B}_s, \bar{C}_s, \bar{D}_s) &= \left(\sqrt{\frac{\bar{\beta}-\alpha}{\bar{\beta}+\alpha}}\bar{B}_s, \bar{B}_s, \sqrt{\frac{\bar{\beta}-\alpha}{\bar{\beta}+\alpha}}\bar{D}_s, \bar{D}_s\right). \end{aligned} \quad (4.16)$$

The critical points of type 2 and 3 describe semi-trivial and non-trivial periodic solutions. As mentioned above for $\bar{\beta} = \alpha$ the critical points of type 2 and 3 are in the (\bar{B}_s, \bar{D}_s) -plane. It follows from (A.18) and (A.19) that the components \tilde{B}_s and \tilde{D}_s of the critical points of type 2 satisfy:

$$\begin{aligned} \tilde{B}_s^2 + \tilde{D}_s^2 &= \frac{2(\bar{\beta} + \alpha)}{3\bar{\beta}_s} (2\bar{\eta} + \sqrt{\bar{\beta}^2 - \alpha^2}), \\ \text{Cond}_4 &= (2\bar{\eta} + \sqrt{\bar{\beta}^2 - \alpha^2}) > 0, \end{aligned} \quad (4.17)$$

whereas the components of the critical points of type 3 for \tilde{B}_s and \tilde{D}_s satisfy:

$$\begin{aligned} \tilde{B}_s^2 + \tilde{D}_s^2 &= 2\frac{\bar{\beta} + \alpha}{3\bar{\beta}} (2\bar{\eta} - \sqrt{\bar{\beta}^2 - \alpha^2}), \\ \text{Cond}_5 &= (2\bar{\eta} - \sqrt{\bar{\beta}^2 - \alpha^2}) > 0. \end{aligned} \quad (4.18)$$

Comparing this case to the case without damping one sees that the presence of positive damping leads to the disappearance of the critical points of type 4. Therefore, in this case the whirling motion does not occur. In the other words, the string only moves in a hyperplane.

For $\bar{\alpha}_1 = \bar{\alpha}_2 = \alpha$ an overview of the existence and the stability of the critical points of type 2 and 3 for system (4.2) is given in Fig. 8. In this figure six domains can be distinguished. The type of critical points and their stability in the different domains and on the boundary curves are completely given in Table 2.

Looking at (4.17) and (4.18) one can see that for given $\bar{\eta}$, $\bar{\beta}$, and α such that $2\bar{\eta} > \bar{\beta} \geq \alpha$, system (4.2) has an infinite number of critical points of type 2 and 3. The collection of these points can be described by two curves, namely \mathbf{S}^α and \mathbf{U}^α , in the four dimensional $(\bar{A}_s, \bar{B}_s, \bar{C}_s, \bar{D}_s)$ -space. For $\bar{\beta} > \alpha$, the projection of the curves \mathbf{S}^α and \mathbf{U}^α are exactly same as the projection of the curves \mathbf{S} and \mathbf{U} in the case without damping, respectively. However,

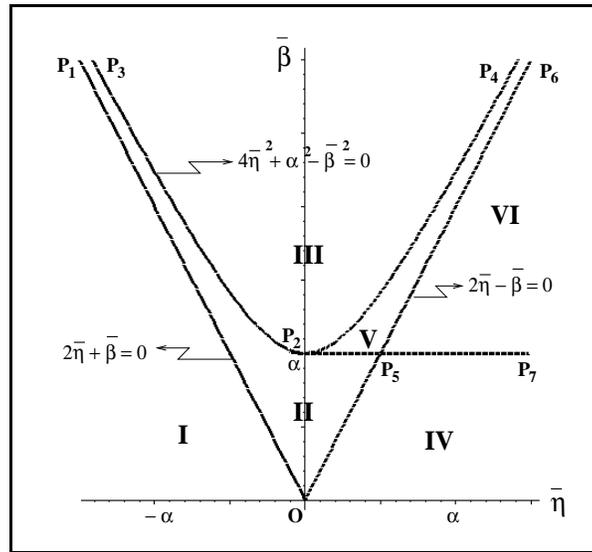


Figure 8: The six domains describing the critical points of system (4.2) in the $(\bar{\eta}, \bar{\beta})$ -plane for $\bar{\alpha}_1 = \bar{\alpha}_2 = \alpha$.

the stability on these curves \mathbf{S}^α and \mathbf{S} is quite different for $\bar{\eta} > \bar{\beta}$. Now we let $\bar{\beta}$ vary while α is kept fixed. If $\bar{\beta}$ decreases to α then the curve \mathbf{S}^α tends to the curve \mathbf{U}^α and when $\bar{\beta}$ is equal to α these curves coincide. The projection of the resulting curve to the (\bar{A}_s, \bar{B}_s) (or (\bar{C}_s, \bar{D}_s))-plane and the (\bar{B}_s, \bar{D}_s) -plane are a segment of a line and a circle with radius $\rho_\alpha = \sqrt{\frac{8}{3}}\bar{\eta}$, respectively, as has been shown in Fig. 9. If $\bar{\beta}$ decreases such that $\bar{\beta} < \alpha$, then the curve \mathbf{U}^α disappears.

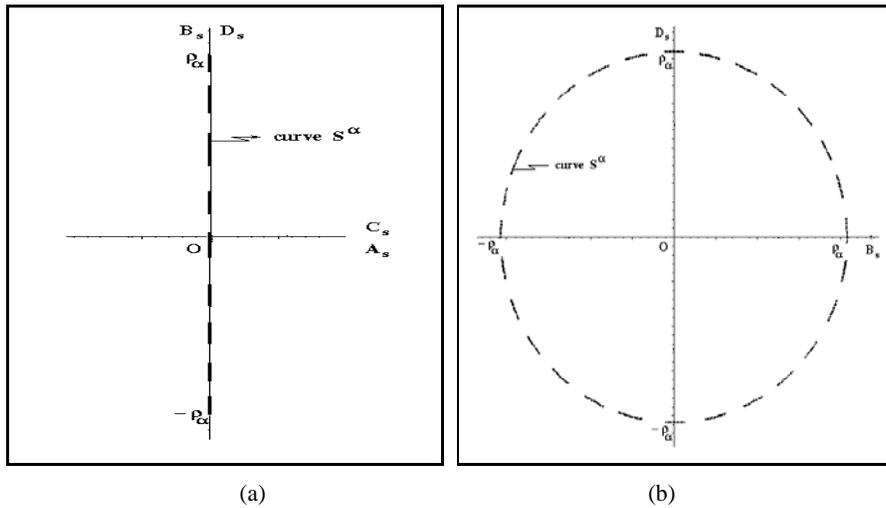


Figure 9: Projection of the curves \mathbf{S}^α ($= \mathbf{U}^\alpha$) for $\bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\beta}$ and $\bar{\eta} > 0$ to: (a) the (\bar{A}_s, \bar{B}_s) (or (\bar{C}_s, \bar{D}_s))-plane ; (b) the (\bar{B}_s, \bar{D}_s) - plane. The dashed line represents unstable solution.

The hyperplanes \mathbf{H}_k , for all k , defined by (4.11) are still invariants of system (4.2) for $\bar{\alpha}_1 = \bar{\alpha}_2$. Therefore, one can study the stability of the critical points in these planes. The

Table 2: The critical points of system (4.2) and their stability in the hyperplane \mathbf{G}_0 for $\bar{\alpha}_1 = \bar{\alpha}_2 = \alpha$ and $\tilde{\beta}_\alpha = \sqrt{\frac{\tilde{\beta}-\alpha}{\tilde{\beta}+\alpha}}$. The stability is determined by using the linearisation method.

Domains/ curves	number of critical points	location of the critical points	behaviour in the hyperplane \mathbf{G}_0
I, IV	1	(0,0,0,0)	stable (focus)
II, $\overline{OP_1}$, $\overline{OP_5}$	1	(0,0,0,0)	stable (node)
III	2	(0,0,0,0) ($-\tilde{\beta}_\alpha \tilde{B}_s, \tilde{B}_s, -\tilde{\beta}_\alpha \tilde{D}_s, \tilde{D}_s$)	unstable (saddle) stable (degenerate)
V, $\overline{P_5P_6}$	3	(0,0,0,0) ($-\tilde{\beta}_\alpha \tilde{B}_s, \tilde{B}_s, -\tilde{\beta}_\alpha \tilde{D}_s, \tilde{D}_s$) ($\tilde{\beta}_\alpha \tilde{B}_s, \tilde{B}_s, \tilde{\beta}_\alpha \tilde{D}_s, \tilde{D}_s$)	stable node stable (degenerate) unstable
VI	3	(0,0,0,0) ($-\tilde{\beta}_\alpha \tilde{B}_s, \tilde{B}_s, -\tilde{\beta}_\alpha \tilde{D}_s, \tilde{D}_s$) ($\tilde{\beta}_\alpha \tilde{B}_s, \tilde{B}_s, \tilde{\beta}_\alpha \tilde{D}_s, \tilde{D}_s$)	stable focus stable (degenerate) unstable
$\overline{P_2P_3}$	1	(0,0,0,0)	stable (degenerate)
$\overline{P_2P_5}$	2	(0,0,0,0) (0, \tilde{B}_s , 0, \tilde{D}_s)	stable node unstable (degenerate)
$\overline{P_2P_4}$	2	(0,0,0,0) ($-\tilde{\beta}_\alpha \tilde{B}_s, \tilde{B}_s, -\tilde{\beta}_\alpha \tilde{D}_s, \tilde{D}_s$)	unstable (degenerate) stable (degenerate)
$\overline{P_5P_7}$	2	(0,0,0,0) (0, \tilde{B}_s , 0, \tilde{D}_s)	stable focus unstable (degenerate)

intersection between the hyperplane \mathbf{H}_k with the curves \mathbf{S}^α and \mathbf{U}^α are:

$$\begin{aligned}
 S_{(\alpha,k)}^\pm & \left(\pm k \tilde{\beta}_\alpha \rho_{(\alpha,k)}^+, \mp k \rho_{(\alpha,k)}^+, \pm \tilde{\beta}_\alpha \rho_{(\alpha,k)}^+, \mp \rho_{(\alpha,k)}^+ \right), & \text{for real } k, \\
 S_{(\alpha,k)}^\pm & \left(\pm \tilde{\beta}_\alpha \rho_{(\alpha,k)}^+, \mp \rho_{(\alpha,k)}^+, 0, 0 \right), & \text{for } k = \pm\infty, \quad (4.19)
 \end{aligned}$$

and

$$\begin{aligned}
 U_{(\alpha,k)}^\pm & \left(\pm k \tilde{\beta}_\alpha \rho_{(\alpha,k)}^-, \mp k \rho_{(\alpha,k)}^-, \pm \tilde{\beta}_\alpha \rho_{(\alpha,k)}^-, \mp \rho_{(\alpha,k)}^- \right), & \text{for real } k, \\
 U_{(\alpha,k)}^\pm & \left(\pm \tilde{\beta}_\alpha \rho_{(\alpha,k)}^-, \mp \rho_{(\alpha,k)}^-, 0, 0 \right), & \text{for } k = \pm\infty, \quad (4.20)
 \end{aligned}$$

respectively, where:

$$\rho_{(\alpha,k)}^{\pm} = \begin{cases} \frac{\rho_{\alpha}^{\pm}}{1+k^2} \sqrt{(1+k^2)} & ; 0 \leq k < \infty, \\ \rho_{\alpha}^{\pm} & ; k = \pm\infty, \end{cases} \quad (4.21)$$

with $\rho_{\alpha}^{\pm} = \sqrt{\frac{2(\bar{\beta}+\alpha)}{3\bar{\beta}}(2\bar{\eta} \pm \sqrt{\bar{\beta}^2 - \alpha^2})}$. In the hyperplane \mathbf{H}_k the points $S_{(\alpha,k)}^{\pm}$ are asymptotically stable while the points $U_{(\alpha,k)}^{\pm}$ are unstable (saddle points).

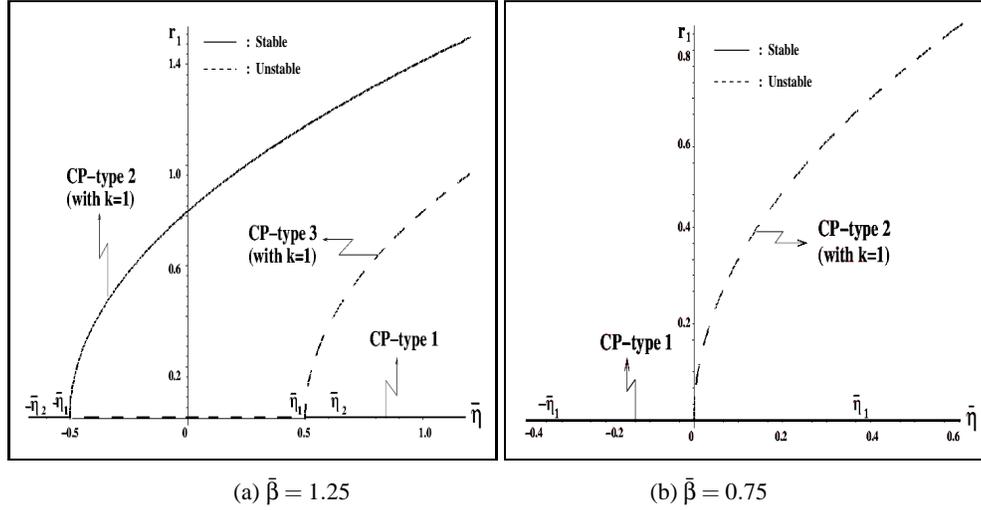


Figure 10: The stability response-curves $r_1 = \sqrt{\tilde{A}_s^2 + \tilde{B}_s^2}$ of system (4.2) as function of $\bar{\eta}$ in the hyperplane \mathbf{H}_k for $k = 1$, with $\alpha = 0.75$.

The response curves of the periodic solutions r_1 as function of $\bar{\eta}$ in the hyperplane \mathbf{H}_1 are given in Fig. 10. Let us suppose that $\bar{\eta}$ increases while $\bar{\beta}_s$ and α are kept fixed such that $\bar{\beta}_s > \alpha$. This process is represented by the line through the points $-\bar{\eta}_2$, $-\bar{\eta}_1$, $\bar{\eta}_1$, and $\bar{\eta}_2$ in Fig. 10(a). Starting with $\bar{\eta} < -\bar{\eta}_1$ only the CP-type 1 (trivial solution) exists, and it is a stable focus for $\bar{\eta} < -\bar{\eta}_2$ and a stable node for $-\bar{\eta}_2 \leq \bar{\eta} < -\bar{\eta}_1$. Between $-\bar{\eta}_1$ and $\bar{\eta}_1$ the CP-type 1 is unstable and two stable critical points of type 2 come in. Beyond the point $\bar{\eta}_1$ an unstable critical point of type 3 occurs while the CP-type 1 is again a stable solution and the critical points of type 2 are also stable. An illustration of the behaviour of the solutions in the hyperplane \mathbf{H}_1 (of course the same results are obtained for the other hyperplanes) is given in Fig. 11 for several values of $\bar{\eta}$.

Now let us consider the case that $\bar{\eta}$ is varied and $\alpha = \bar{\beta}$ (see also Fig. 10(b)). In this case the characteristics of the solutions differ from the case $\bar{\beta} > \alpha$. It can be seen clearly that the critical point of type 1 is always a stable solution, while the critical points of type 2 are unstable solutions. Hence the damping is responsible for the stabilization of the trivial solution. Moreover, the value $\alpha = \bar{\beta}$ is a critical (minimum) value of the damping parameter such that there is eventually no oscillation in the string. The behaviour of the solutions is shown in Fig. 12 for several values of $\bar{\eta}$. In this figure the critical points $U_{(\alpha,1)}^{\pm}$ are saddle-nodes. It can readily be seen that by increasing the values of $\bar{\beta}$ from α the points $U_{(\alpha,1)}^{\pm}$ bifurcate into two stable critical points $S_{(\alpha,1)}^{\pm}$ and two unstable critical points $U_{(\alpha,1)}^{\pm}$ as has been shown in Fig. 11.

If an experiment is done for which the excitation frequency λ is held fixed, but is near a

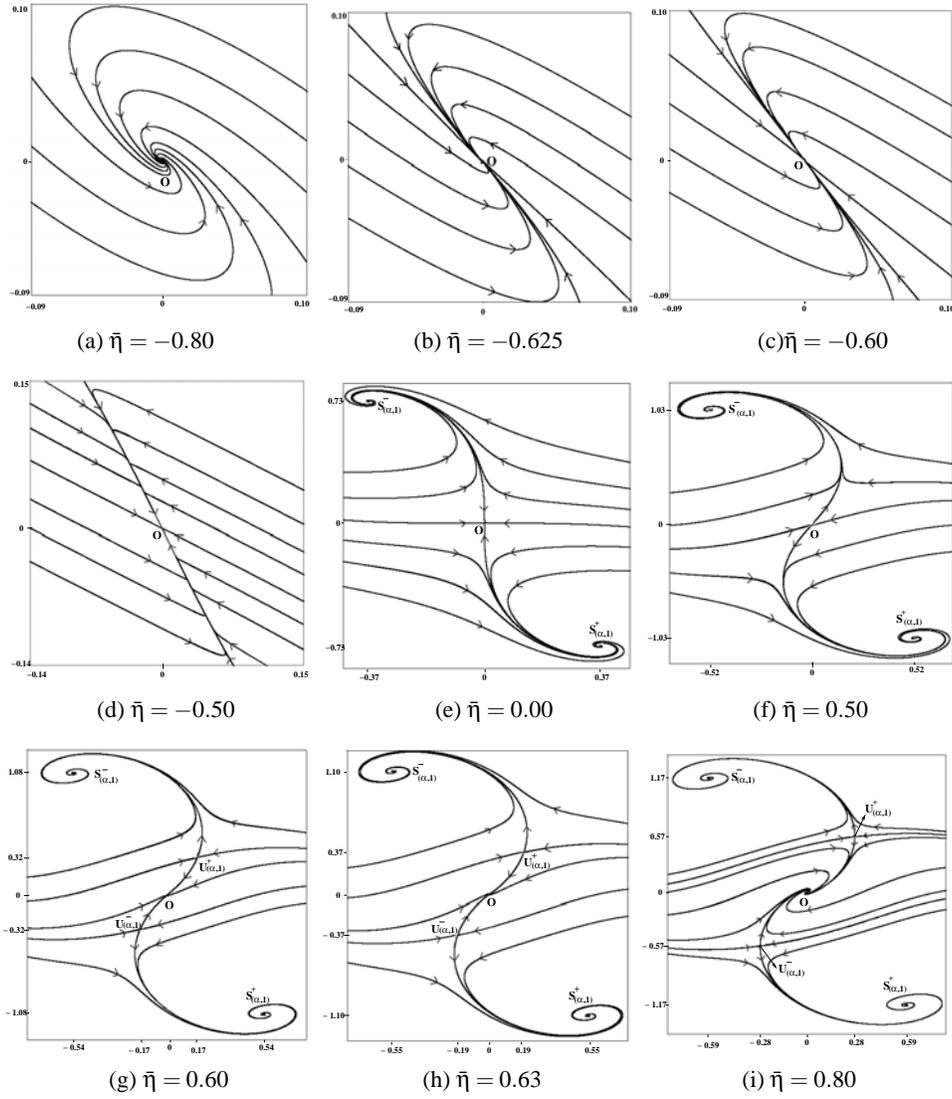


Figure 11: Behaviour of the solutions near the critical points of system (4.2) in the hyperplane \mathbf{H}_k for $k = 1$, $\alpha = 0.75$, and $\bar{\beta} = 1.25$. The horizontal and vertical axis are the \bar{A}_s (or \bar{C}_s) -axis and the \bar{B}_s (or \bar{D}_s) -axis, respectively.

resonance frequency, while the excitation amplitude $\bar{\beta}$ is varied slowly, a jump phenomena from the trivial solution to the periodic solution of type 2 in the plane can be observed. Suppose that the experiment is started at $\bar{\beta} < \bar{\beta}_1$ as indicated in Fig. 13. As $\bar{\beta}$ increases to $\bar{\beta}_3$, the amplitude of the periodic solution in the hyperplane \mathbf{H}_1 is still zero. This means there is no motion in the reference plane \mathbf{H}_1 . When $\bar{\beta}$ passes $\bar{\beta}_3$ a jump upward takes place from the point $\bar{\beta}_3$ to the point $\bar{\beta}'_3$, with an accompanying increase of \mathbf{r}_1 , after which \mathbf{r}_1 increases slowly for increasing $\bar{\beta}$. If this process is reversed, \mathbf{r}_1 decreases slowly as $\bar{\beta}$ decreases from the point $\bar{\beta}'_4$ to the point $\bar{\beta}'_3$. As $\bar{\beta}$ decreases further a jump downward from the point $\bar{\beta}'_1$ to the point $\bar{\beta}_1$ takes places, with an accompanying decrease in \mathbf{r}_1 , after which \mathbf{r}_1 is zero for decreasing $\bar{\beta}$. So it can be concluded that for increasing values of $\bar{\beta}$ the trivial solution becomes unstable. This implies that the string is always moving for large enough excitation amplitudes.

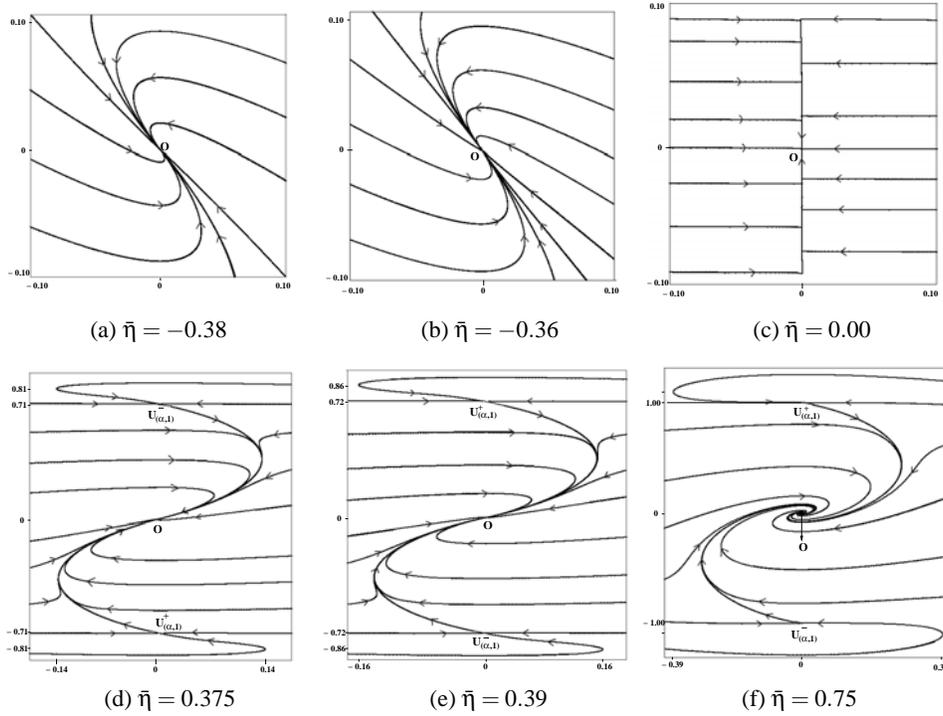


Figure 12: Behaviour of the solutions near the critical points of system (4.2) in the hyperplane \mathbf{H}_k for $k = 1$ and $\alpha = \bar{\beta} = 0.75$. The horizontal and the vertical axis are the \bar{A}_s (or \bar{C}_s) -axis and the \bar{B}_s (or \bar{D}_s) -axis, respectively.

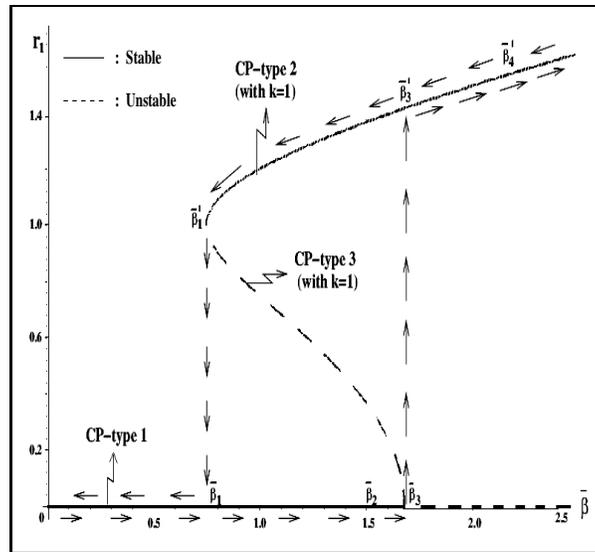


Figure 13: The stability response-curves $r_1 = \sqrt{\bar{A}_s^2 + \bar{B}_s^2}$ of system (4.2) as function of $\bar{\beta}$ in the hyperplane \mathbf{H}_k for $k = 1$, $\alpha = 0.75$, and $\eta = 0.75$.

4.2.2 The case $\alpha_1 \neq \alpha_2$

In this case (4.4) is still a first integral of system (4.2). Therefore the critical points of system (4.2) are only in the hyperplane \mathbf{G}_0 . Moreover, the stability of the critical points of

system (4.2) in the $(\bar{A}_s, \bar{B}_s, \bar{C}_s, \bar{D}_s)$ -space is the same as in \mathbf{G}_0 . By using a similar procedure as in Appendix 5.2 the types of critical points of system (4.2) are:

$$\begin{aligned}
 \text{type 1 : } & (\bar{A}_s, \bar{D}_s, \bar{C}_s, \bar{D}_s) = O(0, 0, 0, 0), \\
 \text{type 2 : } & (\bar{A}_s, \bar{D}_s, \bar{C}_s, \bar{D}_s) = \left(\pm(2-i)\sqrt{\frac{\bar{\beta}-\bar{\alpha}_1}{\bar{\beta}+\bar{\alpha}_1}}\rho_{\bar{\alpha}_1}^+, \mp(2-i)\rho_{\bar{\alpha}_1}^+, \pm(i-1)\sqrt{\frac{\bar{\beta}-\bar{\alpha}_2}{\bar{\beta}+\bar{\alpha}_2}}\rho_{\bar{\alpha}_2}^+, \right. \\
 & \quad \left. \mp(i-1)\rho_{\bar{\alpha}_2}^+ \right), \\
 \text{type 3 : } & (\bar{A}_s, \bar{D}_s, \bar{C}_s, \bar{D}_s) = \left(\pm(2-i)\sqrt{\frac{\bar{\beta}-\bar{\alpha}_1}{\bar{\beta}+\bar{\alpha}_1}}\rho_{\bar{\alpha}_1}^-, \pm(2-i)\rho_{\bar{\alpha}_1}^-, \pm(i-1)\sqrt{\frac{\bar{\beta}-\bar{\alpha}_2}{\bar{\beta}+\bar{\alpha}_2}}\rho_{\bar{\alpha}_2}^-, \right. \\
 & \quad \left. \pm(i-1)\rho_{\bar{\alpha}_2}^- \right), \tag{4.22}
 \end{aligned}$$

where $i = 1$ or 2 . In this case the critical points of type 2 and of type 3 only represent semi-trivial solutions and the existence of them follows from the conditions 4 and 5 in (4.17) and (4.18). This shows that the difference in damping values of $\bar{\alpha}_1$ and $\bar{\alpha}_2$ causes the singular points of system (4.2) to be in the planes \mathbf{H}_∞ and \mathbf{H}_0 . These planes are still invariants of system (4.2). Moreover, it turns out the in-plane and out-of-plane periodic solutions do not interact, and that the damping in both planes influences the stability of the periodic solutions.

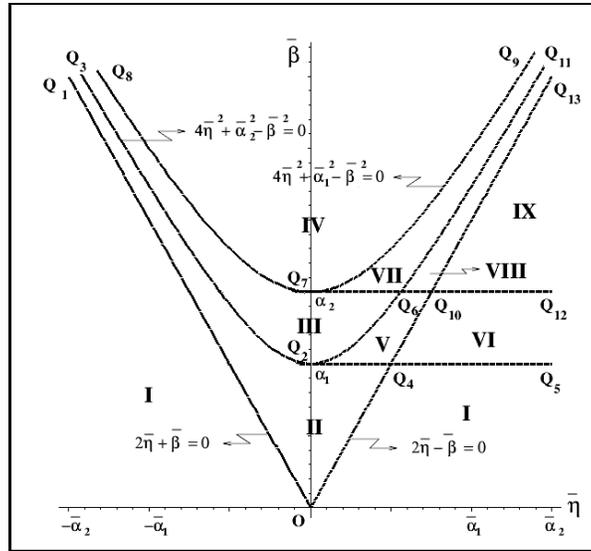


Figure 14: The stability diagram of the critical points of system (4.2) in the $(\bar{\eta}, \bar{\beta})$ -plane for $\bar{\alpha}_1 < \bar{\alpha}_2$.

We assume, without loss of generality, that the values of $\bar{\alpha}_1$ and $\bar{\alpha}_2$ satisfy $0 < \bar{\alpha}_1 < \bar{\alpha}_2$. The critical points in the plane \mathbf{H}_0 , of course, can be expected to appear after increasing the excitation amplitude such that $\bar{\beta} > \bar{\alpha}_2$. The stability diagram of the periodic solutions in the hyperplane \mathbf{G}_0 is given in Fig. 14. In this figure nine domains are defined in which the behaviour of the periodic solutions can be different. The curves in Fig. 14 are found in a similar way as for the case $\bar{\alpha}_1 = \bar{\alpha}_2$. We note that the existence of the critical points in the planes \mathbf{H}_0 and \mathbf{H}_∞ are independent of $\bar{\alpha}_1$ and $\bar{\alpha}_2$, respectively. However, the behaviour of the solutions depends on $\bar{\alpha}_1$ and $\bar{\alpha}_2$.

Let us now analyze in more detail how the behaviour of solutions changes in the $(\bar{\eta}, \bar{\beta})$ -plane by considering Fig. 14. If $(\bar{\eta}, \bar{\beta})$ lies in the domain I, there is only one critical point of type 1 which is a stable focus. For $(\bar{\eta}, \bar{\beta})$ on the curves $\overline{OQ_1}$ and $\overline{OQ_4}$ or in the domain II, there is still one critical point of type 1 but now it is a stable node. For $(\bar{\eta}, \bar{\beta})$ in domain III two new solutions of type 2 in \mathbf{H}_∞ occur and these are stable, while the critical point of type 1 now becomes unstable. For the other domains and curves an overview of the number of critical points and their stability is given in Table 3.

Table 3: The critical points of system (4.2) and their stability in the $(\bar{A}_s, \bar{B}_s, \bar{C}_s, \bar{D}_s)$ -space for $0 < \bar{\alpha}_1 < \bar{\alpha}_2$ and $\tilde{\beta}_{\alpha_i} = \sqrt{\frac{\bar{\beta} - \bar{\alpha}_i}{\bar{\beta} + \bar{\alpha}_i}}, i = 1$ or 2. The stability is determined by using the linearisation method.

Domains/ curves	number of critical points	location of the critical points	behaviour in the hyperplane \mathbf{G}_0
I	1	(0,0,0,0)	stable focus
II, $\overline{OQ_1}$, and $\overline{OQ_4}$	1	(0,0,0,0)	stable node
III	3	(0,0,0,0) $(\pm \tilde{\beta}_{\alpha_1} \rho_{\alpha_1}^+, \mp \rho_{\alpha_1}^+, 0, 0)$	unstable stable
IV	5	(0, 0, 0, 0) $(\pm \tilde{\beta}_{\alpha_1} \rho_{\alpha_1}^+, \mp \rho_{\alpha_1}^+, 0, 0)$ $(0, 0, \pm \tilde{\beta}_{\alpha_2} \rho_{\alpha_2}^+, \mp \rho_{\alpha_2}^+)$	unstable stable unstable
V and $\overline{Q_4Q_{10}}$	5	(0,0,0,0) $(\pm \tilde{\beta}_{\alpha_1} \rho_{\alpha_1}^+, \mp \rho_{\alpha_1}^+, 0, 0)$ $(\pm \tilde{\beta}_{\alpha_1} \rho_{\alpha_1}^-, \pm \rho_{\alpha_1}^-, 0, 0)$	stable 4d-node stable unstable
VI	5	(0, 0, 0, 0) $(\pm \tilde{\beta}_{\alpha_1} \rho_{\alpha_1}^+, \mp \rho_{\alpha_1}^+, 0, 0)$ $(\pm \tilde{\beta}_{\alpha_1} \rho_{\alpha_1}^-, \pm \rho_{\alpha_1}^-, 0, 0)$	stable 4d-focus stable unstable
VII	7	(0, 0, 0, 0) $(\pm \tilde{\beta}_{\alpha_1} \rho_{\alpha_1}^+, \mp \rho_{\alpha_1}^+, 0, 0)$ $(0, 0, \pm \tilde{\beta}_{\alpha_2} \rho_{\alpha_2}^+, \mp \rho_{\alpha_2}^+)$ $(0, 0, \pm \tilde{\beta}_{\alpha_2} \rho_{\alpha_2}^-, \pm \rho_{\alpha_2}^-)$	unstable stable unstable unstable
VIII and $\overline{Q_{10}Q_{13}}$	9	(0,0,0,0) $(\pm \tilde{\beta}_{\alpha_1} \rho_{\alpha_1}^+, \mp \rho_{\alpha_1}^+, 0, 0)$ $(\pm \tilde{\beta}_{\alpha_1} \rho_{\alpha_1}^-, \pm \rho_{\alpha_1}^-, 0, 0)$ $(0, 0, \pm \tilde{\beta}_{\alpha_2} \rho_{\alpha_2}^+, \mp \rho_{\alpha_2}^+)$	stable 4d-node stable unstable unstable

		$(0, 0, \pm\tilde{\beta}_{\alpha_2} \rho_{\alpha_2}^-, \pm\rho_{\alpha_2}^-)$	unstable
IX	9	$(0, 0, 0, 0)$	stable 4d-focus
		$(\pm\tilde{\beta}_{\alpha_1} \rho_{\alpha_1}^+, \mp\rho_{\alpha_1}^+, 0, 0)$	stable
		$(\pm\tilde{\beta}_{\alpha_1} \rho_{\alpha_1}^-, \pm\rho_{\alpha_1}^-, 0, 0)$	unstable
		$(0, 0, \pm\tilde{\beta}_{\alpha_2} \rho_{\alpha_2}^+, \mp\rho_{\alpha_2}^+)$	unstable
		$(0, 0, \pm\tilde{\beta}_{\alpha_2} \rho_{\alpha_2}^-, \pm\rho_{\alpha_2}^-)$	unstable
$\overline{Q_2Q_3}$	1	$(0, 0, 0, 0)$	stable (degenerate)
$\overline{Q_2Q_4}$	3	$(0, 0, 0, 0)$	stable 4d-node
		$(0, \pm\rho_{\alpha_1}^+, 0, 0)$	unstable (degenerate)
$\overline{Q_2Q_6}$	3	$(0, 0, 0, 0)$	unstable (degenerate)
		$(\pm\tilde{\beta}_{\alpha_1} \rho_{\alpha_1}^+, \mp\rho_{\alpha_1}^+, 0, 0)$	stable (degenerate)
$\overline{Q_4Q_5}$	3	$(0, 0, 0, 0)$	stable ("4d-focus")
		$(0, \pm\rho_{\alpha_1}^+, 0, 0)$	stable (degenerate)
$\overline{Q_7Q_8}$	3	$(0, 0, 0, 0)$	unstable (degenerate)
		$(\pm\tilde{\beta}_{\alpha_1} \rho_{\alpha_1}^+, \mp\rho_{\alpha_1}^+, 0, 0)$	stable
$\overline{Q_6Q_7}$	5	$(0, 0, 0, 0)$	unstable
		$(\pm\tilde{\beta}_{\alpha_1} \rho_{\alpha_1}^+, \mp\rho_{\alpha_1}^+, 0, 0)$	stable
		$(0, 0, 0, \pm\rho_{\alpha_2}^+)$	unstable (degenerate)
$\overline{Q_7Q_9}$	5	$(0, 0, 0, 0)$	unstable (degenerate)
		$(\pm\tilde{\beta}_{\alpha_1} \rho_{\alpha_1}^+, \mp\rho_{\alpha_1}^+, 0, 0)$	stable
		$(0, 0, \pm\tilde{\beta}_{\alpha_2} \rho_{\alpha_2}^+, \mp\rho_{\alpha_2}^+)$	unstable (degenerate)
$\overline{Q_6Q_{10}}$	7	$(0, 0, 0, 0)$	stable ("4d-node")
		$(\pm\tilde{\beta}_{\alpha_1} \rho_{\alpha_1}^+, \mp\rho_{\alpha_1}^+, 0, 0)$	stable
		$(\pm\tilde{\beta}_{\alpha_1} \rho_{\alpha_1}^-, \pm\rho_{\alpha_1}^-, 0, 0)$	unstable
		$(0, 0, 0, \pm\rho_{\alpha_2}^+)$	unstable (degenerate)
$\overline{Q_6Q_{11}}$	7	$(0, 0, 0, 0)$	unstable (degenerate)
		$(\pm\tilde{\beta}_{\alpha_2} \rho_{\alpha_2}^+, \mp\rho_{\alpha_2}^+, 0, 0)$	stable (degenerate)
		$(0, 0, \pm\tilde{\beta}_{\alpha_2} \rho_{\alpha_2}^+, \mp\rho_{\alpha_2}^+)$	unstable
		$(0, 0, \pm\tilde{\beta}_{\alpha_2} \rho_{\alpha_2}^-, \pm\rho_{\alpha_2}^-)$	unstable
$\overline{Q_{10}Q_{12}}$	7	$(0, 0, 0, 0)$	stable 4d-focus
		$(\pm\tilde{\beta}_{\alpha_1} \rho_{\alpha_1}^+, \mp\rho_{\alpha_1}^+, 0, 0)$	stable
		$(\pm\tilde{\beta}_{\alpha_1} \rho_{\alpha_1}^-, \pm\rho_{\alpha_1}^-, 0, 0)$	unstable
		$(0, 0, 0, \pm\rho_{\alpha_2}^+)$	unstable (degenerate)

To illustrate these results, we numerically integrate system (4.2) and give several projected

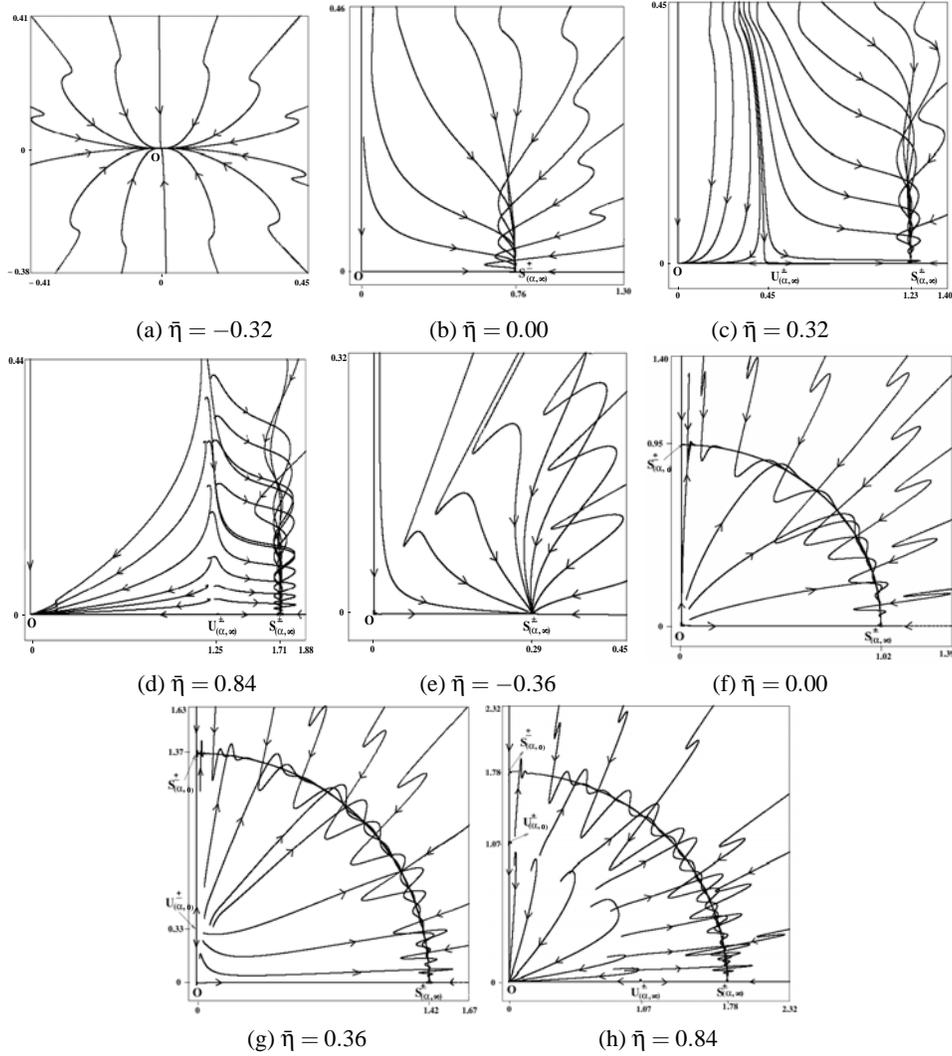


Figure 15: Behaviour of the solutions near the critical points of system (4.2) projected to the (r_1, r_2) -plane; $r_1 = \sqrt{\bar{A}_s^2 + \bar{B}_s^2}$ is the horizontal axis and $r_2 = \sqrt{\bar{C}_s^2 + \bar{D}_s^2}$ is the the vertical axis. The first four figures describe the case $\bar{\alpha}_1 < \bar{\beta} < \bar{\alpha}_2$ ($\bar{\alpha}_1 = 0.65, \bar{\alpha}_2 = 0.90$, and $\bar{\beta} = 0.81$) and the other figures describe the case $\bar{\alpha}_1 < \bar{\alpha}_2 < \bar{\beta}$ ($\bar{\alpha}_1 = 0.65, \bar{\alpha}_2 = 0.77$, and $\bar{\beta} = 1.01$).

trajectories for different values of $\bar{\eta}$ and $\bar{\beta}$. Because a reduction of system (4.2) to a lower dimension seems to be impossible, the trajectories are projected on the (r_1, r_2) -state plane as has been shown in Fig. 15. For $\bar{\alpha}_1 < \bar{\beta} < \bar{\alpha}_2$, the periodic solutions of type 2 and 3 do not exist in the hyperplane \mathbf{H}_0 . Hence, one only expects periodic solutions in the hyperplane \mathbf{H}_∞ (see Figs. 15 (a)-(d)). The stability of the critical points in the $(\bar{A}_s, \bar{B}_s, \bar{C}_s, \bar{D}_s)$ -space is the same as in the hyperplane \mathbf{H}_∞ . Whereas for $\bar{\alpha}_1 < \bar{\alpha}_2 < \bar{\beta}$, one can expect the periodic solutions of type 2 and 3 to exist on both hyperplanes \mathbf{H}_0 and \mathbf{H}_∞ . In this case the stability of the periodic solutions $U_{(\alpha_1, \infty)}^\pm$ and $S_{(\alpha_1, \infty)}^\pm$ in the full system is still similar. However, the stability of the periodic solutions $S_{(\alpha_2, 0)}^\pm$ is really different. We know that the points $S_{(\alpha_2, 0)}^\pm$ are stable foci in the hyperplane \mathbf{H}_0 but become unstable solutions in the full system. This phenomena is caused by the difference in damping. These results are shown in the last four

figures of Fig. 15.

Conclusions

In this paper the dynamics of a stretched string suspended between a fixed support and a vibrating support has been studied. Due to a parametric (longitudinal) excitation the string can vibrate in vertical direction. When the frequency of the parametric excitation is near twice a linear natural frequency the amplitudes of the string oscillation can become large. In this paper the attention is focused on the existence and the stability of (almost) periodic solutions. There are four parameters (i.e. the damping coefficients $\bar{\alpha}_1$ and $\bar{\alpha}_2$, the excitation amplitude $\bar{\beta}$, and a detuning coefficient $\bar{\eta}$), which influence the ultimate oscillations of the string. The study is divided into two cases: no damping (i.e. $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$), and positive damping (i.e. $\bar{\alpha}_1, \bar{\alpha}_2 > 0$). When there is no damping stable whirling motion, and stable planar motion can occur. With positive damping only planar motion can occur. A classification of all (almost) periodic motions and their stability is given.

5 Appendix

On the types of critical points of system (4.2)

5.1 The case : $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$

5.1.1 Identification of the semi-trivial solutions

Starting with $\bar{A}_s = \bar{B}_s = 0$, then equation (4.5) becomes:

$$\begin{aligned} \bar{\beta}\bar{C}_s + \bar{D}_s\left[\frac{3}{4}(\bar{C}_s^2 + \bar{D}_s^2) - 2\bar{\eta}\right] &= 0, \\ \bar{\beta}\bar{D}_s + \bar{C}_s\left[\frac{3}{4}(\bar{C}_s^2 + \bar{D}_s^2) - 2\bar{\eta}\right] &= 0. \end{aligned} \quad (\text{A.1})$$

Since \bar{C}_s and \bar{D}_s are not both zero it follows from (A.1) that $\bar{C}_s = \pm\bar{D}_s \neq 0$ under the condition $\bar{\beta}^2 - \left[\frac{3}{4}(\bar{C}_s^2 + \bar{D}_s^2) - 2\bar{\eta}\right]^2 = 0$. Hence, if $\bar{C}_s = -\bar{D}_s$ then a semi-trivial solution of (4.5) corresponds to a critical point of type 2, whereas for $\bar{C}_s = \bar{D}_s$ a semi-trivial solution corresponds to a critical point of type 3, where $\bar{B}_s = 0$. Similarly it follows from $\bar{C}_s = \bar{D}_s = 0$ that $\bar{A}_s = \pm\bar{B}_s \neq 0$ under the condition $\bar{\beta}^2 - \left[\frac{3}{4}(\bar{A}_s^2 + \bar{B}_s^2) - 2\bar{\eta}\right]^2 = 0$. Again a semi-trivial solution corresponds to a critical point of type 2 or 3.

5.1.2 Identification of the non-trivial solutions

In this case it will turn out to be not so difficult to show that at most one of the solutions \bar{A}_s , \bar{B}_s , \bar{C}_s , and \bar{D}_s is zero. We start with the case that all of the solutions are not zero. Setting $\bar{A}_s = k_1\bar{C}_s$ and $\bar{B}_s = k_2\bar{D}_s$, and then by substituting \bar{A}_s and \bar{B}_s into (4.5), one obtains:

$$\begin{aligned} k_1\bar{\beta}\bar{C}_s + k_2\bar{D}_s\left[\frac{1}{4}\left((3k_1^2 + 1)\bar{C}_s^2 + (3k_2^2 + 1)\bar{D}_s^2\right) - 2\bar{\eta}\right] + \frac{1}{2}(k_1\bar{C}_s^2 + \\ k_2\bar{D}_s^2)\bar{D}_s &= 0, \\ k_2\bar{\beta}\bar{D}_s + k_1\bar{C}_s\left[\frac{1}{4}\left((3k_1^2 + 1)\bar{C}_s^2 + (3k_2^2 + 1)\bar{D}_s^2\right) - 2\bar{\eta}\right] + \frac{1}{2}(k_1\bar{C}_s^2 + \\ k_2\bar{D}_s^2)\bar{C}_s &= 0, \end{aligned}$$

$$\begin{aligned}
\bar{\beta}\bar{C}_s + \bar{D}_s \left[\frac{1}{4} \left((k_1^2 + 3)\bar{C}_s^2 + (k_2^2 + 3)\bar{D}_s^2 \right) - 2\bar{\eta} \right] + \frac{1}{2}k_2(k_1\bar{C}_s^2 + k_2\bar{D}_s^2)\bar{D}_s &= 0, \\
\bar{\beta}\bar{D}_s + \bar{C}_s \left[\frac{1}{4} \left((k_1^2 + 3)\bar{C}_s^2 + (k_2^2 + 3)\bar{D}_s^2 \right) - 2\bar{\eta} \right] + \frac{1}{2}k_1(k_1\bar{C}_s^2 + k_2\bar{D}_s^2)\bar{C}_s &= 0.
\end{aligned} \tag{A.2}$$

Multiplying the third equation of (A.2) with k_2 and then subtracting this from the first equation of (A.2), one obtains:

$$(k_1 - k_2)\bar{C}_s \left[(1 + k_1k_2)\bar{C}_s\bar{D}_s + 2\bar{\beta} \right] = 0. \tag{A.3}$$

Similarly from the second and fourth equation of (A.2) one finds:

$$(k_1 - k_2)\bar{D}_s \left[(1 + k_1k_2)\bar{C}_s\bar{D}_s + 2\bar{\beta} \right] = 0. \tag{A.4}$$

It follows from (A.3) and (A.4) that there are two cases: $k_2 = k_1$ and $k_2 \neq k_1$. If $k_2 = k_1$ (A.2) reduces to:

$$\begin{aligned}
\bar{\beta}\bar{C}_s + \bar{D}_s \left[\frac{3}{4}(1 + k_1^2)(\bar{C}_s^2 + \bar{D}_s^2) - 2\bar{\eta} \right] &= 0, \\
\bar{\beta}\bar{D}_s + \bar{C}_s \left[\frac{3}{4}(1 + k_1^2)(\bar{C}_s^2 + \bar{D}_s^2) - 2\bar{\eta} \right] &= 0.
\end{aligned} \tag{A.5}$$

The non-zero solutions of (A.5) are $\bar{C}_s = \pm\bar{D}_s$ under the condition that

$$\bar{\beta}^2 - \left[\frac{3}{4}(1 + k_1^2)(\bar{C}_s^2 + \bar{D}_s^2) - 2\bar{\eta} \right]^2 = 0. \tag{A.6}$$

This shows that for $k_1 = k_2$ the non-trivial solutions of (4.5) correspond to the critical points of type 2 ($\bar{C}_s = -\bar{D}_s$) and of type 3 ($\bar{C}_s = \bar{D}_s$). For $k_1 \neq k_2$ it follows from (A.3) that the non-trivial solutions of (4.5) satisfy

$$\frac{1}{2}(1 + k_1k_2)\bar{C}_s\bar{D}_s + \bar{\beta} = 0. \tag{A.7}$$

It then follows from (A.7) that system (A.2) reduces to:

$$\begin{aligned}
(1 + k_1^2)\bar{C}_s^2 + 3(1 + k_2^2)\bar{D}_s^2 - 8\bar{\eta} &= 0, \\
3(1 + k_1^2)\bar{C}_s^2 + (1 + k_2^2)\bar{D}_s^2 - 8\bar{\eta} &= 0.
\end{aligned} \tag{A.8}$$

The non-zero solutions of (A.8) are given by:

$$(1 + k_1^2)\bar{C}_s^2 = (1 + k_2^2)\bar{D}_s^2 = 2\bar{\eta}. \tag{A.9}$$

Therefore, non-trivial solutions of (4.5) with $\frac{\bar{A}_s}{\bar{C}_s} \neq \frac{\bar{B}_s}{\bar{D}_s}$ correspond to the critical points of type 4.

Now we consider the case that exactly one of \bar{A}_s , \bar{B}_s , \bar{C}_s , and \bar{D}_s is equal to zero. Starting with $\bar{A}_s = 0$, (4.5) can be simplified to:

$$\begin{aligned}
\bar{\beta} + \frac{1}{2}\bar{C}_s\bar{D}_s &= 0, \\
\bar{B}_s^2 + \bar{C}_s^2 + \bar{D}_s^2 - 4\bar{\eta} &= 0, \\
3\bar{B}_s^2 + \bar{C}_s^2 + 3\bar{D}_s^2 - 8\bar{\eta} &= 0.
\end{aligned} \tag{A.10}$$

From (A.10) one finds:

$$\begin{aligned}\bar{A}_s^2 + \bar{C}_s^2 &= \bar{B}_s^2 + \bar{D}_s^2 = 2\bar{\eta}, \\ \bar{A}_s\bar{B}_s + \bar{C}_s\bar{D}_s &= -2\bar{\beta},\end{aligned}\tag{A.11}$$

where $\bar{A}_s = 0$. In a similar way the other cases can be treated, and one obtains (A.11). This means that non-trivial solutions of (4.5) with one of their components equal to zero, correspond to the critical points of type 4.

5.2 The case : $\bar{\alpha}_1, \bar{\alpha}_2 > 0$

5.2.1 Identification of the semi trivial solutions

Starting with $\bar{A}_s = \bar{B}_s = 0$, (4.13) becomes:

$$\begin{aligned}(\bar{\beta} + \bar{\alpha}_2)\bar{C}_s + \bar{D}_s \left[\frac{3}{4}(\bar{C}_s^2 + \bar{D}_s^2) - 2\bar{\eta} \right] &= 0, \\ (\bar{\beta} - \bar{\alpha}_2)\bar{D}_s + \bar{C}_s \left[\frac{3}{4}(\bar{C}_s^2 + \bar{D}_s^2) - 2\bar{\eta} \right] &= 0.\end{aligned}\tag{A.12}$$

The non-trivial solutions of (A.12) are $\bar{C}_s = \pm \sqrt{\frac{\bar{\beta} - \bar{\alpha}_2}{\bar{\beta} + \bar{\alpha}_2}} \bar{D}_s$ under the condition:

$$(\bar{\beta}^2 - \bar{\alpha}_2^2) - \left[\frac{3}{4}(\bar{C}_s^2 + \bar{D}_s^2) - 2\bar{\eta} \right]^2 = 0.\tag{A.13}$$

If $\bar{C}_s = -\sqrt{\frac{\bar{\beta} - \bar{\alpha}_2}{\bar{\beta} + \bar{\alpha}_2}} \bar{D}_s$, the semi-trivial critical point corresponds to a critical point of type 2, whereas if $\bar{C}_s = \sqrt{\frac{\bar{\beta} - \bar{\alpha}_2}{\bar{\beta} + \bar{\alpha}_2}} \bar{D}_s$, the semi-trivial critical point corresponds to a critical point of type 3. Similarly for $\bar{C}_s = \bar{D}_s = 0$ one finds $\bar{A}_s = \pm \sqrt{\frac{\bar{\beta} - \bar{\alpha}_1}{\bar{\beta} + \bar{\alpha}_1}} \bar{B}_s$ under the condition:

$$(\bar{\beta}^2 - \bar{\alpha}_1^2) - \left[\frac{3}{4}(\bar{A}_s^2 + \bar{B}_s^2) - 2\bar{\eta} \right]^2 = 0.\tag{A.14}$$

Again, one obtains that the semi-trivial critical points correspond to critical points of type 2 or 3.

5.2.2 Identification of the non-trivial solutions

In this case let $\bar{A}_s = k_1\bar{C}_s$ and $\bar{B}_s = k_2\bar{D}_s$ and substitute \bar{A}_s and \bar{B}_s into (4.14) to obtain:

$$\begin{aligned}k_1(\bar{\beta} + \bar{\alpha}_1 + \frac{1}{2}\bar{C}_s\bar{D}_s)\bar{C}_s + k_2\bar{D}_s \left[\frac{1}{4}((3k_1^2 + 1)\bar{C}_s^2 + 3(k_2^2 + 1)\bar{D}_s^2) - 2\bar{\eta} \right] &= 0, \\ k_2(\bar{\beta} - \bar{\alpha}_1 + \frac{1}{2}\bar{C}_s\bar{D}_s)\bar{D}_s + k_1\bar{C}_s \left[\frac{1}{4}(3(k_1^2 + 1)\bar{C}_s^2 + (3k_2^2 + 1)\bar{D}_s^2) - 2\bar{\eta} \right] &= 0, \\ (\bar{\beta} + \bar{\alpha}_2)\bar{C}_s + \bar{D}_s \left[\frac{1}{4}((k_1^2 + 2k_1k_2 + 3)\bar{C}_s^2 + 3(k_2^2 + 1)\bar{D}_s^2) - 2\bar{\eta} \right] &= 0, \\ (\bar{\beta} - \bar{\alpha}_2)\bar{D}_s + \bar{C}_s \left[\frac{1}{4}(3(k_1^2 + 1)\bar{C}_s^2 + (k_2^2 + 2k_1k_2 + 3)\bar{D}_s^2) - 2\bar{\eta} \right] &= 0.\end{aligned}\tag{A.15}$$

Multiply the third equation in (A.15) with k_2 and then subtract this from the first equation in (A.15) to find:

$$(k_1 - k_2) \left[\frac{1}{2}(1 + k_1k_2)\bar{C}_s\bar{D}_s + \bar{\beta} \right] + k_1\bar{\alpha}_1 - k_2\bar{\alpha}_2 = 0.\tag{A.16}$$

Similarly it follows from the second and fourth equation in (A.15) that

$$(k_1 - k_2) \left[\frac{1}{2}(1 + k_1 k_2) \bar{C}_s \bar{D}_s + \bar{\beta} \right] + k_2 \bar{\alpha}_1 - k_1 \bar{\alpha}_2 = 0. \quad (\text{A.17})$$

By subtracting (A.17) from (A.16) one obtains $(k_1 - k_2)(\bar{\alpha}_1 + \bar{\alpha}_2) = 0$. Since $\bar{\alpha}_1, \bar{\alpha}_2 > 0$, it follows that $k_1 = k_2$. Substituting $k_1 = k_2$ into (A.16) or (A.17) and one finds $\bar{\alpha}_1 = \bar{\alpha}_2$ or $k_1 = k_2 = 0$. This means that for a non-trivial solution of (A.14) we should have $\bar{\alpha}_1 = \bar{\alpha}_2$. In other words for $\bar{\alpha}_1 \neq \bar{\alpha}_2$ (4.14) does not have non-trivial solutions. Substitution of $\bar{\alpha}_1 = \bar{\alpha}_2 = \alpha$ and $k_1 = k_2$ into (A.15), one obtains:

$$\begin{aligned} (\bar{\beta} + \alpha) \bar{C}_s + \bar{D}_s \left[\frac{3}{4}(1 + k_2^2)(\bar{C}_s^2 + \bar{D}_s^2) - 2\bar{\eta} \right] &= 0, \\ (\bar{\beta} - \alpha) \bar{D}_s + \bar{C}_s \left[\frac{3}{4}(1 + k_2^2)(\bar{C}_s^2 + \bar{D}_s^2) - 2\bar{\eta} \right] &= 0. \end{aligned} \quad (\text{A.18})$$

The non-trivial solutions of (A.18) are $\bar{C}_s = \pm \sqrt{\frac{\bar{\beta} - \alpha}{\bar{\beta} + \alpha}} \bar{D}_s$ under the condition:

$$(\bar{\beta}^2 - \alpha^2) - \left[\frac{3}{4}(1 + k_2^2)(\bar{C}_s^2 + \bar{D}_s^2) - 2\bar{\eta} \right]^2 = 0. \quad (\text{A.19})$$

Hence, the non-trivial solutions of (4.14) correspond to the critical points of type 2 and 3. For the case $\bar{A}_s = \bar{C}_s = 0$ it can readily be seen from (4.14) that \bar{D}_s satisfies (A.19) with $\alpha = \bar{\beta}$.

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