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Higher order saddlepoint approximations in the Vasicek portfolio credit loss model

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Abstract

This paper utilizes the saddlepoint approximation as an efficient tool to estimate the portfolio credit loss distribution in the Vasicek model. Value at Risk(VaR), the risk measure chosen in the Basel II Accord for the evaluation of capital requirement, can then be found by inverting the loss distribution. VaR Contribution(VaRC), Expected Shortfall(ES) and ES Contribution(ESC) can all be calculated accurately.

Saddlepoint approximation is well known to provide good approximations to very small tail probabilities, which makes it a very suitable technique in the context of portfolio credit loss. The portfolio credit model we employ is the Vasicek one factor model, which has an analytical solution if the portfolio is well diversified. The Vasicek asymptotic formula however fails when the portfolio is dominated by a few loans. We show that saddlepoint approximation is able to handle such exposure concentration.

We also point out that the saddlepoint approximation technique can be readily applied to more general Bernoulli mixture models(possibly multi-factor). It can further handle portfolios with random LGD.

Key Words: Portfolio credit risk, Value at Risk, Expected Shortfall, VaR contribution, saddlepoint approximation

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1 Introduction

The integral issues in the portfolio credit loss modeling are the determination of Value at Risk(VaR) and VaR contribution. VaR is the risk measure chosen in the Basel II Accord (Basel Committee on Bank Supervision, 2004) for the evaluation of capital requirement. The VaR contribution measures how much each obligor in a portfolio contributes to the total VaR. It is equally important as VaR because it is necessary for loan pricing and it can provide limits on large credit exposures. It may also be useful for profitability assessment, asset allocation and portfolio optimization. There are several forms of risk contribution in the literature and we adopt the marginal contribution given in Gourieroux, Laurent and Scaillet (2000), which is the sensitivity of the risk to an infinitesimal fractional change in exposure.

The Vasicek (Vasicek, 2002) portfolio credit loss model is among the most popular models quantifying portfolio credit risk. In particular it is the basis of the Basel II internal ratings based (IRB) approach. The Vasicek model is a one period default-mode model, i.e., loss only occurs when an obligor defaults in a fixed time horizon. Under certain homogeneity conditions, the Vasicek one factor model leads to very simple analytic asymptotic approximation of loss distribution and the VaR. The approximation works very well when the portfolio is of large size and there is no exposure concentration in presence, i.e., the portfolio is not dominated by a few loans. However, the Vasicek one factor model can not detect exposure concentration when it is inherent in the portfolio and it then tends to underestimate risk.

This paper utilizes the saddlepoint approximation as an efficient tool to estimate portfolio credit loss distribution. Saddlepoint approximation method is well known to provide good approximations to very small tail probabilities, which makes it a very suitable technique in the context of portfolio credit loss. The use of saddlepoint approximation in portfolio credit loss is pioneered in a series of articles by Martin, Thompson and Browne (2001a; b). Gordy (2002) showed that saddlepoint approximation is fast and robust when applied to CreditRisk^{+}. This paper differs from them in that (i) we employ the saddlepoint approximation in the Vasicek model and (ii) we apply the saddlepoint approximation to the conditional moment generating function(MGF) of portfolio loss L rather than to the unconditional MGF. We show that this change in implementation of the saddlepoint approximation leads to very accurate results on the portfolio loss distribution, the VaR and VaR contribution, even for small sized portfolios and portfolios with exposure concentration. In addition to the VaR and VaR contribution, we also give the saddlepoint approximations for the Expected Shortfall(ES) and ES contribution.

Although we confine our numerical experiments to the Vasicek one-factor model, the saddlepoint approximation technique can readily be applied to more general Bernoulli mixture models(possibly multi-factor). It can further handle portfolios with random LGD.

The rest of the article is organized as follows. In section 2 we introduce the popular risk measures and risk contributions and we review the Vasicek one-factor model. Section 3 gives a brief introduction in saddlepoint approximation and describes how it can be used in the context of portfolio credit loss modeling. We present numerical results in section 4. The saddlepoint approximation is here applied to the Vasicek one-factor model. Section

5 extends the use of the saddlepoint approximation to more general situations than the Vasicek one-factor model. The 6th section concludes.

2 Portfolio credit loss modeling

2.1 Risk measures and risk contributions

Consider a portfolio consisting of n obligors. Any obligor i can be characterized by three constant quantities: the exposure at default EAD_i, the loss given default LGD_i and the probability of default PD_i. Obligor i is subject to default after a fixed time horizon and the default can be modeled as a Bernoulli random variable D_i such that

$$D_i = \begin{cases} 1 & \text{with probability } \mathrm{PD}_i, \\ 0 & \text{with probability } 1\text{-}\mathrm{PD}_i. \end{cases}$$

Define the effective exposure of obligor i by $w_i = \mathrm{EAD}_i \times \mathrm{LGD}_i$, then the loss incurred by the obligor i is given by

$$L_i = \mathrm{EAD}_i \times \mathrm{LGD}_i \times D_i = w_i D_i.$$

It follows that the portfolio loss is given by

$$L = \sum_{i=1}^{n} L_i = \sum_{i=1}^{n} w_i D_i.$$

Let α be some given confidence level, the α-quantile of the loss distribution of L in this context is called *Value at Risk*(VaR). Thus,

$$\mathrm{VaR}_\alpha = \inf\{x : \mathbb{P}(L \leq x) \geq \alpha\}.$$

Usually the α of interest is close to 1. VaR is the risk measure chosen in the Basel II Accord (Basel Committee on Bank Supervision, 2004) for the evaluation of capital requirement, which means a bank that complies with Basel II needs to reserve capital of amount VaR_α as a cushion for extreme loss. However, VaR is known to be not coherent, in particular not subadditive (see Artzner, Delbaen, Eber and Heath, 1999). So we also consider *Expected Shortfall*(ES), a coherent alternative to the VaR. It is defined as the conditional expectation of the loss given that the loss exceeds the VaR,

$$\mathrm{ES}_\alpha = E[L | L \geq \mathrm{VaR}_\alpha].$$

A risk contribution measures how much each obligor in a portfolio contributes to the total risk. This is equally important as risk measures because it is necessary for loan pricing and it can provide limits on large credit exposures. It may also be useful for profitability assessment, asset allocation and portfolio optimization. A naturally desirable property of the risk contributions is that they sum up to the corresponding risk measure, e.g., for VaR, we want the VaR Contributions(VaRC) add up to total VaR, i.e.,

$$\sum_{i=1}^{n} \mathrm{VaRC}_i = \mathrm{VaR}.$$

A common measure of risk contribution that satisfies this property is the sensitivity of the risk to an infinitesimal fractional change in exposure, as given in Gourieroux et al. (2000). Under some continuity conditions, the VaR contribution coincides with the conditional expectation of L_i given that the portfolio loss L takes value $\mathrm{VaR}_\alpha(L)$, i.e.,

$$\mathrm{VaRC}_{i,\alpha} = w_i \frac{\partial \mathrm{VaR}_\alpha}{\partial w_i}(L) = w_i E[D_i | L = \mathrm{VaR}_\alpha(L)], \tag{1}$$

The sum of the VaR contributions indeed equals the total VaR, i.e.,

$$\begin{aligned} \sum_{i=1}^{n} w_i E\left[D_i | L = \mathrm{VaR}_\alpha(L)\right] &= E\left[\sum_{i=1}^{n} L_i | L = \mathrm{VaR}_\alpha(L)\right] \\ &= E[L | L = \mathrm{VaR}_\alpha(L)] \\ &= \mathrm{VaR}_\alpha(L). \end{aligned}$$

Similarly, the ES contribution(ESC) is given by

$$w_i \frac{\partial ES_\alpha}{\partial w_i}(L) = w_i E[D_i | L \geq \mathrm{VaR}_\alpha(L)]. \tag{2}$$

We also have

$$\sum_{i=1}^{n} w_i E[D_i | L \geq \mathrm{VaR}_\alpha(L)] = \mathrm{ES}_\alpha(L). \tag{3}$$

2.2 The Vasicek portfolio credit loss model

The key issue in portfolio credit loss modeling is the modeling of the default dependence among obligors. A common practice is utilizing the Bernoulli mixture model, such that D_i are independent Bernoulli variables conditional on some common factors Y with $\mathbb{P}(D_i = 1|Y) = p_i(Y)$. The factors Y can represent the state of the economy, different industries and geographical regions, etc.

A broad class of models in the portfolio credit loss modeling can be categorized as Bernoulli mixture models. Examples include almost all popular industrial models like KMV/Vasicek(Vasicek, 2002), CreditRisk$^+$(Credit Suisse Financial Products, 1997) and CreditPortfolioView(Wilson, 1997a; b). For more details see Frey and McNeil (2002a; b). We concentrate on the Vasicek one-factor Gaussian copula model in the sequel.

The Vasicek model is a one period default-mode model, i.e., loss only occurs when an obligor defaults in a fixed time horizon. Based on Merton's firm value model, the Vasicek model evaluates the default of an obligor in terms of the evolution of its asset value. Default occurs when the standardized asset log-return X is less than some pre-specified threshold c where X is normally distributed and $\mathbb{P}(X < c) = \mathrm{PD}$. X is decomposed into a systematic part Y, representing the state of the economy, and an idiosyncratic part Z, such that for obligor i we have

$$X_i = \sqrt{\rho} Y + \sqrt{1-\rho} Z_i, \tag{4}$$

where Y and all Z_i are i.i.d standard normal random variables and ρ is the common pairwise correlation. It is now easily deduced that X_i and X_j are conditionally independent given the realization of Y. This implies that L_i and L_j are also conditionally independent given Y. Further assumptions of the Vasicek model are that all obligors have the same characteristics, such that $\mathrm{PD}_i = p$, $\mathrm{EAD}_i = 1$ and $\mathrm{LGD}_i = 1$, which entails that $w_i = 1$ for all i.

Denote by $p(y) = \mathbb{P}[L_i = 1|Y = y]$, i.e., the probability of default conditional on the common factor $Y = y$. Then

$$p(y) = \mathbb{P}[L_i = 1|Y = y] = \mathbb{P}[X_i < c|Y = y] = \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}y}{\sqrt{1-\rho}}\right), \tag{5}$$

where Φ is the cdf of the standard normal distribution.

As a consequence of the strong law of large numbers, one obtains for $n \to \infty$

$$\mathbb{P}\left[\lim_{n\to\infty} L/n = p(y)\big|Y = y\right] = 1.$$

Equivalently, if we denote by $L(Y)$ the portfolio loss L conditional on Y, we have

$$\lim_{n\to\infty} L(Y)/n = p(Y) = \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}Y}{\sqrt{1-\rho}}\right) \quad a.s. \tag{6}$$

Since $p(Y)$ is strictly monotonically decreasing in Y, the α quantile of L is simply the $1-\alpha$ quantile of Y, i.e.,

$$\mathrm{VaR}_\alpha = np(\Phi^{-1}(1-\alpha)) = n\Phi\left(\frac{\Phi^{-1}(p) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right). \tag{7}$$

As all obligors in the portfolio are equivalent, the VaR contribution of each obligor is simply $\mathrm{VaR}/n = p(\Phi^{-1}(1-\alpha))$.

We note although the assumptions of uniform pairwise correlation ρ and unconditional default probability PD are made in Vasicek (2002), they are not necessary conditions and can be relaxed. Moreover, the convergence in (6) also holds for a portfolio with unequal weights w_i if $\sum w_i^2 \to 0$, in other words, the portfolio is not dominated by a few loans. Summarizing, for a portfolio which is not homogeneous in terms of effective weight, default probability and pairwise correlation, the individual loss variable L_i conditional on Y is given by

$$L_i(Y) = \begin{cases} w_i & \text{with probability } \Phi(\frac{\Phi^{-1}(p_i) - \sqrt{\rho_i}Y}{\sqrt{1-\rho_i}}), \\ 0 & \text{with probability } 1 - \Phi(\frac{\Phi^{-1}(p_i) - \sqrt{\rho_i}Y}{\sqrt{1-\rho_i}}). \end{cases}$$

If the portfolio is not dominated by a few loans, the fraction of loss is given by

$$\tilde{L}(Y) = \lim_{n\to+\infty} \frac{\sum_{i=1}^n L_i(Y)}{\sum_{i=1}^n EAD_i} = \frac{\sum_{i=1}^n w_i \Phi(\frac{\Phi^{-1}(p_i) - \sqrt{\rho_i}Y}{\sqrt{1-\rho_i}})}{\sum_{i=1}^n EAD_i}.$$

Then, VaR and VaR contributions are given by

$$\mathrm{VaR}_\alpha = \sum_{i=1}^n w_i \Phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\rho_i}\Phi^{-1}(\alpha)}{\sqrt{1-\rho_i}}\right), \tag{8}$$

$$\mathrm{VaRC}_{i,\alpha} = w_i \Phi \left(\frac{\Phi^{-1}(p_i) + \sqrt{\rho_i}\Phi^{-1}(\alpha)}{\sqrt{1-\rho_i}} \right). \tag{9}$$

Note the VaR contribution above is a portfolio-invariant linear function of w_i, which implies that the capital contributions of individual exposures only depend on the characteristics of the exposure in question and not on the rest of the portfolio.

The Vasicek asymptotic formula is straightforward but it strongly relies on the assumptions of an infinitely large portfolio and of no exposure concentration. When the two conditions, especially the latter, are violated, which constantly occurs in practice, it tends to underestimate risk. Therefore, the analytic formulas are less suitable when a portfolio is of small size or if it is dominated by a few loans. In the following sections we show that both problems can be handled by the saddlepoint approximation.

3 Saddlepoint approximation

The computation of the probability distribution function of the sum of independent random variables can be facilitated by the use of the moment generating function(MGF), which is defined by $M_X(t) = E(e^{tX})$. For a finite sequence of independent random variables $X_i, i = 1 \dots n$, with known analytic MGF's M_{X_i}, the MGF of the sum $X = \sum_{i=1}^n X_i$ is the product of MGF of X_i, i.e.,

$$M_X(t) = \prod_{i=1}^{n} M_{X_i}(t).$$

Let $K_X(t) = \log M_X(t)$ be the Cumulant Generating Function(CGF) of X. The inverse MGF of X can be written as

$$f_X(x) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \exp(K_X(t) - tx)dt. \tag{10}$$

Saddlepoint approximation arises in this setting to give an accurate analytic approximation. It is usually highly accurate in the tail of a distribution. The saddlepoint approximation can be thought of as the Edgeworth expansion at the center of an Esscher transformed density. For a detailed exposition of saddlepoint approximations, see Jensen (1995).

The saddle point, i.e., the point at which $K_X(t) - tx$ is stationary, is a $t = \tilde{t}$ such that

$$K'_X(\tilde{t}) = x. \tag{11}$$

The density $f_X(x)$ and the tail probability $\mathbb{P}(X > x)$ can be approximated by $K_X(t)$ and its derivative up to second order at $\tilde{t}$.

The Taylor expansion of $K(t) - tx$ (function of t) around $\tilde{t}$ gives

$$K(t) - tx = K(\tilde{t}) - \tilde{t}x + \frac{1}{2}(t - \tilde{t})^2 K''(\tilde{t}) + ... \tag{12}$$

Substitute (12) into (10), we get

$$f_X(x) \approx \frac{\exp(K(\tilde{t}) - \tilde{t}x)}{2\pi i} \int_{-i\infty}^{+i\infty} \exp\left(\frac{1}{2}(t-\tilde{t})^2 K''(\tilde{t})\right) dt = \frac{\exp(K(\tilde{t}) - \tilde{t}x)}{\sqrt{2\pi K''(\tilde{t})}}. \tag{13}$$

The tail probability is approximated as

$$\begin{aligned} \mathbb{P}(X > x) &= \frac{1}{2\pi i} \int_{-i\infty,(0+)}^{+i\infty} \frac{\exp(K(t) - tx)}{t} dt \\ &\approx \begin{cases} \exp\left(K(\tilde{t}) - \tilde{t}x + \frac{1}{2}\tilde{t}^2 K''(\tilde{t})\right) \Phi\left(-\sqrt{\tilde{t}^2 K''(\tilde{t})}\right) & x > E(X), \\ \frac{1}{2} & x = E(X), \\ 1 - \exp\left(K(\tilde{t}) - \tilde{t}x + \frac{1}{2}\tilde{t}^2 K''(\tilde{t})\right) \Phi\left(-\sqrt{\tilde{t}^2 K''(\tilde{t})}\right) & x < E(X). \end{cases} \end{aligned} \tag{14}$$

If all the X_i are identically distributed, the relative errors of both approximations in (13) and (14) are known to be $O(n^{-1})$. Higher order approximations of the density and the tail probability are given by the Daniels (Daniels, 1987) formula

$$f_X(x) = \frac{\phi(z_l)}{\sqrt{K''(\tilde{t})}} \left(\left\{ 1 + \left[-\frac{5K'''(\tilde{t})^2}{24K''(\tilde{t})^3} + \frac{K^{(4)}(\tilde{t})}{8K''(\tilde{t})^2} \right] \right\} + O(n^{-2}) \right), \tag{15}$$

and the Lugannani-Rice (Lugannani and Rice, 1980) formula

$$\mathbb{P}(X > x) = 1 - \Phi(z_l) + \phi(z_l) \left[\frac{1}{z_w} - \frac{1}{z_l} + O\left(n^{-3/2}\right) \right], \tag{16}$$

where $z_w = \tilde{t}\sqrt{K''(\tilde{t})}$ and $z_l = sgn(\tilde{t})\sqrt{2[x\tilde{t} - K(\tilde{t})]}$.

3.1 Saddlepoint approximation in portfolio credit risk modeling

The use of saddlepoint approximation in portfolio credit loss is pioneered by a series of articles by Martin et al. (2001a; b). Gordy (2002) showed that saddlepoint approximation is fast and robust when applied to CreditRisk$^+$. All of them apply saddlepoint approximation to the unconditional MGF of loss L, despite the fact that L_i are not independent. Annaert, Garcia, Lamoot and Lanine (2005) show that the procedure described in Gordy (2002) may give inaccurate results in case of portfolios with high skewness and kurtosis in exposure size. This paper differs substantially from them in that we apply the saddlepoint approximation to the *conditional* MGF of L given the common factor Y, so that L_i are independent, which is the situation the saddlepoint approximation will work well. We employ the saddlepoint approximation in the Vasicek model. In section 4 we will show by a numerical example that the accuracy of our procedure is not impaired by high skewness and kurtosis in exposure size.

In the Vasicek model, obligors are independent conditional on the common factor. The application of the saddlepoint approximation is then straightforward. We write the conditional MGF of L as

$$M(t, Y) = \prod_{i=1}^{n} \left(1 - p_i(Y) + p_i(Y)e^{w_i t}\right). \tag{17}$$

The conditional CGF and its derivatives up to fourth order are defined as follows:

$$K(t,Y) = \sum_{i=1}^{n} \log\left(1 - p_i(Y) + p_i(Y)e^{w_i t}\right), \tag{18}$$

$$K'(t,Y) = \sum_{i=1}^{n} \frac{w_i p_i(Y)e^{w_i t}}{1 - p_i(Y) + p_i(Y)e^{w_i t}}, \tag{19}$$

$$K''(t,Y) = \sum_{i=1}^{n} \frac{(1 - p_i(Y))w_i^2 p_i(Y)e^{w_i t}}{[1 - p_i(Y) + p_i(Y)e^{w_i t}]^2}. \tag{20}$$

$$K'''(t,Y) = \sum_{i=1}^{n} \left\{ \frac{(1 - p_i(Y))w_i^3 p_i(Y)e^{w_i t}}{[1 - p_i(Y) + p_i(Y)e^{w_i t}]^2} - \frac{2(1 - p_i(Y))w_i^3 p_i^2(Y)e^{2w_i t}}{[1 - p_i(Y) + p_i(Y)e^{w_i t}]^3} \right\} \tag{21}$$

$$\begin{aligned} K^{(4)}(t,Y) \quad = \quad & \sum_{i=1}^{n} \left\{ \frac{(1 - p_i(Y))w_i^4 p_i(Y)e^{w_i t}}{[1 - p_i(Y) + p_i(Y)e^{w_i t}]^2} - \frac{6(1 - p_i(Y))w_i^4 p_i^2(Y)e^{2w_i t}}{[1 - p_i(Y) + p_i(Y)e^{w_i t}]^3} + \right. \\ & \left. + \frac{6(1 - p_i(Y))w_i^4 p_i^3(Y)e^{3w_i t}}{[1 - p_i(Y) + p_i(Y)e^{w_i t}]^4} \right\} \end{aligned} \tag{22}$$

With $K(t,Y)$ available, we are able to calculate the conditional loss density and the conditional tail probability by the saddlepoint approximation. Since $K'(t,Y)$ is a monotonically increasing function of t and it is bounded in the interval $[0, \sum w_i]$, the equation $K'(t,Y) = x$ admits a unique solution $\tilde{t}$ for $x \in [0, \sum w_i]$. Integrating over Y gives the unconditional loss density and tail probability. For example, the tail probability is given by

$$\mathbb{P}(L > x) = \int \mathbb{P}(L > x|Y)d\mathbb{P}(Y). \tag{23}$$

The VaR can then be found by inverting the loss distribution. Moreover, to obtain the VaR contribution, we differentiate $\mathbb{P}(L > x)$ with respect to the effective exposure:

$$\frac{\partial}{\partial w_i}\mathbb{P}(L > x) = E_Y \left\{ \frac{1}{2\pi i} \int_{-i\infty,(0+)}^{+i\infty} \left[\frac{1}{t}\frac{\partial K(t,Y)}{\partial w_i} - \frac{\partial x}{\partial w_i} \right] \exp(K(t,Y) - tx)dt \right\}. \tag{24}$$

Here we replace x by VaR_α. Since the tail probability $\mathbb{P}(L > \text{VaR}_\alpha)$ is fixed at $1-\alpha$, the left hand side should vanish and we obtain

$$\begin{aligned} w_i \frac{\partial \text{VaR}_\alpha}{\partial w_i} \quad &= \quad w_i \frac{E_Y\left[\int_{-i\infty,(0+)}^{+i\infty} \frac{\partial K(t,Y)}{\partial w_i}\frac{1}{t} \exp\left(K(t,Y) - t\text{VaR}_\alpha\right)dt\right]}{E_Y\left[\int_{-i\infty}^{+i\infty} \exp(K(t,Y) - t\text{VaR}_\alpha)dt\right]} \\ &= \quad w_i \frac{E_Y\left[\int_{-i\infty}^{+i\infty} \frac{p_i(Y)e^{w_i t}}{1 - p_i(Y) + p_i(Y)e^{w_i t}} \exp\left(K(t,Y) - t\text{VaR}_\alpha\right)dt\right]}{E_Y\left[f_L(\text{VaR}_\alpha|Y)\right]}. \end{aligned} \tag{25}$$

If we define

$$\hat{K}^i(t,Y) = \log\left(p_i(Y)e^{w_i t}\right) + \sum_{j \neq i} \log\left(1 - p_j(Y) + p_j(Y)e^{w_j t}\right),$$

which can be thought of as the CGF of L given Y and $D_i = 1$, (25) is rewritten as

$$w_i \frac{\partial \text{VaR}_\alpha}{\partial w_i} = w_i \frac{E_Y \left[\int_{-i\infty}^{+i\infty} \exp\left(\hat{K}^i(t,Y) - t\text{VaR}_\alpha \right) dt \right]}{E_Y \left[f_L(\text{VaR}_\alpha | Y) \right]}. \tag{26}$$

Both the numerator and the denominator can be approximated by the saddlepoint method.

The VaR contribution can also be derived in another way. With $\hat{L}^i = \sum_{j \neq i} w_j D_j$, we have

$$\begin{aligned} w_i E(D_i | L = \text{VaR}_\alpha) &= w_i \frac{f(L = \text{VaR}_\alpha; D_i = 1)}{f_L(\text{VaR}_\alpha)} \\ &= w_i \frac{E_Y \left[f\left(\hat{L}^i = \text{VaR}_\alpha - w_i \middle| Y \right) p_i(Y) \right]}{E_Y \left[f_L(\text{VaR}_\alpha | Y) \right]}. \end{aligned} \tag{27}$$

The conditional density in the numerator is the conditional loss density of a portfolio excluding obligor i and can again be calculated by the saddlepoint approximation. We note that (26) and (27) are essentially the same because both formulas use the saddlepoint $\tilde{t}$ that solves

$$\sum_{j \neq i} \frac{w_j p_j(Y) e^{w_j t}}{1 - p_j(Y) + p_j(Y) e^{w_j t}} = \text{VaR}_\alpha - w_i. \tag{28}$$

Similarly, the ES contributions are given by

$$w_i E(D_i | L \geq \text{VaR}_\alpha) = w_i \frac{E_Y \left[\mathbb{P}\left(\hat{L}^i \geq \text{VaR}_\alpha - w_i \middle| Y \right) p_i(Y) \right]}{E_Y \left[\mathbb{P}(L \geq \text{VaR}_\alpha | Y) \right]}, \tag{29}$$

and ES can be obtained by simply summing up all the ES contributions, i.e.,

$$ES_\alpha = \sum w_i E(D_i | L \geq \text{VaR}_\alpha). \tag{30}$$

Remarks:

- Although the obligors in a portfolio are assumed to be completely heterogeneous, for the sake of computational efficiency, it is advisable to group obligors as much as possible into homogeneous buckets with similar characteristics, esp. for large portfolios. The main advantages of doing this are (i) the expedition of the calculation of conditional CGF and its partial derivatives and (ii) a reduction of the amount of risk contributions that need to be computed.

- Martin et al. (2001b) proposed a simple estimate to the VaR contribution, which reads

$$\text{VaRC}_{i,\alpha} \approx \frac{E_Y \left[f_L(\text{VaR}_\alpha | Y) \frac{w_i}{\tilde{t}} \frac{\partial K(t,Y)}{\partial w_i} \Big|_{t=\tilde{t}} \right]}{E_Y \left[f_L(\text{VaR}_\alpha | Y) \right]} \tag{31}$$

  in the Bernoulli mixture models. In our numerical examples we show, however, this approximation may be inaccurate.

4 Numerical results

We now illustrate the performance of the saddlepoint approximation in the Vasicek one factor model. For the implementation of the saddlepoint approximation, we always employ the Lugannani-Rice formula (16) for the tail probability. We truncate the common factor Y in the interval $[-5, 5]$ so that the probability of Y falling out of this interval is merely 5.7×10^{-7}. Discretization of Y is done by Gauss-Legendre quadrature, generating 1000 abscissas and weights. The three examples evaluated are:

- Example 1: A homogeneous portfolio with 1000 obligors, each with EAD=1, LGD=1, PD=0.01 and $\rho = 0.2$.
- Example 2: A portfolio consisting of 100 obligors with $\text{EAD}_k = k$, $k = 1, 2, \ldots 100$, PD=0.1, $\rho = 0.2$.
- Example 3: A portfolio consisting of 1 obligor with $\text{EAD}_1 = 100$ and 10,000 obligors with $\text{EAD}_2 = 1$. All obligors have PD=0.005 and $\rho = 0.2$.

We compare the loss ditribution from the saddlepoint approximation to results from the analytic Vasicek formula and from Monte Carlo simulation in the first two examples. We use Monte Carlo simulation with 4 million scenarios as the benchmark method. The loss distribution corresponding to the Vasicek model is obtained by inverting the VaR given by Vasicek's formula (8) for a series of quantile α.

Example 1 is an ideal case for the Vasicek formula to be accurate. The loss distributions from different methods are presented in Figure 1. The x-axis represents the loss percentage, i.e., the loss amount in proportion to the total exposure. The y-axis, the tail probability $P(L > x)$, is in log-scale. It can be seen that saddlepoint approximation is highly accurate and follows our benchmark very well. The Vasicek formula only slightly underestimates the risk.

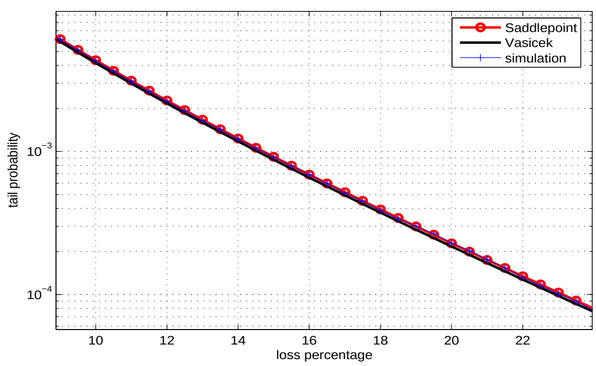

Figure 1: Comparison of the saddlepoint approximation, Vasicek's formula and Monte Carlo simulation for the tail probability in Example 1.

In Example 2, the assumptions of the Vasicek asymptotic approximation are not satisfied any longer: the size of the portfolio is small and there is exposure concentration (the weight of the largest loan is about 2%). Vasicek's formula is not suitable in this situation, which is confirmed in Figure 2. We observe however that the saddlepoint approximation gives results comparable to simulation in this example.

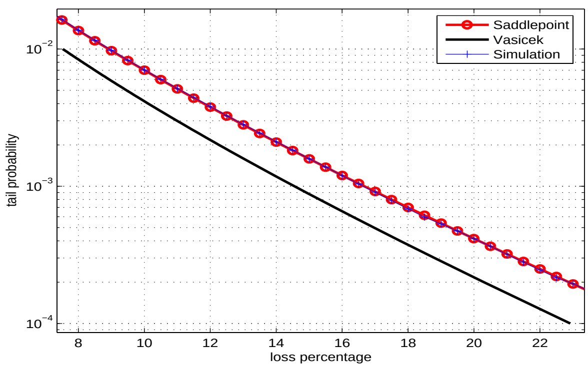

Figure 2: Comparison of the Saddlepoint approximation, Vasicek's formula and Monte Carlo simulation for the tail probability in Example 2.

Example 3 is a particular case when the VaR and VaR contributions can be computed almost exactly by the Binomial Expansion Method(BEM) if we treat the portfolio loss as a discrete variable. It is therefore a suitable test portfolio for the calculation of VaR contributions. BEM will serve as the benchmark for both the VaR and the VaR contributions. More details on BEM can be found in the Appendix. The loss distribution of this portfolio given by the saddlepoint approximation and the BEM are shown in Figure 3. The saddlepoint approximations again follow our benchmark very well.

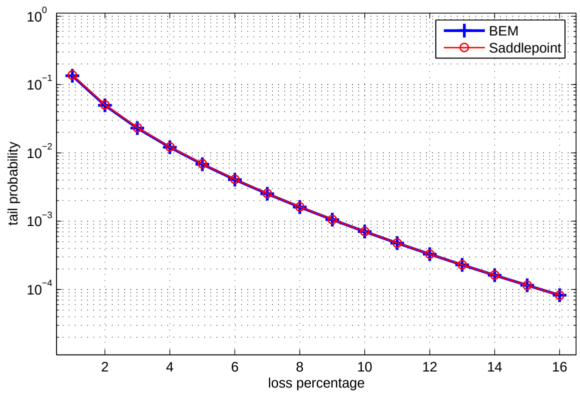

Figure 3: Tail probability given by the saddlepoint approximation and the BEM for portfolio in Example 3.

As for the VaR contribution, we first consider a fixed loss level $L = 922$, which lies around the 99.9% quantile. We compute the VaR contributions of both the large obligor(VaRC_1) and any small obligor(VaRC_2). We use both the standard and higher order saddlepoint approximation given by (13) and (15), respectively (denoted by SA2 and

SA4). Results are shown in Table 1 and in parenthesis are the relative errors of saddlepoint approximation to the benchmark. Besides, we compute the Vasicek VaR contribution, the saddlepoint approximation for the VaR contribution as given by (31) (denoted by SA-Martin) for comparison.

	VaRC_1	VaRC_2	$\sum$ VaRC
Vasicek	9.13	0.0913	922
SA-Martin	21.82	0.0900	921.99
SA2	12.24(2.93%)	0.0904(0.55%)	916.64(0.58%)
SA4	12.65(0.32%)	0.0907(0.22%)	920.00(0.21%)
BEM	12.61	0.0909	921.95

Table 1: VaR contributions at the loss level $L = 922$ and their relative errors. The portfolio is given in Example 3.

The results given by the benchmark BEM show that the VaR contribution increases non-linearly with the size of the exposure. Both the standard and higher order saddlepoint method successfully capture this feature and give the VaR contributions with small relative errors. The higher order approximation, with relative error less that 1%, outperforms the standard approximation. The only (negligible) problem is that the VaR contributions do not add up to the total VaR exactly. It is also clear that the VaR contributions of the large obligor(VaRC_1) obtained from Vasicek and SA-Martin are both relatively far from the true value. The Vasicek contribution is proportional to the effective exposure and therefore it underestimates the large obligor's risk contribution. SA-Martin penalizes large exposure too much.

Next we consider a fixed confidence level $\alpha = 99.99\%$ in example 3, which is truly far in the tail. The Lugannani-Rice formula will be used to compute the loss distribution and subsequently the higher order saddlepoint approximation is used for the loss density. Results are shown in Table 2. The accuracy of the higher order saddlpoint approximation is very satisfactory.

	$\text{VaR}_{99.99\%}$	VaRC_1	VaRC_2	$\sum$ VaRC
SA4	1558	19.71(0.4%)	0.1537(0.06%)	1556.27(0.1%)
BEM	1558	19.79	0.1538	1557.87

Table 2: VaR contributions at the loss level $\text{VaR}_{99.99\%}$ and their relative errors. The portfolio is given in Example 3.

Finally we compute the ES contributions and ES as in (29) and (30) at the confidence level $\alpha = 99.99\%$. The tail probabilities are computed by the Lugannani-Rice formula. The results are presented in Table 3. The table suggests that the approximation is more accurate for the VaR contribution than for the ES contribution. This can be understood roughly because the relative error of Daniels formula is $O(n^{-2})$ and that of Lugannani-Rice formula is $O(n^{-3/2})$, with n being the number of i.i.d random variables, (although in our example L_i are not really identically distributed). The approximations are, however, satisfactory for both the ES contributions and the ES.

	$\text{VaR}_{99.99\%}$	ESC_1	ESC_2	ES
SA4	1558	23.18(0.17%)	0.1848(0.49%)	1871(0.46%)
BEM	1558	23.14	0.1839	1862.51

Table 3: ES contributions and ES at the loss level $\text{VaR}_{99.99\%}$ and their relative errors. The portfolio is given in Example 3.

We remark that in example 3 the skewness and kurtosis in exposure size are 99.985 and 9998, respectively. They are much higher than in the portfolios 4 and 5 given in Annaert et al. (2005), where it is shown that the accuracy and reliability of the saddlepoint approximation obtained from Gordy's (2002) procedure may deteriorate. In our approach high skewness and kurtosis do not pose any problem with respect to accuracy.

5 Further Extensions

Although we confine our numerical experiments to the Vasicek one-factor model here, the saddlepoint approximation technique can be readily applied to all other Bernoulli mixture models. A different choice of mixture model only gives a difference in the form of the conditional default probability $p_i(Y)$, eg., in the Vasicek one factor model we have

$$p_i(Y) = \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho}Y}{\sqrt{1-\rho}}\right),$$

and in CreditRisk$^+$

$$p_i(Y) = p_i\left(w_{i0} + \sum Y_k w_{ik}\right),$$

where Y_k are assumed to be independently gamma distributed (see Gordy, 2002).

In case of multi-factor models, esp. when the number of factors is larger than three, instead of Gaussian quadrature, Monte Carlo simulation or low discrepancy sequences can be employed for the integration. The saddlepoint approximation itself is however not affected, since all the information of the common factors is encapsulated in $p_i(Y)$ before starting the approximation procedure. We note that when x is large and Y is large and positive, $\mathbb{P}(L > x|Y)$ will tend to zero and thus the integration in (23) will not be efficient. This is particularly an issue for multi-factor models, since for one-factor models we can simply increase the number of samples of Y. The idea of importance sampling can be used for a significant improvement in such cases. By choosing a $\mathbb{P}$-equivalent probability measure $\mathbb{Q}$, the tail probability can be rewritten as

$$\mathbb{P}(L > x) = \int \mathbb{P}(L > x|Y)d\mathbb{P}(Y) = \int \mathbb{P}(L > x|Y)\frac{d\mathbb{P}(Y)}{d\mathbb{Q}(Y)}d\mathbb{Q}(Y).$$

Several procedures to find the optimal measure $\mathbb{Q}$ are suggested by Glasserman and Li (2005), Glasserman (2006).

Remarks: The above mentioned hybrid method of saddlepoint approximation and importance sampling is more efficient than pure importance sampling, since in the simulation only the common factors need to be generated and not the specific risks. This is more advantageous for large portfolios. Moreover for the calculation of the VaR contribution importance sampling can only use the few replications $L = x$, whereas the hybrid method need not condition on this rare event.

Furthermore the saddlepoint approximation can also handle LGD volatility. When the LGD is random, the conditional CGF becomes

$$K(t, Y) = \sum \log \left[1 - p_i(Y) + p_i(Y) E(e^{w_i t} | Y)\right]. \tag{32}$$

Various forms of distribution of LGD can be found in the literature. For example, in Frye's (2000) model, the LGD is modeled as a normal random variable with mean μ and standard deviation σ such that

$$LGD_i = \mu + \sigma \left(-b_i Y + \sqrt{1 - b_i^2}\epsilon_i\right).$$

Here the ϵ_i, independent to Y, are assumed to be i.i.d standard normal variables and the b_i are assumed to be positive to insure the correct qualitative effect of LGD, which is mostly determined by the value of collateral. It should tend to be higher when the economy is weak and lower when the economy is strong. It follows that

$$\begin{aligned} E(e^{w_i t}|Y) &= e^{EAD_i(\mu - \sigma b_i Y)t} E(e^{EAD_i \sigma \sqrt{1-b_i^2}\epsilon_i t}) \\ &= \exp\left(EAD_i(\mu - \sigma b_i Y)t + EAD_i^2 \sigma^2 (1 - b_i^2) t^2/2\right). \end{aligned} \tag{33}$$

After substitution of (33) into (32), we see that a random LGD will not complicate the problem further.

6 Conclusions

We have described a new procedure to embed the saddlepoint approximation as a useful tool in portfolio credit loss modeling. We apply the saddlepoint approximation in the Vasicek one-factor model. The saddlepoint approximations, esp. the higher order approximations, are able to produce accurate results on both the VaR and the VaR contribution. The ES and ES contribution can also be computed satisfactorily. We have also illustrated that the saddlepoint approximation works well for small sized portfolios and portfolios with exposure concentration, where Vasicek's asymptotic formulas fail. We further point out that the saddlepoint approximation is a flexible method that it can be applied in quite general situations, for example, other Bernoulli mixture (possibly multi-factor) models and portfolios with random LGD.

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A Binomial expansion method

Consider a portfolio consisting of 1 obligor with $\text{EAD}_1 = k$, PD$= p_1$ and n obligors with $\text{EAD}_2 = 1$, PD$= p_2$. In a Bernoulli mixture model, the losses of the obligors are conditionally independent given the common factor Y. Let $p_1(Y)$ and $p_2(Y)$ be the conditional default probabilities, we have

$$\begin{aligned} \mathbb{P}(L = m) &= \int \mathbb{P}(L = m|Y) d\mathbb{P}(Y) \\ &= \int p_1(Y)\mathbb{P}(L^n = m - k|Y) + (1 - p_1(Y))\mathbb{P}(L^n = m|Y)\, d\mathbb{P}(Y), \end{aligned}$$

where

$$\mathbb{P}(L^n = m|Y) = \binom{n}{m} (p_2(Y))^m (1 - p_2(Y))^{n-m}.$$

The VaR and VaR contributions are then given, respectively, by

$$\text{VaR}_\alpha = \inf \left\{ x \Big| \sum_{m=0}^{x} \mathbb{P}(L = m) \geq \alpha \right\},$$

and

$$\text{VaRC}_1 = \frac{\int p_1(Y)\mathbb{P}(L^n = \text{VaR}_\alpha - k|Y) d\mathbb{P}(Y)}{\mathbb{P}(L = \text{VaR}_\alpha)},$$

$$\begin{aligned} \text{VaRC}_2 &= \frac{1}{\mathbb{P}(L = \text{VaR}_\alpha)} \left\{ \int p_2(Y)p_1(Y)\mathbb{P}(L^{n-1} = \text{VaR}_\alpha - k - 1|Y) d\mathbb{P}(Y) + \right. \\ &\quad \left. + \int p_2(Y)(1 - p_1(Y))\mathbb{P}(L^{n-1} = \text{VaR}_\alpha - 1|Y) d\mathbb{P}(Y) \right\}. \end{aligned}$$

The ES contributions are computed according to (29) with

$$\mathbb{P}(L \geq x) = 1 - \sum_{m=0}^{x-1} \mathbb{P}(L = m)$$

and ES is obtained by (30).

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