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EQUATIONS

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Abstract

In the classical multiple scales perturbation method for ordinary difference equations ($O\Delta Es$) as developed in 1977 by Hoppensteadt and Miranker, difference equations are replaced at a certain moment in the perturbation procedure by ordinary differential equations ($ODEs$). Taking into account the possibly different behaviour of the solutions of an $O\Delta E$ and of the solutions of a nearby ODE , one can not always be sure that the constructed approximations by the Hoppensteadt-Miranker method indeed reflect the behaviour of the exact solutions of the $O\Delta Es$. For that reason an improved version of the multiple scales perturbation method for $O\Delta Es$ will be presented and formulated in this paper completely in terms of difference equations. The goal of this paper is not only to present this improved method but also to show how this method can be applied to regularly perturbed $O\Delta Es$ and to singularly perturbed, linear $O\Delta Es$.

Keywords. ordinary difference equation, multiple scales perturbation method, regularly and singularly perturbed problems, asymptotic validity.

1 Introduction

Nowadays the multiple time-scales perturbation method for differential equations is a well-developed, well accepted, and very popular method to approximate solutions of weakly nonlinear differential equations. This method was developed in the period 1935-1962 by Krylov and Bogoliubov, Kuzmak, Kevorkian and Cole, Cochran, and Mahony. In the early 1970s Nayfeh popularized this method by writing many papers and books on this subject (see for instance [18]). More recent books on this method and its historical development are for instance the books by Andrianov and Manevitch [4], Holmes [5], Kevorkian and Cole [10], Murdock [17], and Verhulst [22]. The development of the multiple scales perturbation method for ordinary difference equations ($O\Delta Es$) started in 1960 with the work of Torng [21]. In this paper a second order $O\Delta E$ is reduced to a system of two first order $O\Delta Es$ by means of the method of variation of parameters for $O\Delta Es$. Then, nonlinear terms are expanded in discrete Fourier series, and a Krylov-Bogoliubov method (or equivalently, an averaging method) is applied to obtain the equations that describe the slow dynamics of the problem approximately. A similar method was presented in 1970 by Huston in [7]. From the results in [7] and in [21] it is clear that the solution of a weakly perturbed (non-) linear $O\Delta E$ behaves differently on different iteration scales. In 1977 Hoppensteadt and Miranker introduced in [6] the multiple scales perturbation method for $O\Delta Es$. For a problem with two significant iteration scales these authors assume that the solution x_n of an $O\Delta E$ for instance depends on n and $s = \epsilon n$ (that is, depends on a fast iteration scale and on a slow iteration scale). In the $O\Delta E$ x_{n+1} is then replaced by $x(n+1, \epsilon(n+1)) = x(n+1, s+\epsilon)$. In the perturbation scheme $x(n+1, s+\epsilon)$ is expanded in a Taylor polynomial, that is, $x(n+1, s+\epsilon) = x(n+1, s) + \epsilon \frac{\partial x}{\partial s}(n+1, s) + O(\epsilon^2)$, and to avoid unbounded terms (or secular terms) in the perturbation expansion for x_n one finally has to

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solve ordinary differential equations (*ODEs*) due to the derivatives in the Taylor expansions. So, in the perturbation procedure *OΔEs* are replaced (partly) by *ODEs*. A slightly different multiple scales perturbation method for *OΔEs* was introduced in [20] by Subramanian and Krishnan in 1979. In their approach the difference operator Δ is replaced by partial difference operators. For a problem with two significant iteration scales the authors of [20] introduced:

$$x_{n+1} - x_n = \Delta x_n = \Delta x(n, s) = \Delta_n(n, s) + \epsilon \Delta_s x(n, s), \quad (1)$$

where $\Delta_n x(n, s) = x(n+1, s) - x(n, s)$, and $\Delta_s x(n, s) = x(n, s+\epsilon) - x(n, s)$. This replacement is based on the two-timescales perturbation method for *ODEs*, where $x(t)$ is replaced by $\tilde{x}(t, \tau)$ with $\tau = \epsilon t$ and

$$\frac{dx(t)}{dt} = \frac{\partial \tilde{x}(t, \tau)}{\partial t} + \epsilon \frac{\partial \tilde{n}(t, \tau)}{\partial \tau}.$$

Nowadays the method of Hoppensteadt and Miranker is assumed to be the standard form of the multiple scales perturbation method for *OΔEs* (see for instance [5, 14, 15, 16]). Also recently this method was “rediscovered” by Luongo [11] and by Maccari [12]. It should be observed, however, that many results concerning *ODEs* carry over quite easily to corresponding results for *OΔEs*, while other results are completely different from their continuous counterparts. To illustrate some of these differences the following examples can be considered.

Example 1:

Consider the following *ODE* for $x = x(t)$:

$$\frac{dx}{dt} = ax(1-x), \quad t > 0, \quad (2)$$

where a is a positive constant, and $x(0) = x_0$. A forward Euler method is applied with positive stepsize h , and x_n is an approximation of $x(t)$ at $t = nh$. The corresponding *OΔE* becomes:

$$x_{n+1} = x_n + ahx_n(1-x_n)$$

for $n = 0, 1, 2, \dots$. Introducing $x_n = \frac{(1+ah)}{ah} y_n$, and $b = (1+ah) > 0$ the following *OΔE* for y_n is obtained:

$$y_{n+1} = by_n(1-y_n). \quad (3)$$

For all $a > 0$ the *ODE* (2) has two equilibrium points (one stable point at $x = 1$, and an unstable one at $x = 0$), and the dynamics is simple. Whereas the logistic difference equation (3) is well-known for its period-doublings and its chaotic behaviour for b -values between 3 and 4. So, for certain parameter values the solution of the *ODE* and the solution of the (nearby) *OΔE* behave quite differently.

Example 2:

Consider the following *ODE* for $x = x(t)$:

$$\epsilon \frac{dx}{dt} = -x, \quad t > 0, \quad (4)$$

where ϵ is a small positive parameter, i.e. $0 < \epsilon \ll 1$, and $x(0) = 1$. The solution of the *ODE* (4) is: $x(t) = \exp(-\frac{t}{\epsilon})$. When an explicit, forward Euler method is applied to (4) with positive stepsize h , and where x_n is an approximation of $x(t)$ at $t = nh$, then the following *OΔE* for x_n will be obtained: (for $n = 0, 1, 2, \dots$, and $x_0 = 1$)

$$x_{n+1} = (1 - \frac{h}{\epsilon})^{x_n} \Rightarrow x_n = (1 - \frac{h}{\epsilon})^n.$$

For a fixed value of h and for $\epsilon \downarrow 0$ x_n obviously blows up, whereas $x(nh)$ rapidly tends to zero. When an implicit Euler method is applied to (4) then the following *OΔE* for x_n is obtained (again h is a positive stepsize, x_n is an approximation of $x(t)$ at $t = nh, n = 0, 1, 2, \dots$, and $x_0 = 1$):

$$x_{n+1} - x_n = -\frac{h}{\epsilon}x_{n+1} \Rightarrow x_{n+1} = \frac{\epsilon}{h+\epsilon}x_n \Rightarrow x_n = \left(\frac{\epsilon}{h+\epsilon}\right)^n.$$

For a fixed value of h and for $\epsilon \downarrow 0$ x_n behaves like $(\frac{\epsilon}{h})^n = \exp(n \ln(\frac{\epsilon}{h}))$, whereas $x(nh)$ behaves like $\exp(-\frac{nh}{\epsilon})$. From the asymptotical point of view for $\epsilon \downarrow 0$ the behaviour of x_n and $x(nh)$ is quite different. When the trapezoidal formula is applied to (4) then the following $O\Delta E$ for x_n is obtained (again h is a positive stepsize, x_n is an approximation of $x(t)$ at $t = nh$, $n = 0, 1, 2, \dots$, and $x_0 = 1$):

$$\begin{aligned} x_{n+1} - x_n &= -\frac{h}{2\epsilon}(x_{n+1} + x_n) \Rightarrow x_{n+1} = -\frac{(h-2\epsilon)}{h+2\epsilon}x_n \\ \Rightarrow x_n &= (-1)^n \frac{(h-2\epsilon)^n}{(h+2\epsilon)^n}. \end{aligned}$$

Now x_n behaves like $(-1)^n \exp(n(\frac{-4\epsilon}{h} + O(\epsilon^3)))$, that is, x_n is oscillatory (whereas $x(nh)$ is non-oscillatory), and the amplitudes of x_n and $x(nh)$ behave differently for $\epsilon \downarrow 0$.

These examples clearly indicate that the solution of an ODE and the solution of an (nearby) $O\Delta E$ can behave quite differently. The reader is referred to [1,2,3,5,8,9,15] for some further striking differences (and similarities) in the theory for $ODEs$ and for $O\Delta Es$. In the multiple scales perturbation method for $O\Delta Es$ as developed in [6] by Hoppensteadt and Miranker difference equations are replaced at a certain moment by differential equations. Taking into account the possibly different behaviour of the solutions of an $O\Delta E$ and of the solutions of an (nearby) ODE one can not always be sure that the constructed approximations by the Hoppensteadt-Miranker method indeed reflect the behaviour of the exact solutions of the $O\Delta E$. For that reason an improved version of the multiple scales perturbation method for $O\Delta Es$ will be presented and formulated in this paper completely in terms of difference equations.

The goal of this paper is not only to present this improved method, but also to show how this method can be applied to regularly perturbed $O\Delta Es$ and to singularly perturbed $O\Delta Es$.

This paper is organized as follows. In section 2 of this paper the multiple scales perturbation method for $O\Delta Es$ will be presented completely in terms of difference operators. How this method can be applied to a second order regularly perturbed, weakly nonlinear $O\Delta Es$ will be shown in section 3 of this paper. The asymptotic validity of the constructed approximations on sufficiently long iteration scales will be discussed in section 4. How solutions of singularly perturbed, linear $O\Delta Es$ can be approximated will be shown in section 5. Compared to the existing rescaling procedures for singularly perturbed $ODEs$ and $O\Delta Es$ (see for instance [10,13,19,22]) also a slightly revised rescaling procedure will be presented in section 5 of this paper to find the significant scalings for some singularly perturbed, linear $O\Delta Es$. Finally, in section 6 of this paper some conclusions will be drawn, and some remarks on future research will be made.

2 The multiple scales perturbation method for $O\Delta Es$

In this section the multiple scales perturbation method for $O\Delta Es$ will be presented in a complete ‘‘difference operator’’ setting. Before introducing this method several operators have to be defined (and motivated). The well-known shift operator E , the difference operator Δ , and the identity operator I are defined as follows:

$$Ex_n = x_{n+1}, \quad \Delta x_n = x_{n+1} - x_n, \quad \text{and} \quad Ix_n = x_n. \quad (5)$$

The relationship between these operators easily follows from (5):

$$E = \Delta + I \quad \Leftrightarrow \quad \Delta = E - I. \quad (6)$$

The solution of a weakly perturbed $O\Delta E$ usually contains a rapidly changing part in n , and a slowly changing part in n . This is usually referred to as multiple scales behaviour. Consider the following functions:

$$\begin{aligned}
 a_n = 3^n &\Rightarrow \Delta a_n = 3^{n+1} - 3^n = (3-1)3^n = 2a_n = O(a_n), \\
 b_n = e^{\epsilon n} &\Rightarrow \Delta b_n = e^{\epsilon(n+1)} - e^{\epsilon n} = O(\epsilon b_n), \\
 c_n = (1+\epsilon)^n &\Rightarrow \Delta c_n = (1+\epsilon)^{n+1} - (1+\epsilon)^n = O(\epsilon c_n), \\
 d_n = 3^n(1+\epsilon)^n &\Rightarrow \Delta d_n = 3^{n+1}(1+\epsilon)^{n+1} - 3^n(1+\epsilon)^n = (2+3\epsilon)d_n.
 \end{aligned} \tag{7}$$

From (7) it is obvious that a_n only has a rapidly changing part in n , that b_n and c_n only have a slowly changing part in n , and that d_n has a rapidly changing part in n and a slowly one. To make this behaviour more clear in notation the following notations are proposed: $a_n = a(n)$, $b_n = b(\epsilon n)$, $c_n = c(\epsilon n)$, and $d_n = d(n, \epsilon n)$. It should be observed that these notations are similar to the ones used in the multiple timescales perturbation method for $ODEs$. Now it is assumed that $x_n = x(n, \epsilon n)$. This assumption implies that the solution of the $O\Delta E$ depends on two variables. So, the $O\Delta E$ actually becomes a partial difference equation. For that reason also partial shift operators and partial difference operators have to be defined. The following definitions are proposed:

$$\begin{aligned}
 E_1 x(n, \epsilon n) &= x(n+1, \epsilon n), \\
 E_\epsilon x(n, \epsilon n) &= x(n, \epsilon(n+1)), \\
 \Delta_1 x(n, \epsilon n) &= x(n+1, \epsilon n) - x(n, \epsilon n) = (E_1 - I)x(n, \epsilon n), \\
 \Delta_\epsilon x(n, \epsilon n) &= x(n, \epsilon(n+1)) - x(n, \epsilon n) = (E_\epsilon - I)x(n, \epsilon n).
 \end{aligned} \tag{8}$$

From (5), (6), and (8) it follows that (assuming $x_n = x(n, \epsilon n)$):

$$\begin{aligned}
 \Delta x_n &= x_{n+1} - x_n = x(n+1, \epsilon(n+1)) - x(n, \epsilon n) = E_1 E_\epsilon x(n, \epsilon n) - Ix(n, \epsilon n) \\
 &= (\Delta_1 + I)(\Delta_\epsilon + I)x(n, \epsilon n) - Ix(n, \epsilon n) = \\
 &= (\Delta_1 + \Delta_\epsilon + \Delta_1 \Delta_\epsilon)x(n, \epsilon n).
 \end{aligned}$$

And so, it follows that

$$\Delta = \Delta_1 + \Delta_\epsilon + \Delta_1 \Delta_\epsilon, \quad \text{and} \quad E = E_1 E_\epsilon. \tag{9}$$

Furthermore, for the partial difference operators Δ_1 and Δ_ϵ it is assumed that (also based on (7)):

$$\begin{aligned}
 \Delta_1 x(n, \epsilon n) &= O(x(n, \epsilon n)), \quad \text{and} \\
 \Delta_\epsilon x(n, \epsilon n) &= O(\epsilon x(n, \epsilon n)).
 \end{aligned} \tag{10}$$

From (9) it is obvious that in 1) the operator $\Delta_1 \Delta_\epsilon$ is missing (see also [20]). When x_n depends on $m+1$ scales the given definitions can readily be generalized, yielding: (for $j = 0, 1, \dots, m$)

$$\begin{aligned}
 x_n &= x(n, \epsilon n, \epsilon^2 n, \dots, \epsilon^m n), \\
 E_{\epsilon^j} x(n, \dots, \epsilon^m n) &= x(n, \epsilon n, \dots, \epsilon^j(n+1), \dots, \epsilon^m n), \\
 \Delta_{\epsilon^j} x(n, \dots, \epsilon^m n) &= (E_{\epsilon^j} - I)x(n, \dots, \epsilon^m n), \\
 E &= E_1 E_\epsilon E_{\epsilon^2} \dots E_{\epsilon^m}, \\
 \Delta &= (\Delta_1 + I)(\Delta_\epsilon + I) \dots (\Delta_{\epsilon^m} + I) - I, \\
 \Delta_{\epsilon^j} x(n, \dots, \epsilon^m n) &= O(\epsilon^j x(n, \dots, \epsilon^m n)).
 \end{aligned} \tag{11}$$

Now it will be shown how these operators can be used. For that reason a simple example will be treated. Consider the weakly perturbed, linear, second order $O\Delta E$

$$x_{n+2} + \epsilon x_{n+1} + x_n = 0, \tag{12}$$

where ϵ is a small parameter with $0 < \epsilon \ll 1$. Using (5) and (6) it follows that (12) can be rewritten in:

$$\begin{aligned}
 E^2 x_n + \epsilon E x_n + I x_n &= 0 && \Leftrightarrow \\
 (\Delta + I)^2 x_n + \epsilon (\Delta + I) x_n + I x_n &= 0 && \Leftrightarrow \\
 \Delta^2 x_n + (\epsilon + 2) \Delta x_n + (2 + \epsilon) x_n &= 0.
 \end{aligned} \tag{13}$$

Assuming that x_n depends on two scales (a fast scale n , and a slow scale ϵn) it follows that $x_n = x(n, \epsilon n)$ and that (12) or (13) becomes

$$\begin{aligned}
 & (\Delta_1 + \Delta_\epsilon + \Delta_1 \Delta_\epsilon)^2 x(n, \epsilon n) + (\epsilon + 2)(\Delta_1 + \Delta_\epsilon + \Delta_1 \Delta_\epsilon) x(n, \epsilon n) + \\
 & + (2 + \epsilon) x(n, \epsilon n) = 0 \quad \Leftrightarrow \\
 & (\Delta_1^2 + 2\Delta_1 + 2) x(n, \epsilon n) + (2\Delta_1(\Delta_\epsilon + \Delta_1 \Delta_\epsilon) + 2(\Delta_\epsilon + \Delta_1 \Delta_\epsilon) \\
 & + \epsilon \Delta_1 + \epsilon) x(n, \epsilon n) + O(\epsilon^2 x(n, \epsilon n)) = 0 \quad \Leftrightarrow \\
 & (\Delta_1^2 + 2\Delta_1 + 2) x(n, \epsilon n) + (2(\Delta_1 + I)(\Delta_\epsilon + \Delta_1 \Delta_\epsilon) + \epsilon(\Delta_1 + I)) x(n, \epsilon n) \\
 & + O(\epsilon^2 x(n, \epsilon n)) = 0.
 \end{aligned} \tag{14}$$

To construct an approximation for $x_n = x(n, \epsilon n)$ one now has to substitute into (14) a formal power series (in ϵ) for x_n , that is,

$$x(n, \epsilon n) = x_0(n, \epsilon n) + \epsilon x_1(n, \epsilon n) + \epsilon^2 x_2(n, \epsilon n) + \dots \tag{15}$$

Then, by taking together those terms of equal powers in ϵ one obtains as $O(1)$ -problem

$$(\Delta_1^2 + 2\Delta_1 + 2)x_0(n, \epsilon n) = 0 \quad \Leftrightarrow \quad x_0(n+2, \epsilon n) + x_0(n, \epsilon n) = 0, \tag{16}$$

and as $O(\epsilon)$ -problem

$$\epsilon(\Delta_1^2 + 2\Delta_1 + 2)x_1(n, \epsilon n) + \left(2(\Delta_1 + I)(\Delta_\epsilon + \Delta_1 \Delta_\epsilon + \frac{\epsilon}{2})\right) x_0(n, \epsilon n) = 0, \tag{17}$$

and so on. The $O(1)$ -problem (16) can readily be solved, yielding

$$x_0(n, \epsilon n) = f_0(\epsilon n) \cos\left(\frac{n\pi}{2}\right) + g_0(\epsilon n) \sin\left(\frac{n\pi}{2}\right), \tag{18}$$

where $f_0(\epsilon n)$ and $g_0(\epsilon n)$ are still arbitrary functions, which can be used to avoid unbounded behaviour in $x_1(n, \epsilon n)$ on the $O(\frac{1}{\epsilon})$ iteration scale.

The $O(\epsilon)$ -problem (17) now becomes:

$$\begin{aligned}
 & \epsilon(x_1(n+2, \epsilon n) + x_1(n, \epsilon n)) + 2(x_0(n+2, \epsilon(n+1)) - x_0(n+2, \epsilon n)) + \epsilon x_0(n, \epsilon n) = 0 \\
 \Leftrightarrow & \epsilon(x_1(n+2, \epsilon n) + x_1(n, \epsilon n)) = (2\Delta_\epsilon f_0(\epsilon n) - \epsilon g_0(\epsilon n)) \cos\left(\frac{n\pi}{2}\right) + \\
 & + (2\Delta_\epsilon g_0(\epsilon n) + \epsilon f_0(\epsilon n)) \sin\left(\frac{n\pi}{2}\right).
 \end{aligned} \tag{19}$$

In the $O\Delta E$ (19) for $x_1(n, \epsilon n)$ it is obvious that the righthand side contains terms (i.e., $\cos(\frac{n\pi}{2})$ and $\sin(\frac{n\pi}{2})$), which are solutions of the homogeneous $O\Delta E$. Then, to avoid unbounded or secular behaviour in $x_1(n, \epsilon n)$ it follows that $f_0(\epsilon n)$ and $g_0(\epsilon n)$ have to satisfy:

$$\begin{aligned}
 & 2\Delta_\epsilon f_0(\epsilon n) - \epsilon g_0(\epsilon n) = 0, \\
 & 2\Delta_\epsilon g_0(\epsilon n) + \epsilon f_0(\epsilon n) = 0.
 \end{aligned} \tag{20}$$

System (20) for $f_0(\epsilon n)$ and $g_0(\epsilon n)$ can readily be solved, yielding

$$\begin{aligned}
 & f_0(\epsilon n) = a_0 \left(1 + \frac{\epsilon^2}{4}\right)^{\frac{n}{2}} \cos(n\mu(\epsilon)) + b_0 \left(1 + \frac{\epsilon^2}{4}\right)^{\frac{n}{2}} \sin(n\mu(\epsilon)), \\
 & g_0(\epsilon n) = -a_0 \left(1 + \frac{\epsilon^2}{4}\right)^{\frac{n}{2}} \sin(n\mu(\epsilon)) + b_0 \left(1 + \frac{\epsilon^2}{4}\right)^{\frac{n}{2}} \cos(n\mu(\epsilon)),
 \end{aligned} \tag{21}$$

where a_0 and b_0 are arbitrary constants, and where $\mu(\epsilon)$ is given by $\cos(\mu(\epsilon)) = \left(1 + \frac{\epsilon^2}{4}\right)^{-\frac{1}{2}}$, and $\sin(\mu(\epsilon)) = \frac{\epsilon}{2} \left(1 + \frac{\epsilon^2}{4}\right)^{-\frac{1}{2}}$. From these expressions $\mu(\epsilon)$ can be approximated by

$$\mu(\epsilon) = \frac{1}{2}\epsilon - \frac{1}{24}\epsilon^3 + O(\epsilon^5), \tag{22}$$

and from (19) $x_1(n, \epsilon n)$ can be determined, yielding

$$x_1(n, \epsilon n) = f_1(\epsilon n) \cos\left(\frac{n\pi}{2}\right) + g_1(\epsilon n) \sin\left(\frac{n\pi}{2}\right), \tag{23}$$

where $f_1(\epsilon n)$ and $g_1(\epsilon n)$ are still arbitrary functions which can be used to avoid secular terms in $x_2(n, \epsilon n)$. At this moment, however, we are not interested in the higher order approximations. For that reason we will take in (23) $f_1(\epsilon n)$ and $g_1(\epsilon n)$ equal to the constants a_1 and b_1 respectively. So far we have constructed an approximation for the solution of the $O\Delta E$ (12). In this case the approximation $x_0(n, \epsilon n)$ can be compared with the exact solution of the $O\Delta E$ (12). The exact solution is given by

$$x_n = a 1^n \cos(n\theta(\epsilon)) + b 1^n \sin(n\theta(\epsilon)), \quad (24)$$

where a and b are arbitrary constants, and where $\theta(\epsilon)$ is given by $\cos(\theta(\epsilon)) = -\frac{\epsilon}{2}$ and $\sin(\theta(\epsilon)) = (1 - \frac{\epsilon^2}{4})^{\frac{1}{2}}$, and $\theta(\epsilon)$ can be approximated by $\theta(\epsilon) = \frac{\pi}{2} + \frac{\epsilon}{2} + \frac{\epsilon^3}{48} + O(\epsilon^5)$. The approximation $x_0(n, \epsilon n)$ is given by (18), (21), and (22). This approximation can be rewritten in the following form

$$x_0(n, \epsilon n) = a_0 \left(1 + \frac{\epsilon^2}{4}\right)^{\frac{n}{2}} \cos\left(\frac{n\pi}{2} + n\mu(\epsilon)\right) + b_0 \left(1 + \frac{\epsilon^2}{4}\right)^{\frac{n}{2}} \sin\left(\frac{n\pi}{2} + n\mu(\epsilon)\right). \quad (25)$$

From (24) and (25) it can readily be deduced that the difference between the exact solution x_n and the approximation $x_0(n, \epsilon n)$ is of order ϵ for $n \sim \frac{1}{\epsilon}$. So, the constructed approximation is $O(\epsilon)$ accurate on an iteration scale of order $\frac{1}{\epsilon}$. Usually of course the exact solution of a weakly (non)linearly perturbed $O\Delta E$ will not be available. In section 4 of this paper it will be shown how for such cases the asymptotic validity of an approximation can be obtained on a sufficiently long iteration scale. In the next section of this paper it will be shown how the multiple scales perturbation method can be applied to a second order, weakly nonlinear, regularly perturbed $O\Delta E$.

3 On a weakly nonlinear, regularly perturbed $O\Delta E$

In this section an approximation of the solution of a second order, weakly nonlinear, regularly perturbed $O\Delta E$ with a Van der Pol type of nonlinearity will be constructed. The $O\Delta E$ can be obtained by using a central finite difference approximation of the continuous Van der Pol equation. The $O\Delta E$ is given by

$$x_{n+2} - 2 \cos(\theta)x_{n+1} + x_n = \epsilon(1 - x_{n+1}^2)(x_{n+2} - x_n), \quad (26)$$

where ϵ is a small parameter, that is, $0 < \epsilon \ll 1$, and where θ is constant (which is related to the stepsize in making the continuous Van der Pol equation discrete). It turns out that a straightforward, naive perturbation expansion for x_n (that is, $x_0(n) + \epsilon x_1(n) + \epsilon^2 x_2(n) + \dots$) will lead to secular behaviour in the expansion. To avoid this a two scales perturbation method will be used with $x_n = x(n, \epsilon n)$. To apply the perturbation method the $O\Delta E$ (26) first has to be rewritten with difference operators (see also (5), (6)), yielding

$$\Delta^2 x_n + (2 - 2 \cos(\theta))(\Delta x_n + x_n) = \epsilon(1 - (\Delta x_n + x_n)^2)(\Delta^2 x_n + 2\Delta x_n). \quad (27)$$

Then, the operator Δ in (27) is replaced by $\Delta_1 + \Delta_\epsilon + \Delta_1 \Delta_\epsilon$ (see (9) or (11)), and the function x_n is replaced by $x(n, \epsilon n)$. The $O\Delta E$ (27) then becomes

$$\begin{aligned} & (\Delta_1^2 + 2\Delta_1(\Delta_\epsilon + \Delta_1 \Delta_\epsilon) + \Delta_\epsilon^2 + 2\Delta_1 \Delta_\epsilon^2 + \Delta_1^2 \Delta_\epsilon^2)x(n, \epsilon n) + \\ & (2 - 2 \cos(\theta))(\Delta_1 + \Delta_\epsilon + \Delta_1 \Delta_\epsilon + I)x(n, \epsilon n) = \\ & \epsilon(1 - ((\Delta_1 + \Delta_\epsilon + \Delta_1 \Delta_\epsilon + I)x(n, \epsilon n))^2)((\Delta_1^2 + 2\Delta_1(\Delta_\epsilon + \Delta_1 \Delta_\epsilon) + \\ & \Delta_\epsilon^2 + 2\Delta_1 \Delta_\epsilon^2 + \Delta_1^2 \Delta_\epsilon^2)x(n, \epsilon n) + 2(\Delta_1 + \Delta_\epsilon + \Delta_1 \Delta_\epsilon)x(n, \epsilon n)). \end{aligned} \quad (28)$$

Then, the function $x(n, \epsilon n)$ is expanded in a formal power series in ϵ , that is,

$$x(n, \epsilon n) = x_0(n, \epsilon n) + \epsilon x_1(n, \epsilon n) + \epsilon^2 x_2(n, \epsilon n) + \dots, \quad (29)$$

and is substituted into (28). By taking together those terms of equal powers in ϵ one obtains as $O(1)$ -problem:

$$\Delta_1^2 x_0(n, \epsilon n) + (2 - 2 \cos(\theta))(\Delta_1 + I)x_0(n, \epsilon n) = 0, \quad (30)$$

and as $O(\epsilon)$ -problem

$$\begin{aligned} \epsilon \Delta_1^2 x_1(n, \epsilon n) + \epsilon(2 - 2 \cos(\theta))(\Delta_1 + I)x_1(n, \epsilon n) = \\ -2\Delta_1(\Delta_\epsilon + \Delta_1 \Delta_\epsilon)x_0(n, \epsilon n) - (2 - 2 \cos(\theta))(\Delta_\epsilon + \Delta_1 \Delta_\epsilon)x_0(n, \epsilon n) \\ + \epsilon(1 - ((\Delta_1 + I)x_0(n, \epsilon n))^2)((\Delta_1^2 + 2\Delta_1)x_0(n, \epsilon n)), \end{aligned} \quad (31)$$

and so on. The $O(1)$ -problem (30) can readily be solved, yielding

$$x_0(n, \epsilon n) = f_0(\epsilon n) \cos(n\theta) + g_0(\epsilon n) \sin(n\theta), \quad (32)$$

where $f_0(\epsilon n)$ and $g_0(\epsilon n)$ are still arbitrary functions which can be used to avoid secular terms in $x_1(n, \epsilon n)$. Then, by substituting (32) into the $O(\epsilon)$ -problem (31), and after rearranging terms, one finally obtains as $O(\epsilon)$ -problem

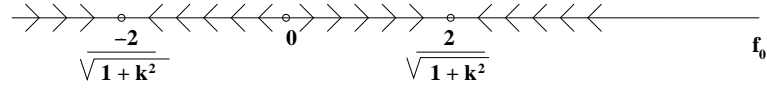
$$\begin{aligned} \epsilon \Delta_1^2 x_1(n, \epsilon n) + \epsilon(2 - 2 \cos \theta)(\Delta_1 + I)x_1(n, \epsilon n) = \\ \cos(n\theta) \left\{ (1 - \cos(2\theta))(\Delta_\epsilon f_0(\epsilon n) - \epsilon f_0(\epsilon n)(1 - \frac{1}{4}f_0^2(\epsilon n) - \frac{1}{4}g_0^2(\epsilon n))) \right. \\ \left. - \sin(2\theta)(\Delta_\epsilon g_0(\epsilon n) - \epsilon g_0(\epsilon n)(1 - \frac{1}{4}f_0^2(\epsilon n) - \frac{1}{4}g_0^2(\epsilon n))) \right\} + \\ \sin(n\theta) \left\{ \sin(2\theta)(\Delta_\epsilon f_0(\epsilon n) - \epsilon f_0(\epsilon n)(1 - \frac{1}{4}f_0^2(\epsilon n) - \frac{1}{4}g_0^2(\epsilon n))) \right. \\ \left. + (1 - \cos(2\theta))(\Delta_\epsilon g_0(\epsilon n) - \epsilon g_0(\epsilon n)(1 - \frac{1}{4}f_0^2(\epsilon n) - \frac{1}{4}g_0^2(\epsilon n))) \right\} + \\ \epsilon \left\{ \cos(3n\theta + 2\theta) \left[(\cos(2\theta) - 1)(-\frac{1}{4}f_0^3(\epsilon n) + \frac{3}{4}f_0(\epsilon n)g_0^2(\epsilon n)) + \right. \right. \\ \left. \left. \sin(2\theta)(-\frac{3}{4}f_0^2(\epsilon n)g_0(\epsilon n) + \frac{1}{4}g_0^3(\epsilon n)) \right] + \right. \\ \left. \sin(3n\theta + 2\theta) \left[(\cos(2\theta) - 1)(-\frac{3}{4}f_0^2(\epsilon n)g_0(\epsilon n) + \frac{1}{4}g_0^3(\epsilon n)) + \right. \right. \\ \left. \left. \sin(2\theta)(\frac{1}{4}f_0^3(\epsilon n) - \frac{3}{4}f_0(\epsilon n)g_0^2(\epsilon n)) \right] \right\}. \end{aligned} \quad (33)$$

In the $O\Delta E$ (33) for $x_1(n, \epsilon n)$ it is obvious that the righthand side contains terms (that is, $\cos(n\theta)$ and $\sin(n\theta)$), which are solutions of the corresponding homogeneous $O\Delta E$. It should be observed that the terms involving $\cos(3n\theta + 2\theta)$ and $\sin(3n\theta + 2\theta)$ are no solutions of the homogeneous $O\Delta E$. So, to avoid unbounded or secular behaviour in $x_1(n, \epsilon n)$ it follows from (33) that $f_0(\epsilon n)$ and $g_0(\epsilon n)$ have to satisfy:

$$\begin{aligned} \Delta_\epsilon f_0(\epsilon n) &= \epsilon f_0(\epsilon n)(1 - \frac{1}{4}f_0^2(\epsilon n) - \frac{1}{4}g_0^2(\epsilon n)), \\ \Delta_\epsilon g_0(\epsilon n) &= \epsilon g_0(\epsilon n)(1 - \frac{1}{4}f_0^2(\epsilon n) - \frac{1}{4}g_0^2(\epsilon n)). \end{aligned} \quad (34)$$

From (34) it is obvious that when $f_0(\epsilon n)$ (or $g_0(\epsilon n)$) is equal to zero for some $n = n_0$ then $f_0(\epsilon n)$ (or $g_0(\epsilon n)$) is zero for all $n \geq n_0$. It also follows from (34) that $g_0(\epsilon n)\Delta_\epsilon f_0(\epsilon n) - f_0(\epsilon n)\Delta_\epsilon g_0(\epsilon n) = 0$ or equivalently (assuming that $f_0(\epsilon n) \neq 0$):

$$\Delta \left(\frac{g_0(\epsilon n)}{f_0(\epsilon n)} \right) = 0 \quad \Leftrightarrow \quad g_0(\epsilon n) = k f_0(\epsilon, n) \quad (35)$$


 Figure 1: The dynamics of the solutions of the $O\Delta E$ (36) for $0 < \varepsilon < 1$.

for some constant k , which is determined by the initial conditions. The system of two first order $O\Delta E$ s (34) then reduces by using (35) to the following first order $O\Delta E$ for $f_0(\varepsilon n)$.

$$\Delta_\varepsilon f_0(\varepsilon n) = \varepsilon f_0(\varepsilon n) \left(1 - \frac{1}{4}(1+k^2)f_0^2(\varepsilon n) \right). \quad (36)$$

As far as we know there are no exact solutions available for the $O\Delta E$ (36). However, for $0 < \varepsilon < 1$ the $O\Delta E$ (36) has three equilibrium points: an unstable one for $f_0 = 0$, and two stable ones: one for $f_0 = \frac{-2}{\sqrt{1+k^2}}$ and one for $f_0 = \frac{2}{\sqrt{1+k^2}}$. The dynamics of the solutions of the $O\Delta E$ (36) is depicted in Figure 3.1.

For $\varepsilon \geq 1$ the dynamics becomes much more complicated (three unstable equilibrium points, period doublings, chaotic behaviour), but since $0 < \varepsilon \ll 1$ that case is beyond the scope of the analysis. From (35), (36), and Figure 3.1 it now follows that (for $f_0^2(\varepsilon n) + g_0^2(\varepsilon n) \neq 0$) $f_0^2(\varepsilon n) + g_0^2(\varepsilon n)$ tends (slowly) to 4 for $n \rightarrow \infty$. From (33) $x_1(n, \varepsilon n)$ can now be determined such that $x_1(n, \varepsilon n)$ is bounded for $n \sim \frac{1}{\varepsilon}$, and from (32) and (35) it follows that

$$x_0(n, \varepsilon n) = f_0(\varepsilon n) \sqrt{1+k^2} \sin(n\theta + \varphi), \quad (37)$$

where φ is given by $\sin(\varphi) = \frac{1}{\sqrt{1+k^2}}$ and $\cos(\varphi) = \frac{k}{\sqrt{1+k^2}}$, and where $f_0(\varepsilon n)$ is a solution of the $O\Delta E$ (36) (see also Figure 3.1). So far an approximation $x_{approx}(n, \varepsilon n) = x_0(n, \varepsilon n) + \varepsilon x_1(n, \varepsilon n)$ has been constructed, where $x_1(n, \varepsilon n)$ still contains some arbitrary functions which can be used to avoid secular terms in $x_2(n, \varepsilon n)$. Since we are not interested in the higher order approximations there arbitrary functions will or can be chosen equal to their initial values. The approximation $x_{approx}(n, \varepsilon n)$ satisfied the $O\Delta E$ (26) accurately, that is, up to $O(\varepsilon^2)$. In fact it can be shown that

$$x_{approx}(n+2, \varepsilon(n+2)) - 2 \cos(\theta) x_{approx}(n+1, \varepsilon(n+1)) + x_{approx}(n, \varepsilon n) - \varepsilon(1 - x_{approx}^2(n+1, \varepsilon(n+1)))(x_{approx}(n+2, \varepsilon(n+2)) - x_{approx}(n, \varepsilon n)) = \varepsilon^2 R(n, \varepsilon n), \quad (38)$$

where $R(n, \varepsilon n)$ depends on $x_0(n, \varepsilon n)$ and on $x_1(n, \varepsilon n)$, and where $R(n, \varepsilon n)$ is bounded for $n \sim \frac{1}{\varepsilon}$. It requires an additional analysis to show that $x_{approx}(n, \varepsilon n)$ and $x_0(n, \varepsilon n)$ are both $O(\varepsilon)$ accurate approximations of the solution of (26) for $n \sim \frac{1}{\varepsilon}$. The proof of asymptotic validity of the approximation(s) on long iteration scales (that is, for $n \sim \frac{1}{\varepsilon}$) will be given in the next section of this paper.

4 On the asymptotic validity of approximations

In this section a justification of the multiple scales perturbation method for $O\Delta E$ s will be given, which covers all the examples that have been presented in this paper. Consider the following system of k first order $O\Delta E$ s:

$$\underline{u}(n+1) = A\underline{u}(n) + \varepsilon \underline{f}(\underline{u}(n), n; \varepsilon), \quad (39)$$

where $\underline{u}(n)$ is a $(k \times 1)$ -vector, A is a $(k \times k)$ -matrix with constant and ϵ -independent elements, ϵ is a small parameter with $0 < \epsilon \ll 1$, and where \underline{f} is $(k \times 1)$ -vector, with $\underline{f} : \mathbb{R}^k \times \mathbb{R} \rightarrow \mathbb{R}^k$. Moreover, \underline{f} satisfies the following Lipschitz condition, that is,

$$\| \underline{f}(\underline{u}(n), n; \epsilon) - \underline{f}(\tilde{\underline{u}}(n), n; \epsilon) \| \leq L \| \underline{u}(n) - \tilde{\underline{u}}(n) \| \quad (40)$$

for some constant L , and where $\tilde{\underline{u}}(n)$ is a $(k \times 1)$ -vector, and where $\|\cdot\|$ is assumed to be the maximum norm. The vector function \underline{f} is not necessarily linear in $\underline{u}(n)$. Let $\tilde{\underline{u}}(n)$ be an approximation of $\underline{u}(n)$, which has been constructed by some kind of perturbation method (for instance a multiple scales perturbation method or another perturbation method), and let $\tilde{\underline{u}}(n)$ satisfy

$$\tilde{\underline{u}}(n+1) = A\tilde{\underline{u}}(n) + \epsilon \underline{f}(\tilde{\underline{u}}(n), n; \epsilon) + \epsilon^{m+1} \underline{R}(n; \epsilon), \quad (41)$$

where $\underline{R}(n; \epsilon)$ is a $(k \times 1)$ -vector, and where m is a positive integer (usually $m = 1$). Now it will be assumed that matrix A is similar to a diagonal matrix, that is, it will be assumed that there exists a nonsingular matrix P (with $\|P\|$ and $\|P^{-1}\|$ bounded by a constant M_0) such that $P^{-1}AP$ is a diagonal matrix with on the diagonal the eigenvalue $\lambda_1, \lambda_2, \dots, \lambda_k$ of matrix A . When A is not similar to a diagonal matrix (that is, when the algebraic multiplicity of an eigenvalue is not equal to the geometric multiplicity of this eigenvalue) then a similar proof can be given by using the Jordan form of matrix A . This almost similar proof will be omitted in this paper. Now let $\underline{u}(n) = P\underline{v}(n)$ and $\tilde{\underline{u}}(n) = P\tilde{\underline{v}}(n)$. Then, (4.1) and (4.3) become

$$\begin{aligned} \underline{v}(n+1) &= P^{-1}AP\underline{v}(n) + \epsilon P^{-1}\underline{f}(P\underline{v}(n), n; \epsilon), \\ \tilde{\underline{v}}(n+1) &= P^{-1}AP\tilde{\underline{v}}(n) + \epsilon P^{-1}\underline{f}(P\tilde{\underline{v}}(n), n; \epsilon) + \epsilon^{m+1}P^{-1}\underline{R}(n; \epsilon), \end{aligned} \quad (42)$$

where $P^{-1}AP$ is a diagonal matrix with on the diagonal the eigenvalues $\lambda_1, \dots, \lambda_k$ of matrix A . Now let

$$\lambda = \max_{1 \leq i \leq k} |\lambda_i|. \quad (43)$$

If $\lambda \leq 1$ it will be assumed that for $n \sim \frac{1}{\epsilon}$

$$\|\underline{R}(n; \epsilon)\| \leq M_1, \quad (44)$$

where M_1 is a positive constant. And for $\lambda > 1$ it will be assumed that for $n \sim \frac{1}{\epsilon}$

$$\|\underline{R}(n; \epsilon)\| \leq M_2 \lambda^n (1 + M_3 \epsilon)^n, \text{ and} \quad (45)$$

$$\|\underline{f}(\underline{u}(n), n; \epsilon)\| \leq M_4 \|\underline{u}(n)\| + M_5 \text{ for } \|\underline{u}(n)\| \rightarrow \infty, \quad (46)$$

where M_2, M_3, M_4 , and M_5 are positive constants. Condition (4.8) for $\lambda > 1$ indicates that the linear part in (4.1) dominates the dynamics of the problem (or equivalently, the term $\epsilon \underline{f}$ in (4.1) remains relatively small compared to the other terms in (4.1)). Now first the case $0 < \lambda \leq 1$ will be considered. The degenerate case $\lambda = 0$ will not be considered in this paper. By subtracting the two equations in (4.4), and by using (4.2), (4.5), and (4.6) the following estimate can be obtained.

$$\begin{aligned} \|\underline{v}(n+1) - \tilde{\underline{v}}(n+1)\| &\leq (\lambda + \epsilon LM_0) \|\underline{v}(n) - \tilde{\underline{v}}(n)\| + \epsilon^{m+1} M_0 M_1 \Rightarrow \\ \|\underline{v}(n) - \tilde{\underline{v}}(n)\| &\leq (\lambda + \epsilon LM_0)^n \|\underline{v}(0) - \tilde{\underline{v}}(0)\| + \epsilon^{m+1} M_0 M_1 \left(\frac{(\lambda + \epsilon LM_0)^n - 1}{\lambda + \epsilon LM_0 - 1} \right). \end{aligned} \quad (47)$$

From (4.9) it follows that for $0 < \lambda < 1$

$$\|\underline{v}(n) - \tilde{\underline{v}}(n)\| \leq \|\underline{v}(0) - \tilde{\underline{v}}(0)\| O(\lambda^n e^{\epsilon n \frac{LM_0}{\lambda}}) + O(\epsilon^{m+1} \lambda^n e^{\epsilon n \frac{LM_0}{\lambda}}), \quad (48)$$

and for $\lambda = 1$ that

$$\|\underline{v}(n) - \tilde{\underline{v}}(n)\| \leq \|\underline{v}(0) - \tilde{\underline{v}}(0)\| O(e^{\epsilon n LM_0}) + O(\epsilon^m e^{\epsilon n LM_0}). \quad (49)$$

Now if $\|\underline{u}(0) - \tilde{\underline{u}}(0)\| = O(\epsilon^{m+1})$ for $0 < \lambda < 1$, and if $\|\underline{u}(0) - \tilde{\underline{u}}(0)\| = O(\epsilon^m)$ for $\lambda = 1$ it follows from (4.10) that for $0 < \lambda < 1$

$$\|\underline{u}(n) - \tilde{\underline{u}}(n)\| = \|P^{-1}\underline{v}(n) - P^{-1}\tilde{\underline{v}}(n)\| \leq M_0 \|\underline{v}(n) - \tilde{\underline{v}}(n)\| = O(\epsilon^{m+1}\lambda^n) \quad (50)$$

for $n = O(\frac{1}{\epsilon})$, and it follows from (4.11) that for $\lambda = 1$

$$\|\underline{u}(n) - \tilde{\underline{u}}(n)\| = O(\epsilon^m) \quad (51)$$

for $n = O(\frac{1}{\epsilon})$.

Now the case $\lambda > 1$ will be considered. Again consider (4.4) and let $\underline{v}(n) = \lambda^n \underline{w}(n)$ and $\tilde{\underline{v}}(n) = \lambda^n \tilde{\underline{w}}(n)$. System (4.4) then becomes

$$\begin{aligned} \underline{w}(n+1) &= \frac{1}{\lambda} P^{-1} A P \underline{w}(n) + \frac{\epsilon}{\lambda^{n+1}} P^{-1} \underline{f}(\lambda^n P \underline{w}(n), n; \epsilon), \\ \tilde{\underline{w}}(n+1) &= \frac{1}{\lambda} P^{-1} A P \tilde{\underline{w}}(n) + \frac{\epsilon}{\lambda^{n+1}} P^{-1} \underline{f}(\lambda^n P \tilde{\underline{w}}(n), n; \epsilon) + \frac{\epsilon^{m+1}}{\lambda^{n+1}} P^{-1} \underline{R}(n; \epsilon). \end{aligned} \quad (52)$$

In (4.14) it should be observed that $\frac{1}{\lambda} P^{-1} A P$ is a diagonal matrix for which the modulus of the largest diagonal element in modulus is equal to one. Then, by subtracting the two equations in (4.14), and by using (4.2), (4.5), (4.7), and (4.8) it follows that (assuming that $\|\underline{w}(0) - \tilde{\underline{w}}(0)\| = O(\epsilon^m)$)

$$\begin{aligned} \|\underline{w}(n+1) - \tilde{\underline{w}}(n+1)\| &\leq 1 \cdot \|\underline{w}(n) - \tilde{\underline{w}}(n)\| + \frac{\epsilon LM_0}{\lambda} \|\underline{w}(n) - \tilde{\underline{w}}(n)\| + \frac{\epsilon^{m+1}}{\lambda^{n+1}} M_0 \|\underline{R}(n; \epsilon)\| \\ \Rightarrow \|\underline{w}(n) - \tilde{\underline{w}}(n)\| &\leq (1 + \frac{\epsilon LM_0}{\lambda})^n \{ \|\underline{w}(0) - \tilde{\underline{w}}(0)\| + O(\epsilon^m e^{\epsilon n M_3}) \} \\ \Rightarrow \|\underline{w}(n) - \tilde{\underline{w}}(n)\| &= O(\epsilon^m) \quad \text{for } n \sim \frac{1}{\epsilon}. \end{aligned} \quad (53)$$

From $\underline{u}(n) = P \underline{v}(n) = \lambda^n P \underline{w}(n)$ and $\tilde{\underline{u}}(n) = \lambda^n P \tilde{\underline{w}}(n)$, and from (4.15) it can then be deduced that for $\lambda > 1$

$$\|\underline{u}(n) - \tilde{\underline{u}}(n)\| = O(\epsilon^m \lambda^n) \quad (54)$$

for $n = O(\frac{1}{\epsilon})$. The results which have been proved so far are summarized in the following theorem.

Theorem 4.1 *Let $\underline{u}(n)$ and $\tilde{\underline{u}}(n)$ satisfy (39) and (41) respectively, where \underline{f} satisfies (40) and where matrix A is assumed to be similar to a diagonal matrix. Let λ be defined by (43), and let \underline{R} and \underline{f} additionally satisfy (44) or (45)-(46) for $0 < \lambda \leq 1$ or $\lambda > 1$ respectively. Furthermore, let $\|\underline{u}(0) - \tilde{\underline{u}}(0)\|$ be $O(\epsilon^{m+1})$ for $0 < \lambda < 1$ and $O(\epsilon^m)$ for $\lambda \geq 1$. Then, for $n = O(\frac{1}{\epsilon})$ it follows that*

$$\begin{aligned} \|\underline{u}(n) - \tilde{\underline{u}}(n)\| &= O(\epsilon^{m+1}\lambda^n) \quad \text{for } 0 < \lambda < 1, \text{ and} \\ \|\underline{u}(n) - \tilde{\underline{u}}(n)\| &= O(\epsilon^m \lambda^n) \quad \text{for } \lambda \geq 1. \end{aligned}$$

Since each k -th order $O\Delta E$ can be rewritten as a system of k first order $O\Delta E$ s it follows that Theorem 4.1 directly can be applied to the examples as treated in the previous sections (see the linear $O\Delta E$ (12) and the weakly nonlinear $O\Delta E$ (26)). For both examples it can be simply shown that $\lambda = 1$ and that $|x_n - (x_0(n, \epsilon n) + \epsilon x_1(n, \epsilon n))| = O(\epsilon)$ for $n = O(\frac{1}{\epsilon})$. It also follows for $n = O(\frac{1}{\epsilon})$ that

$$\begin{aligned}
|x_n - x_0(n, \epsilon n)| &= |x_n - (x_0(n, \epsilon n) + \epsilon x_1(n, \epsilon n)) + \epsilon x_1(n, \epsilon n)| \leq \\
&\leq |x_n - (x_0(n, \epsilon n) + \epsilon x_1(n, \epsilon n))| + \epsilon |x_1(n, \epsilon n)| \\
&= O(\epsilon) + O(\epsilon) = O(\epsilon)
\end{aligned}$$

since $x_1(n, \epsilon n)$ is bounded for $n = O(\frac{1}{\epsilon})$. So, in both examples the functions $x_0(n, \epsilon n)$ are $O(\epsilon)$ accurate approximation of x_n for $0 \leq n \leq \frac{K}{\epsilon}$, where K is an ϵ -independent constant.

5 On singularly perturbed, linear $O\Delta E$ s

In this section the following three singularly perturbed, linear, second order $O\Delta E$ s will be studied for $n = 0, 1, 2, \dots$:

$$(\epsilon - \frac{1}{4})x_{n+2} + (\frac{1}{2} - 2\epsilon)x_{n+1} + \epsilon x_n = 0, \quad (55)$$

$$(\epsilon + 2)y_{n+2} - (1 + 2\epsilon)y_{n+1} + \epsilon y_n = 0, \text{ and} \quad (56)$$

$$z_{n+2} + \epsilon z_{n+1} + \epsilon z_n = 0, \quad (57)$$

where ϵ is a small parameter with $0 < \epsilon \ll 1$. A singularly perturbed $O\Delta E$ is characterised by the fact that the order of the $O\Delta E$ is reduced when the small parameter ϵ is taken equal to zero (in this case). In the $O\Delta E$ s (55) and (56) the order of the $O\Delta E$ s will be reduced by one, whereas for $O\Delta E$ s (57) with $\epsilon = 0$ one can hardly speak of an $O\Delta E$. When a naive, straight forward perturbation approach is used to approximate the solution of the $O\Delta E$ it is usually impossible to satisfy all the initial conditions because of this order reduction in the $O\Delta E$. For ODE s rescaling procedures are used to tackle this problem. In this section a rescaling procedure for $O\Delta E$ s like (55)-(57) will be proposed to solve the aforementioned problem for $O\Delta E$ s. When the significant scalings in the $O\Delta E$ s are known the multiple scales perturbation method for $O\Delta E$ s (as developed in section 2 of this paper) will be applied to approximate the solutions of the $O\Delta E$ s. In fact a two scales method will be used for (55), and a three scales method will be used for (56) and for (57). It will turn out that the constructed approximations of the solutions are accurate ones on long iteration scales. This can be shown by using the theory as developed in the previous section or by comparing the approximations directly with the exact solutions. Now the $O\Delta E$ (55) will be considered firstly.

5.1 The $O\Delta E$ (55)

To determine the significant scalings in (55) the following rescaling procedure is proposed. Let

$$x_n = \delta_n(\epsilon)a_n, \quad (58)$$

where $\delta_n(\epsilon)$ is a rescaling function which might depend on ϵ , and where a_n is a function which remains $O(1)$ on the iteration scale under consideration (usually this will be an iteration scale of $O(\frac{1}{\epsilon})$). Then, (58) is substituted into (55), yielding

$$(\epsilon - \frac{1}{4})\delta_{n+2}(\epsilon)a_{n+2} + (\frac{1}{2} - 2\epsilon)\delta_{n+1}(\epsilon)a_{n+1} + \epsilon\delta_n(\epsilon)a_n = 0 \quad (59)$$

and a balancing procedure will be followed to determine the rescaling function(s) $\delta_n(\epsilon)$. When the first term and the second term in (59) are assumed to be the most significant ones then $\delta_n(\epsilon)$ has to satisfy

$$\frac{1}{4}\delta_{n+2}(\epsilon) + \frac{1}{2}\delta_{n+1}(\epsilon) = 0 \quad \Rightarrow \quad \delta_n(\epsilon) = \delta_0 2^n. \quad (60)$$

It can readily be verified that the third term in (59) (that is, $\epsilon\delta_n(\epsilon)a_n$) is indeed smaller. So, $\delta(\epsilon) = 2^n$ is indeed a significant rescaling. When the second term and the third term in (59) are assumed to be the most significant ones then $\delta_n(\epsilon)$ has to satisfy

$$\frac{1}{2}\delta_{n+1}(\epsilon) + \epsilon\delta_n(\epsilon) = 0 \quad \Rightarrow \quad \delta_n(\epsilon) = \delta_0(-2\epsilon)^n. \quad (61)$$

It can be verified that the first term in (59) is indeed smaller, and so, $\delta_n(\epsilon) = (-2\epsilon)^n$ is another significant rescaling. Similarly it can be checked that the first term and the third term in (59) can not be the most significant ones simultaneously, since the second term would be in that case larger. Now two functionally independent approximations of the solutions of the linear $O\Delta E$ (55) will be constructed. The sum of these two approximations will be an approximation of the general solution of the $O\Delta E$ (55). Firstly let $x_n = 2^n a_n$. The $O\Delta E$ (55) then becomes

$$(-1 + 4\epsilon)a_{n+2} + (1 - 4\epsilon)a_{n+1} + \epsilon a_n = 0. \quad (62)$$

A two scales perturbation method will be used to approximate the solution a_n of (62) since the straightforward perturbation expansion will lead to secular terms. The following expansion for a_n

$$a_n = a(n, \epsilon n) = a_0(n, \epsilon n) + \epsilon a_1(n, \epsilon n) + \epsilon^2 a_2(n, \epsilon n) + \dots \quad (63)$$

is substituted into (62), and terms of equal powers in ϵ are taken together (see also section 2 of this paper), yielding as $O(1)$ -problem

$$\Delta_1^2 a_0(n, \epsilon n) + \Delta_1 a_0(n, \epsilon n) = 0 \quad \Leftrightarrow \quad a_0(n+2, \epsilon n) - a_0(n+1, \epsilon n) = 0, \quad (64)$$

and as $O(\epsilon)$ -problem

$$\begin{aligned} \epsilon \Delta_1^2 a_1(n, \epsilon n) + \epsilon \Delta_1 a_1(n, \epsilon n) &= -2\Delta_1(\Delta_1 \Delta_\epsilon + \Delta_\epsilon) a_0(n, \epsilon n) + \\ &+ 4\epsilon \Delta_1^2 a_0(n, \epsilon n) - (\Delta_1 \Delta_\epsilon + \Delta_\epsilon) a_0(n, \epsilon n) + 4\epsilon \Delta_1 a_0(n, \epsilon n) + \epsilon x_0(n, \epsilon n), \end{aligned} \quad (65)$$

and so on. The $O(1)$ -problem (64) can readily be solved, yielding

$$a_0(n, \epsilon n) = f_0(\epsilon n), \quad (66)$$

where $f_0(\epsilon n)$ is an arbitrary function which will be used to avoid secular terms in $a_1(n, \epsilon n)$. The $O(\epsilon)$ -problem (65) then becomes

$$\epsilon \Delta_1^2 a_1(n, \epsilon n) + \epsilon \Delta_1 a_1(n, \epsilon n) = -\Delta_\epsilon f_0(\epsilon n) + \epsilon f_0(\epsilon n). \quad (67)$$

To avoid secular terms in $a_1(n, \epsilon n)$ it is obvious from (67) that $f_0(\epsilon n)$ should satisfy

$$-\Delta_\epsilon f_0(\epsilon n) + \epsilon f_0(\epsilon n) = 0 \quad \Rightarrow \quad f_0(\epsilon n) = c_0(1 + \epsilon)^n, \quad (68)$$

and

$$a_1(n, \epsilon n) = f_1(\epsilon n),$$

where c_0 is an arbitrary constant, and where $f_1(\epsilon n)$ is an arbitrary function which can be used to avoid secular terms in $a_2(n, \epsilon n)$. Since we are not interested in the higher order approximation $f_1(\epsilon n)$ will now be taken equal to its initial value c_1 . So far the following approximation for a_n has been constructed (see (63), (66), and (68)): $c_0(1 + \epsilon)^n + \epsilon c_1$, where c_0 and c_1 are constants. And so, it follows that an approximation of a solution of (55) is

$$2^n(c_0(1 + \epsilon)^n + \epsilon c_1). \quad (69)$$

Another functionally independent approximation of the solution of (55) can be constructed by using the rescaling given by (61), that is, let $x_n = (-2\epsilon)^n b_n$. The $O\Delta E$ (55) then becomes

$$\epsilon(1 - 4\epsilon)b_{n+2} + (1 - 4\epsilon)b_{n+1} - b_n = 0. \quad (70)$$

The following expansion for b_n

$$b_n = b(n, \epsilon n) = b_0(n, \epsilon n) + \epsilon b_1(n, \epsilon n) + \epsilon^2 b_2(n, \epsilon n) + \dots \quad (71)$$

is substituted into (70), and terms of equal powers in ϵ are taken together, yielding as $O(1)$ -problem

$$\Delta_1 b_0(n, \epsilon n) = 0, \quad (72)$$

and as $O(\epsilon)$ -problem

$$\begin{aligned} \epsilon \Delta_1 b_1(n, \epsilon n) = & -\epsilon \Delta_1^2 b_0(n, \epsilon n) - (\Delta_1 \Delta_\epsilon + \Delta_\epsilon) b_0(n, \epsilon n) \\ & + 2\epsilon \Delta_1 b_0(n, \epsilon n) + 3\epsilon b_0(n, \epsilon n), \end{aligned} \quad (73)$$

and so on. The $O(1)$ -problem (72) can readily be solved, yielding

$$b_0(n, \epsilon n) = g_0(\epsilon n), \quad (74)$$

where $g_0(\epsilon n)$ is an arbitrary function which will be used to avoid secular terms in $b_1(n, \epsilon n)$. The $O(\epsilon)$ (73) now becomes

$$\epsilon \Delta_1 b_1(n, \epsilon n) = -\Delta_\epsilon g_0(\epsilon n) + 3\epsilon g_0(\epsilon n). \quad (75)$$

To avoid secular terms in $b_1(n, \epsilon n)$ it is clear from (75) that $g_0(\epsilon n)$ should satisfy

$$-\Delta_\epsilon g_0(\epsilon n) + 3\epsilon g_0(\epsilon n) = 0 \quad \Rightarrow \quad g_0(\epsilon n) = d_0(1 + 3\epsilon)^n,$$

and

$$b_1(n, \epsilon n) = g_1(\epsilon n), \quad (76)$$

where d_0 is an arbitrary constant, and where $g_1(\epsilon n)$ is an arbitrary function which can be used to avoid secular terms in $b_2(n, \epsilon n)$. Since we are not interested in the higher order approximations $g_1(\epsilon n)$ will now be taken equal to a constant d_1 . So far the following approximation for b_n has been constructed (see (71), (74), and (76)): $d_0(1 + 3\epsilon)^n + \epsilon d_1$, where d_0 and d_1 are constants. And so, it follows that an approximation of a solution of (55) is

$$(-2\epsilon)^n (d_0(1 + 3\epsilon)^n + \epsilon d_1). \quad (77)$$

Since the $O\Delta E$ (55) is linear the superposition principle can be used, and then it follows from (69) and (77) that an approximation $x(n, \epsilon n)$ of the solution x_n of (55) is given by

$$x(n, \epsilon n) = 2^n (c_0(1 + \epsilon)^n + \epsilon c_1) + (-2\epsilon)^n (d_0(1 + 3\epsilon)^n + \epsilon d_1), \quad (78)$$

where c_0, c_1, d_0 , and d_1 are constants which are determined by the initial conditions for $n = 0, 1$ (that is, when x_0 and x_1 , and their expansions in ϵ are given). How well $x(n, \epsilon n)$ approximates the exact solution x_n can now be determined in two ways. The first way is to apply Theorem 4.1 (see the previous section). For $O\Delta E$ (55) we have that $\lambda = 2$ and $x(n, \epsilon n)$ satisfies the $O\Delta E$ (55) upto $(2 + 2\epsilon)^n O(\epsilon^2) + (-2\epsilon - 6\epsilon)^n O(\epsilon^3)$. So, it follows from Theorem 4.1 that

$$|x_n - x(n, \epsilon n)| = 2^n O(\epsilon) \quad \text{for } n = O\left(\frac{1}{\epsilon}\right). \quad (79)$$

From (79) it follows that the absolute error becomes large on the iteration scale of $O(\frac{1}{\epsilon})$, but that the relative error is small for $n \sim \frac{1}{\epsilon}$, that is,

$$\left| \frac{x_n - x(n, \epsilon n)}{x(n, \epsilon n)} \right| = O(\epsilon) \quad \text{for } n \sim \frac{1}{\epsilon}. \quad (80)$$

Since the linear $O\Delta E$ (55) has constant coefficients also the exact solution x_n can be computed directly, yielding

$$\begin{aligned} x_n &= a(1 + (1 - 4\epsilon)^{-\frac{1}{2}})^n + b(1 - (1 - 4\epsilon)^{-\frac{1}{2}})^n \\ &= a(2 + 2\epsilon + 6\epsilon^2 + O(\epsilon^3))^n + b(-2\epsilon - 6\epsilon^2 + O(\epsilon^3))^n, \end{aligned} \quad (81)$$

where a and b are constants. It can also readily be verified from (81) that $x(n, \epsilon n)$ satisfies (79) and (80).

5.2 The $O\Delta E$ (56)

The rescaling and balancing procedure as introduced in section 5.1 can be repeated for the $O\Delta E$ (56), and it turns out that the significant scalings are

$$\delta_n(\epsilon) = \left(\frac{1}{2}\right)^n, \quad \text{and } \delta_n(\epsilon) = \epsilon^n. \quad (82)$$

As in the previous example two functionally independent approximations of the solutions of the linear $O\Delta E$ (56) will be constructed but in this example by using a three scales perturbation method instead of a two scales method. By putting $y_n = \left(\frac{1}{2}\right)^n a_n$ the $O\Delta E$ (56) becomes

$$(\epsilon + 2)a_{n+2} - (2 + 4\epsilon)a_{n+1} + 4\epsilon a_n = 0. \quad (83)$$

Then, the following expansion for a_n

$$a_n = a_0(n, \epsilon n, \epsilon^2 n) + \epsilon a_1(n, \epsilon n, \epsilon^2 n) + \epsilon^2 a_2(n, \epsilon n, \epsilon^2 n) + \dots \quad (84)$$

is substituted into (83), and as usual the following $O(\epsilon^m)$ -problems are obtained for $m = 0, 1, 2, \dots$ (see also section 2 of this paper):

the $O(1)$ -problem:

$$\begin{aligned} 2\Delta_1^2 a_0(n, \epsilon n, \epsilon^2 n) + 2\Delta_1 a_0(n, \epsilon n, \epsilon^2 n) &= 0 \quad \Leftrightarrow \\ a_0(n + 2, \epsilon n, \epsilon^2 n) - a_0(n + 1, \epsilon n, \epsilon^2 n) &= 0, \end{aligned} \quad (85)$$

the $O(\epsilon)$ -problem:

$$\begin{aligned} 2\epsilon\Delta_1^2 a_1(n, \epsilon n, \epsilon^2 n) + 2\Delta_1 a_1(n, \epsilon n, \epsilon^2 n) &= -(4\Delta_1 + 2)(\Delta_1\Delta_\epsilon + \Delta_\epsilon)a_0(n, \epsilon n, \epsilon^2 n) \\ -\epsilon\Delta_1^2 a_0(n, \epsilon n, \epsilon^2 n) + 2\epsilon\Delta_1 a_0(n, \epsilon n, \epsilon^2 n) - \epsilon a_0(n, \epsilon n, \epsilon^2 n), \end{aligned} \quad (86)$$

the $O(\epsilon^2)$ -problem:

$$\begin{aligned} 2\epsilon^2\Delta_1 a_2(n, \epsilon n, \epsilon^2 n) + 2\epsilon^2\Delta_1 a_1(n, \epsilon n, \epsilon^2 n) &= -\epsilon(4\Delta_1 + 2)(\Delta_1\Delta_\epsilon + \Delta_\epsilon)a_1(n, \epsilon n, \epsilon^2 n) \\ -\epsilon^2\Delta_1^2 a_1(n, \epsilon n, \epsilon^2 n) + 2\epsilon^2\Delta_1 a_1(n, \epsilon n, \epsilon^2 n) - \epsilon^2 a_1(n, \epsilon n, \epsilon^2 n) \\ - (4\Delta_1(\Delta_1\Delta_{\epsilon^2} + \Delta_{\epsilon^2}) + 2(\Delta_1\Delta_\epsilon + \Delta_\epsilon)^2)a_0(n, \epsilon n, \epsilon^2 n) \\ - \epsilon(2\Delta_1 - 2)(\Delta_1\Delta_\epsilon + \Delta_\epsilon)a_0(n, \epsilon n, \epsilon^2 n) - 2(\Delta_1\Delta_{\epsilon^2} + \Delta_{\epsilon^2})a_0(n, \epsilon n, \epsilon^2 n), \end{aligned} \quad (87)$$

and so on. The $O(1)$ -problem (85) readily can be solved, yielding

$$a_0(n, \epsilon n, \epsilon^2 n) = f_0(\epsilon n, \epsilon^2 n), \quad (88)$$

where $f_0(\epsilon n, \epsilon^2 n)$ is still an arbitrary function which will be used to avoid secular terms in $a_1(n, \epsilon n, \epsilon^2 n)$ and in $a_2(n, \epsilon n, \epsilon^2 n)$. The $O(\epsilon)$ -problem (86) now becomes

$$2\epsilon\Delta_1^2 a_1(n, \epsilon n, \epsilon^2 n) + 2\epsilon\Delta_1 a_1(n, \epsilon n, \epsilon^2 n) = -2\Delta_\epsilon f_0(\epsilon n, \epsilon^2 n) - \epsilon f_0(\epsilon n, \epsilon^2 n). \quad (89)$$

To avoid secular terms in $a_1(n, \epsilon n, \epsilon^2 n)$ it is obvious from (89) that $f_0(\epsilon n, \epsilon^2 n)$ has to satisfy

$$-2\Delta_\epsilon f_0(\epsilon n, \epsilon^2 n) - \epsilon f_0(\epsilon n, \epsilon^2 n) = 0 \quad \Rightarrow \quad f_0(\epsilon n, \epsilon^2 n) = (1 - \frac{\epsilon}{2})^n g_0(\epsilon^2 n), \quad (90)$$

where $g_0(\epsilon^2 n)$ is still an arbitrary function which will be used to avoid secular terms in the $O(\epsilon^2)$ -problem (87). From (89) and (90) it now also follows that

$$a_1(n, \epsilon n, \epsilon^2 n) = f_1(\epsilon n, \epsilon^2 n), \quad (91)$$

where $f_1(\epsilon n, \epsilon^2 n)$ is still an arbitrary function which can be used to avoid secular terms in the higher order problems. The $O(\epsilon^2)$ -problem (87) now becomes (observe that $\Delta_1 a_0 = 0$ and $\Delta_1 a_1 = 0$):

$$\begin{aligned} 2\epsilon^2\Delta_1 a_2(n, \epsilon n, \epsilon^2 n) + 2\epsilon^2\Delta_1 a_2(n, \epsilon n, \epsilon^2 n) &= -2\epsilon\Delta_\epsilon f_1(\epsilon n, \epsilon^2 n) - \epsilon^2 f_1(\epsilon n, \epsilon^2 n) \\ -2\Delta_\epsilon^2 f_0(\epsilon n, \epsilon^2 n) - 2\Delta_{\epsilon^2} f_0(\epsilon n, \epsilon^2 n) + 2\epsilon\Delta_\epsilon f_0(\epsilon n, \epsilon^2 n) & \end{aligned} \quad (92)$$

Now it should be observed that all terms in the righthand side of (92) will lead to secular terms in $a_2(n, \epsilon n, \epsilon^2 n)$. To avoid these secular terms the righthand side of (92) should be equal to zero, or equivalently (by using (90)):

$$2\epsilon\Delta_\epsilon f_1(\epsilon n, \epsilon^2 n) + \epsilon^2 f_1(\epsilon n, \epsilon^2 n) = (1 - \frac{\epsilon}{2})^n (-2\Delta_{\epsilon^2} g_0(\epsilon^2 n) - \frac{3}{2}\epsilon^2 g_0(\epsilon^2 n)). \quad (93)$$

Now all terms in the righthand side of (93) will lead to secular terms in $f_1(\epsilon n, \epsilon^2 n)$. To avoid this secular behaviour it follows from (93) that $g_0(\epsilon^2 n)$ has to satisfy:

$$-2\Delta_{\epsilon^2} g_0(\epsilon^2 n) - \frac{3}{2}\epsilon^2 g_0(\epsilon^2 n) = 0 \quad \Rightarrow \quad g_0(\epsilon^2 n) = c_0 (1 - \frac{3}{4}\epsilon^2)^n, \quad (94)$$

where c_0 is a constant. From (93) it then also follows that

$$f_1(\epsilon n, \epsilon^2 n) = (1 - \frac{\epsilon}{2})^n g_1(\epsilon^2 n), \quad (95)$$

where $g_1(\epsilon^2 n)$ is an arbitrary function. And from (92) it then follows that

$$a_2(n, \epsilon n, \epsilon^2 n) = f_2(\epsilon n, \epsilon^2 n), \quad (96)$$

where $f_2(\epsilon n, \epsilon^2 n)$ is an arbitrary function. From $y_n = (\frac{1}{2})^n a_n$, (84), (88), (90)-(96) it follows that an approximation of y_n is given by

$$c_0 (\frac{1}{2})^n (1 - \frac{\epsilon}{2})^n (1 - \frac{3}{4}\epsilon^2)^n + O(\epsilon) (\frac{1}{2})^n. \quad (97)$$

In a completely similar way a second, functionally independent approximation of y_n can be constructed (starting with $y_n = \epsilon^n b_n$, and so on), yielding

$$d_0 \epsilon^n (1 + \epsilon^2)^n + \epsilon^n O(\epsilon), \quad (98)$$

where d_0 is a constant. The computations to obtain (98) are left to the reader as an exercise. From (97) and (98) it follows by using the superposition principle that an approximation $y(n, \epsilon n, \epsilon^2 n)$ of the general solution y_n of (56) is given by

$$y(n, \epsilon n, \epsilon^2 n) = c_0 \left(\frac{1}{2}\right)^n \left(1 - \frac{\epsilon}{2}\right)^n \left(1 - \frac{3}{4}\epsilon^2\right)^n + d_0 \epsilon^n (1 + \epsilon^2)^n. \quad (99)$$

How accurate this approximation is, can directly be seen by comparing $y(n, \epsilon n, \epsilon^2 n)$ with the exact solution y_n which is given by

$$\begin{aligned} y_n &= a \left(\frac{1+2\epsilon+(1-4\epsilon)^{\frac{1}{2}}}{2(\epsilon+2)} \right)^n + b \left(\frac{1+2\epsilon-(1-4\epsilon)^{\frac{1}{2}}}{2(\epsilon+2)} \right)^n \\ &= a \left(\frac{1}{2} - \frac{1}{4}\epsilon - \frac{3}{8}\epsilon^2 + O(\epsilon^3) \right)^n + b \left(\epsilon + \epsilon^3 + O(\epsilon^4) \right)^n. \end{aligned} \quad (100)$$

From (99) and (100) it can readily be deduced that for $n \sim \frac{1}{\epsilon^2}$

$$|y_n - y(n, \epsilon n, \epsilon^2 n)| = \left(\frac{1}{2}\right)^n O(\epsilon).$$

5.3 The $O\Delta E$ (57)

The rescaling and balancing procedure as introduced in section 5.1 can be repeated again for the $O\Delta E$ (57), and it turns out that only one significant scaling is present:

$$\delta_n(\epsilon) = (\sqrt{\epsilon})^n. \quad (101)$$

As in the previous examples approximations of the solution of the linear $O\Delta E$ (57) will be constructed. In this example a three scales perturbation method will be applied. By putting $z_n = (\sqrt{\epsilon})^n a_n$ the $O\Delta E$ (57) becomes

$$a_{n+2} + \sqrt{\epsilon} a_{n+1} + a_n = 0. \quad (102)$$

Then, the following expansion for a_n

$$a_n = a_0(n, \sqrt{\epsilon}n, \epsilon n) + \sqrt{\epsilon} a_1(n, \sqrt{\epsilon}n, \epsilon n) + \epsilon a_2(n, \sqrt{\epsilon}n, \epsilon n) + \dots \quad (103)$$

is substituted into (102), and as usual the following $O(\epsilon^{\frac{m}{2}})$ -problems are obtained for $m = 0, 1, 2, \dots$:

the $O(1)$ -problem:

$$\begin{aligned} \Delta_1^2 a_0(n, \sqrt{\epsilon}n, \epsilon n) + 2\Delta_1 a_0(n, \sqrt{\epsilon}n, \epsilon n) + 2a_0(n, \sqrt{\epsilon}n, \epsilon n) &= 0 \Leftrightarrow \\ a_0(n+2, \sqrt{\epsilon}n, \epsilon n) + a_0(n, \sqrt{\epsilon}n, \epsilon n) &= 0, \end{aligned} \quad (104)$$

the $O(\sqrt{\epsilon})$ -problem:

$$\begin{aligned} \sqrt{\epsilon} \Delta_1^2 a_1(n, \sqrt{\epsilon}n, \epsilon n) + 2\sqrt{\epsilon} \Delta_1 a_1(n, \sqrt{\epsilon}n, \epsilon n) + 2\sqrt{\epsilon} a_1(n, \sqrt{\epsilon}n, \epsilon n) &= \\ -2(\Delta_1 + I)(\Delta_1 \Delta_{\sqrt{\epsilon}} + \Delta_{\sqrt{\epsilon}}) a_0(n, \sqrt{\epsilon}n, \epsilon n) - \sqrt{\epsilon}(\Delta_1 + I) a_0(n, \sqrt{\epsilon}n, \epsilon n), \end{aligned} \quad (105)$$

the $O(\epsilon)$ -problem:

$$\begin{aligned} \epsilon \Delta_1^2 a_2(n, \sqrt{\epsilon}n, \epsilon n) + 2\epsilon \Delta_1 a_2(n, \sqrt{\epsilon}n, \epsilon n) + 2\epsilon a_2(n, \sqrt{\epsilon}n, \epsilon n) &= \\ -\sqrt{\epsilon}(2\Delta_{\sqrt{\epsilon}} E_1^2 + \sqrt{\epsilon} E_1) a_1(n, \sqrt{\epsilon}n, \epsilon n) & \\ -(2\Delta_3 E_1^2 + \Delta_{\sqrt{\epsilon}}^2 E_1^2 + \sqrt{\epsilon} \Delta_{\sqrt{\epsilon}} E_1) a_0(n, \sqrt{\epsilon}n, \epsilon n), \end{aligned} \quad (106)$$

and so on. The $O(1)$ -problem (104) can easily be solved, yielding

$$a_0(n, \sqrt{\epsilon}n, \epsilon n) = f_0(\sqrt{\epsilon}n, \epsilon n) \cos\left(\frac{n\pi}{2}\right) + g_0(\sqrt{\epsilon}n, \epsilon n) \sin\left(\frac{n\pi}{2}\right), \quad (107)$$

where $f_0(\sqrt{\epsilon n}, \epsilon n)$ and $g_0(\sqrt{\epsilon n}, \epsilon n)$ are still arbitrary functions which will be used to avoid secular terms in $a_1(n, \sqrt{\epsilon n}, \epsilon n)$ and in $a_2(n, \sqrt{\epsilon n}, \epsilon n)$. The $O(\sqrt{\epsilon})$ -problem (106) now becomes

$$\begin{aligned} & \sqrt{\epsilon} \Delta_1^2 a_1(n, \sqrt{\epsilon n}, \epsilon n) + 2\sqrt{\epsilon} \Delta_1 a_1(n, \sqrt{\epsilon n}, \epsilon n) + 2\sqrt{\epsilon} a_1(n, \sqrt{\epsilon n}, \epsilon n) = \\ & \cos\left(\frac{n\pi}{2}\right) (2\Delta_{\sqrt{\epsilon}} f_0(\sqrt{\epsilon n}, \epsilon n) - \sqrt{\epsilon} g_0(\sqrt{\epsilon n}, \epsilon n)) + \\ & \sin\left(\frac{n\pi}{2}\right) (2\Delta_{\sqrt{\epsilon}} g_0(\sqrt{\epsilon n}, \epsilon n) + \sqrt{\epsilon} f_0(\sqrt{\epsilon n}, \epsilon n)). \end{aligned} \quad (108)$$

Obviously the righthand side of (108) contains terms (i.e. $\cos(\frac{n\pi}{2})$ and $\sin(\frac{n\pi}{2})$) which are solutions of the corresponding homogeneous $O\Delta E$ and which consequently lead to secular terms in $a_1(n, \sqrt{\epsilon n}, \epsilon n)$. To avoid this secular behaviour in $a_1(n, \sqrt{\epsilon n}, \epsilon n)$ it follows from (108) that $f_0(\sqrt{\epsilon n}, \epsilon n)$ and $g_0(\sqrt{\epsilon n}, \epsilon n)$ have to satisfy

$$\begin{aligned} 2\Delta_{\sqrt{\epsilon}} f_0(\sqrt{\epsilon n}, \epsilon n) - \sqrt{\epsilon} g_0(\sqrt{\epsilon n}, \epsilon n) &= 0, \\ 2\Delta_{\sqrt{\epsilon}} g_0(\sqrt{\epsilon n}, \epsilon n) + \sqrt{\epsilon} f_0(\sqrt{\epsilon n}, \epsilon n) &= 0. \end{aligned} \quad (109)$$

System (109) can readily be solved, yielding

$$\begin{aligned} f_0(\sqrt{\epsilon n}, \epsilon n) &= \alpha_0(\epsilon n) \left(1 + \frac{\epsilon}{4}\right)^{\frac{n}{2}} \cos(n\mu(\epsilon)) + \beta_0(\epsilon n) \left(1 + \frac{\epsilon}{4}\right)^{\frac{n}{2}} \sin(n\mu(\epsilon)), \\ g_0(\sqrt{\epsilon n}, \epsilon n) &= -\alpha_0(\epsilon n) \left(1 + \frac{\epsilon}{4}\right)^{\frac{n}{2}} \sin(n\mu(\epsilon)) + \beta_0(\epsilon n) \left(1 + \frac{\epsilon}{4}\right)^{\frac{n}{2}} \cos(n\mu(\epsilon)), \end{aligned} \quad (110)$$

where $\alpha_0(\epsilon n)$ and $\beta_0(\epsilon n)$ are arbitrary functions which will be used to avoid secular terms in $a_2(n, \sqrt{\epsilon n}, \epsilon n)$, and where $\mu(\epsilon)$ is given by

$$\begin{aligned} \cos(\mu(\epsilon)) &= \left(1 + \frac{\epsilon}{4}\right)^{-\frac{1}{2}}, \text{ and } \sin(\mu(\epsilon)) = \frac{1}{2}\sqrt{\epsilon} \left(1 + \frac{\epsilon}{4}\right)^{-\frac{1}{2}}, \text{ and} \\ \mu(\epsilon) &= \frac{1}{2}\sqrt{\epsilon} - \frac{1}{24}\epsilon\sqrt{\epsilon} + O(\epsilon^2\sqrt{\epsilon}) \text{ for } \epsilon \rightarrow 0. \end{aligned} \quad (111)$$

Then, it follows from (108) that $a_1(n, \sqrt{\epsilon n}, \epsilon n)$ is given by

$$a_1(n, \sqrt{\epsilon n}, \epsilon n) = f_1(\sqrt{\epsilon n}, \epsilon n) \cos\left(\frac{n\pi}{2}\right) + g_1(\sqrt{\epsilon n}, \epsilon n) \sin\left(\frac{n\pi}{2}\right), \quad (112)$$

where $f_1(\sqrt{\epsilon n}, \epsilon n)$ and $g_1(\sqrt{\epsilon n}, \epsilon n)$ are arbitrary functions which can be used to avoid secular terms in the solutions of the higher order problems. By using (107), (110), and (112) it follows that the $O(\epsilon)$ -problem (106) now becomes

$$\begin{aligned} & \epsilon \Delta_1^2 a_2(n, \sqrt{\epsilon n}, \epsilon n) + 2\epsilon \Delta_1 a_2(n, \sqrt{\epsilon n}, \epsilon n) + 2\epsilon a_2(n, \sqrt{\epsilon n}, \epsilon n) = \\ & \cos\left(\frac{n\pi}{2}\right) \left\{ 2\sqrt{\epsilon} \Delta_{\sqrt{\epsilon}} f_1(\sqrt{\epsilon n}, \epsilon n) - \epsilon g_1(\sqrt{\epsilon n}, \epsilon n) + 2\Delta_{\epsilon} f_0(\sqrt{\epsilon n}, \epsilon n) + \frac{\epsilon}{4} f_0(\sqrt{\epsilon n}, \epsilon n) \right\} + \\ & \sin\left(\frac{n\pi}{2}\right) \left\{ 2\sqrt{\epsilon} \Delta_{\sqrt{\epsilon}} g_1(\sqrt{\epsilon n}, \epsilon n) + \epsilon f_1(\sqrt{\epsilon n}, \epsilon n) + 2\Delta_{\epsilon} g_0(\sqrt{\epsilon n}, \epsilon n) + \frac{\epsilon}{4} g_0(\sqrt{\epsilon n}, \epsilon n) \right\}. \end{aligned} \quad (113)$$

To avoid secular terms in $a_2(n, \sqrt{\epsilon n}, \epsilon n)$ it is obvious from (113) that f_1, g_1, f_0 , and g_0 have to satisfy

$$\begin{aligned} 2\sqrt{\epsilon} \Delta_{\sqrt{\epsilon}} f_1(\sqrt{\epsilon n}, \epsilon n) - \epsilon g_1(\sqrt{\epsilon n}, \epsilon n) &= -(2\Delta_{\epsilon} \alpha_0(\epsilon n) + \frac{\epsilon}{4} \alpha_0(\epsilon n)) \left(1 + \frac{\epsilon}{4}\right)^{\frac{n}{2}} \cos(n\mu(\epsilon)) \\ &\quad - (2\Delta_{\epsilon} \beta_0(\epsilon n) + \frac{\epsilon}{4} \beta_0(\epsilon n)) \left(1 + \frac{\epsilon}{4}\right)^{\frac{n}{2}} \sin(n\mu(\epsilon)), \\ 2\sqrt{\epsilon} \Delta_{\sqrt{\epsilon}} g_1(\sqrt{\epsilon n}, \epsilon n) + \epsilon g_1(\sqrt{\epsilon n}, \epsilon n) &= (2\Delta_{\epsilon} \alpha_0(\epsilon n) + \frac{\epsilon}{4} \alpha_0(\epsilon n)) \left(1 + \frac{\epsilon}{4}\right)^{\frac{n}{2}} \sin(n\mu(\epsilon)) \\ &\quad - (2\Delta_{\epsilon} \beta_0(\epsilon n) + \frac{\epsilon}{4} \beta_0(\epsilon n)) \left(1 + \frac{\epsilon}{4}\right)^{\frac{n}{2}} \cos(n\mu(\epsilon)). \end{aligned} \quad (114)$$

Since $\left(1 + \frac{\epsilon}{4}\right)^{\frac{n}{2}} \cos(n\mu(\epsilon))$, and $\left(1 + \frac{\epsilon}{4}\right)^{\frac{n}{2}} \sin(n\mu(\epsilon))$ are solutions of the homogeneous system (114) it is obvious that these terms lead to secular behaviour in $f_1(\sqrt{\epsilon n}, \epsilon n)$, and in $g_1(\sqrt{\epsilon n}, \epsilon n)$. To avoid this secular behaviour it follows from (114) that $\alpha_0(\epsilon n)$ and $\beta_0(\epsilon n)$ have to satisfy

$$\begin{aligned} 2\Delta_{\epsilon} \alpha_0(\epsilon n) + \frac{\epsilon}{4} \alpha_0(\epsilon n) &= 0, \\ 2\Delta_{\epsilon} \beta_0(\epsilon n) + \frac{\epsilon}{4} \beta_0(\epsilon n) &= 0. \end{aligned} \quad (115)$$

System (115) can readily be solved, yielding

$$\begin{aligned}\alpha_0(\epsilon n) &= k_0(1 - \frac{\epsilon}{8})^n, \\ \beta_0(\epsilon n) &= l_0(1 - \frac{\epsilon}{8})^n,\end{aligned}\tag{116}$$

where k_0 and l_0 are arbitrary constants. From (114) f_1 and g_1 can now be determined, and from (113) a_2 can be determined, yielding

$$\begin{aligned}f_1(\sqrt{\epsilon}n, \epsilon n) &= \alpha_1(\epsilon n)(1 + \frac{\epsilon}{4})^{\frac{n}{2}} \cos(n\mu(\epsilon)) + \beta_1(\epsilon n)(1 + \frac{\epsilon}{4})^{\frac{n}{2}} \sin(n\mu(\epsilon)), \\ g_1(\sqrt{\epsilon}n, \epsilon n) &= -\alpha_1(\epsilon n)(1 + \frac{\epsilon}{4})^{\frac{n}{2}} \sin(n\mu(\epsilon)) + \beta_1(\epsilon n)(1 + \frac{\epsilon}{4})^{\frac{n}{2}} \cos(n\mu(\epsilon)), \\ a_2(n, \sqrt{\epsilon}n, \epsilon n) &= f_2(\sqrt{\epsilon}n, \epsilon n) \cos(\frac{n\pi}{2}) + g_2(\sqrt{\epsilon}n, \epsilon n) \sin(\frac{n\pi}{2}),\end{aligned}$$

where α_1, β_1, f_2 , and g_2 are arbitrary functions. To determine these functions completely the $O(\epsilon\sqrt{\epsilon})$ -problem and the $O(\epsilon^2)$ -problem have to be solved. Since we are not interested in the higher order approximations at this moment we will take α_1, β_1, f_2 , and g_2 equal to constants (such that the initial conditions (if present) can be satisfied up to $O(\epsilon)$). From (107), (110), (111), and (116) it follows that $a_0(n, \sqrt{\epsilon}n, \epsilon n)$ is given by

$$\begin{aligned}a_0(n, \sqrt{\epsilon}n, \epsilon n) &= k_0(1 - \frac{\epsilon}{8})^n(1 + \frac{\epsilon}{4})^{\frac{n}{2}} \cos(\frac{n\pi}{2} + n\mu(\epsilon)) + \\ & l_0(1 - \frac{\epsilon}{8})^n(1 + \frac{\epsilon}{4})^{\frac{n}{2}} \sin(\frac{n\pi}{2} + n\mu(\epsilon)),\end{aligned}\tag{117}$$

where k_0 and l_0 are constants. The exact solution of the $O\Delta E$ (102) is given by

$$a_n = k 1^n \cos(n\theta(\epsilon)) + l 1^n \sin(n\theta(\epsilon)),\tag{118}$$

where k and l are constants, and where $\theta(\epsilon)$ is given by

$$\begin{aligned}\cos(\theta(\epsilon)) &= -\frac{1}{2}\sqrt{\epsilon}, \text{ and } \sin(\theta(\epsilon)) = (1 - \frac{\epsilon}{4})^{\frac{1}{2}}, \text{ and} \\ \theta(\epsilon) &= \frac{\pi}{2} + \frac{1}{2}\sqrt{\epsilon} + \frac{1}{48}\epsilon\sqrt{\epsilon} + O(\epsilon^2\sqrt{\epsilon}) \text{ for } \epsilon \rightarrow 0.\end{aligned}$$

From (117) and (118) it is not difficult to deduce that

$$|a_n - a_0(n, \sqrt{\epsilon}n, \epsilon n)| = O(\sqrt{\epsilon}) \quad \text{for } n = O(\frac{1}{\epsilon}).$$

The examples as presented in this section and in section 2 and 3 of this paper clearly prove that the multiple scales perturbation method gives accurate approximations on long iteration scales.

6 Conclusions and remarks

In this paper an improved version of the multiple scales perturbation method for $O\Delta Es$ has been presented and formulated completely in terms of difference equations. It has been shown how this improved method can be applied to regularly perturbed $O\Delta Es$ and to singularly perturbed, linear $O\Delta Es$. The relative and/or absolute errors in the constructed approximations of the solutions of the $O\Delta Es$ have been determined, and it has been shown that these approximations are valid on long iteration scales.

It is to be expected that the presented perturbation method also can be applied successfully to weakly perturbed partial difference equations, and to singularly perturbed, weakly nonlinear $O\Delta Es$. Of course, these extensions will be interesting subjects for future research. Finally, it should be remarked that the presented perturbation method also can be used in the numerical analysis of certain classes of regularly or singularly perturbed differential equations to see whether the solutions of the discretized equations (i.e. the difference equations) have the same type of behaviour as the solutions of the differential equations or not.

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