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Extension of Stochastic Volatility Models with Hull-White Interest Rate Process

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Abstract

In recent years the financial world has focused on accurate pricing of exotic and hybrid products that are based on a combination of underlyings from different asset classes. In this paper we present an extension of the stochastic volatility models by a stochastic Hull-White interest rate component. It is our goal to include this system of stochastic differential equations in the class of affine jump diffusion - linear quadratic jump-diffusion processes (Duffie, Pan and Singleton [11], Cheng and Scaillet [8]) so that the pricing of European products can be efficiently done within the pricing framework of Carr-Madan [7].

Key words: hybrid products, Schöbel-Zhu-Hull-White framework, stochastic volatility and interest rate model, affine jump-diffusion process

1 Introduction

In this paper we present a flexible multi-factor stochastic volatility (SV) model which includes the term structure of the stochastic interest rates (IR). Our aim is to combine an arbitrage-free Hull-White IR model in which the parameters are consistent with the bond prices implied by the zero coupon yield curve. In order to perform option valuation using the fast Fourier transform we aim to fit this process in the class of Affine Jump Diffusion (AJD) processes [11] (although jump processes are not included in this work). We specify under which conditions such a general model can fall in the class of AJD processes. We also apply the model to price some hybrid structured derivatives, which combine the different asset classes: equity and interest rate.

A major step, away from the assumption of constant volatility in derivatives pricing, was made by Hull and White [17], Stein and Stein [33] and Heston [16], who defined the volatility as a diffusion process. This improved the pricing of derivatives under heavy-tailed return distributions significantly and allowed a trader to quantify the uncertainty in the pricing. Since the development of the stochastic volatility models, they have become very popular for derivative pricing and hedging, see, for example, [13], however financial engineers have developed more complex exotic products, that require additionally the modeling of a stochastic interest rate component. A derivative pricing tool in which all these features are explicitly modeled has the potential of generating more accurate option prices for hybrid products. These products can be designed to provide capital or income protection, diversification for portfolios and customized solutions for both institutional and retail markets. Because of these features, the hybrid market expands rapidly.

The approach proposed by Carr and Madan in [7] for pricing European options with the Fast Fourier Transform (FFT) technique is state-of-the-art for parameter calibration in the financial industry. It relies heavily on the availability of the characteristic function of the price process, which is guaranteed if we stay within the AJD class, see Duffie-Pan-Singleton [11], Lee [20] and Lewis [21]. We examine the effect of correlated processes for asset, stochastic volatility and interest rate on the option prices through a comparison with, for example, the Heston model. In the evaluation of the empirical performance of the

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alternative option pricing models, we use both the relative difference and the implied Black-Scholes [4] (BS) volatility as measures of systematic error.

The plan of this paper is as follows: In Section 2 we perform the analysis for the Schöbel-Zhu-Hull-White stochastic volatility and interest rate model; In Subsection 2.1 we explain the general AJD framework and in Subsection 2.2 we show that the Hull-White process fits nicely into the framework. Subsection 2.3 is the heart of this paper, where we show that the hybrid model of interest admits a semi-closed form for the characteristic function. Subsection 2.4 discusses an alternative hybrid model, the Heston-Hull-White model. In Section 3, numerical experiments are performed with the Heston, Heston-Hull-White and the Schöbel-Zhu-Hull-White models. We start with model calibration in Subsection 3.1, after which a variety of hybrid products are priced in subsequent subsections. Section 4 concludes. The lengthy proofs of the lemmas are placed in the appendices.

2 Extension of stochastic volatility models

In this section we present a hybrid stochastic volatility model which includes a stochastic interest rate process. In particular, we add to the SV model the well-known Hull-White stochastic interest rate process [18], which is a generalization of the Vasiček model [34].

We consider a three-dimensional system of stochastic differential equations, of the following form:

$$\begin{aligned} dS_t &= r_t S_t dt + \sigma_t^p S_t dW_t^S && (1)\\ dr_t &= \lambda(\theta_t - r_t)dt + \eta dW_t^r && (2)\\ d\sigma_t &= -\kappa\left(\sigma_t - \overline{\sigma}\right)dt + \gamma\sigma_t^{1-p} dW_t^\sigma && (3)\end{aligned}$$

where p is an exponent, κ and λ control the speed of mean reversion, η represents the interest rate volatility, and $\gamma\sigma^{1-p}$ determines the variance of the σ_t process. Parameters $\overline{\sigma}$ and θ_t are the long-run mean of the volatility and the interest rate processes, respectively. W^i are correlated Wiener processes, also governed by an instantaneous covariance matrix,

$$\Sigma = \begin{bmatrix} dW_t^S dW_t^S & dW_t^S dW_t^\sigma & dW_t^S dW_t^r \\ dW_t^\sigma dW_t^S & dW_t^\sigma dW_t^\sigma & dW_t^\sigma dW_t^r \\ dW_t^r dW_t^S & dW_t^r dW_t^\sigma & dW_t^r dW_t^r \end{bmatrix} = \begin{bmatrix} 1 & \rho_{s,\sigma} & \rho_{s,r} \\ \rho_{\sigma,s} & 1 & \rho_{\sigma,r} \\ \rho_{r,s} & \rho_{r,\sigma} & 1 \end{bmatrix} dt. \quad (4)$$

If we keep r_t constant and $p = \frac{1}{2}$, we have the Heston model [16],

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{\sigma_t} S_t dW_t^S, \\ d\sigma_t &= -\kappa^H\left(\sigma_t - \overline{\sigma}^H\right)dt + \gamma^H\sqrt{\sigma_t} dW_t^\sigma. \end{aligned}$$

For $p = 1$ our model is, in fact, the generalized Stein-Stein [33] model, which is also called the Schöbel-Zhu [30] model,

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{v_t} S_t dW_t^S, \\ dv_t &= -2\kappa\left(v_t + \sigma_t\overline{\sigma} + \frac{\gamma^2}{2\kappa}\right)dt + 2\gamma\sqrt{v_t} dW_t^\sigma, \end{aligned}$$

in which the squared volatility, $v_t = \sigma_t^2$, represents the variance of the instantaneous stock return.

It was already indicated in [16] and [30] that the plain Schöbel-Zhu model is a particular case of the original Heston model. We can see that, if $\overline{\sigma} = 0$, the Schöbel-Zhu model equals the Heston model in which $\kappa^H = 2\kappa$, $\overline{\sigma}^H = \gamma^2/2\kappa$, and $\gamma^H = 2\gamma$. This relation gives a direct connection between their discounted characteristic functions (see [23]). Finally, if we set r_t constant, $p = 0$ in equations (1), (3) and zero correlations, the model collapses to the standard Black-Scholes model. In the following we will choose the parameters in the equations (1), (2) and (3) such that we deal with a Schöbel-Zhu-Hull-White model. In [14] and [8] it is was shown that the so-called linear-quadratic jump-diffusion (LQJD) models are equivalent to the AJD models with an augmented state vector.

2.1 Affine jump-diffusion processes

The AJD class refers to a fixed probability space $(\Omega, \mathcal{F}, P)$ and a Markovian n-dimensional affine process $\mathbf{X}_t$ in some space $D \subset \mathbb{R}^n$. It can be expressed in the following stochastic differential form:

$$d\mathbf{X}_t = \mu(\mathbf{X}_t)dt + \sigma(\mathbf{X}_t)d\mathbf{W}_t + d\mathbf{Z}_t,$$

where $\mathbf{W}_t$ is $\mathcal{F}_t$-standard Brownian motion in $\mathbb{R}^n$, $\mu(\mathbf{X_t}) : D \to \mathbb{R}^n$, $\sigma(\mathbf{X}_t) : D \to \mathbb{R}^{n\times n}$, $\mathbf{Z}_t$ is a pure jump process with a fixed probability distribution ν on $\mathbb{R}^n$ and arrival intensity $\{\lambda(\mathbf{X_t}) : t \geq 0\}$ for arbitrary $\lambda : D \to [0, \infty)$ with jump transform $\varsigma(c) = \int_{\mathbb{R}^n} e^{c\times z} d\nu(z)$ for any $\varsigma \in \mathbb{C}^n$. Moreover, for processes in the AJD class it is assumed that drift, volatility, jump intensities and interest rate components are of the affine form, i.e.

$$\begin{aligned}
\mu(\mathbf{X}_t) &= a_0 + a_1\mathbf{X}_t \text{ for any } (a_0, a_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n},\\
\lambda(\mathbf{X}_t) &= b_0 + b_1^T\mathbf{X}_t, \text{ for any } (b_0, b_1) \in \mathbb{R} \times \mathbb{R}^n,\\
\sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^T &= (c_0)_{ij} + (c_1)_{ij}^T\mathbf{X}_t, \text{ for arbitrary } (c_0, c_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n},\\
r(\mathbf{X}_t) &= r_0 + r_1^T\mathbf{X}_t, \text{ for } (r_0, r_1) \in \mathbb{R} \times \mathbb{R}^n.
\end{aligned}$$

Then for a state vector, $\mathbf{X}_t$, the discounted characteristic function (CF) is of the following form:

$$\phi(\mathbf{u}, \mathbf{X}_T, t, T) = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds + i\mathbf{u}^T\mathbf{X}_T}|\mathcal{F}_t\right) = e^{A(\mathbf{u},\tau)+\mathbf{B^T}(\mathbf{u},\tau)\mathbf{X}_t}$$

where the expectation is taken under the risk neutral measure, $\mathbb{Q}$. For a time lag, $\tau := T - t$, the coefficients $A(\mathbf{u}, \tau)$ and $\mathbf{B}^T(\mathbf{u}, \tau)$ have to satisfy certain complex-valued ordinary differential equations (ODEs) [11]:

$$\frac{d}{d\tau}A(\mathbf{u}, \tau) = -r_0 + \mathbf{B}^T a_0 + \frac{1}{2}\mathbf{B}^T c_0\mathbf{B} + b_1(\varsigma\mathbf{B} - 1) \tag{5}$$

$$\frac{d}{d\tau}\mathbf{B}(\mathbf{u}, \tau) = -r_1 + a_1^T\mathbf{B} + \frac{1}{2}\mathbf{B}^T c_1\mathbf{B} + b_0(\varsigma\mathbf{B} - 1). \tag{6}$$

The dimension of these ODEs corresponds to the dimension of the state vector, $\mathbf{X}_t$. Typically, multi-factor models provide a better fit to the observed market data than the one-factor models. However, as the dimension of SDE system increases, the ODEs to be solved to get the CF are increasingly complex. If an analytical solution to the ODEs cannot be obtained, one can apply well-known numerical ODE techniques. This may require substantial computational effort, which essentially makes the model useless for practical applications. Therefore, in this paper we will set up a model for which an analytic solution to most of the ODEs appearing can be obtained. We will not consider jumps in this paper, so the b_0- and b_1-parts in (5),(6) disappear.

For several years now the pricing of plain vanilla options is common practice in the Fourier domain. These solution methods rely on the availability of the CF of the logarithm of the stock price. Although originally based on the Gil-Palaez inversion formula [35], the popularity of numerical integration and Fourier transformation increased with the efficient pricing method by Carr and Madan [7], with which vanilla options for a whole range of strikes can be priced in one computation. In 1D, a damped version of the European call price with damping factor α, strike K, and maturity T can be expressed in the following form:

$$\Pi(t, T, K, \alpha) = \frac{e^{-\alpha\log K}}{\pi}\Re\left(\int_0^{+\infty} e^{-iu\log K}\psi_T(u)du\right),$$

with i the imaginary unit, $\Re(\cdot)$ denotes taking the real part of the expression in brackets, and

$$\psi_T(u) = \frac{\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}e^{\log(S_T)(1+\alpha+iu)}|\mathcal{F}_t\right)}{\alpha + \alpha^2 - u^2 + iu(2\alpha + 1)}.$$

Here the expectation under the risk neutral measure, $\mathbb{Q}$, can be recognized as a discounted CF, i.e.,

$$\phi((u - i(1 + \alpha)), S_T, t, T) \equiv \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}e^{(1+\alpha+iu)\log(S_T)}|\mathcal{F}_t\right).$$

The pricing algorithm assumes the existence of the CF of the log stock price, which is often available in analytic form (although the density is not known in closed form). The trapezoidal rule formula for the Fourier transform of the option price reads,

$$\Pi(t,T,k_j) \approx \frac{e^{-\alpha k_j}}{\pi} \Re\left\{ \Delta_u \left(\sum_{n=1}^{N} \omega_N^{(n-1)(j-1)} e^{i(n-1)\Delta_u b} \psi_T(u_n) - \frac{1}{2}\left(g(u_1)+g(u_N)\right) \right) \right\},$$

with the imposed condition, $\Delta_u \Delta_k = 2\pi/N$, and where $k_j = -b+\Delta_k(j-1)$, $j = 1\dots N$ and $b = N\Delta_k/2 \in \mathbb{R}$, the lower boundary of the log-strike domain. Further, $g(u) \equiv e^{-iuk}\psi_T(u)$, and $\omega_N = \exp(-\frac{2\pi i}{N})$ provided that $g(u_1) = g(0)$.

The expression can be easily computed with the help of the FFT. The availability of such a pricing formula is particularly useful in a calibration procedure, in which the parameters of the stochastic processes need to be approximated. In practice, option pricing models are calibrated to a large number of market observed call option prices. It is therefore desirable for such a parameter estimation procedure to be highly efficient. A (semi-)closed form for an option pricing formula is mandatory.

In fact, for the CF resulting from hybrid SDE system (1), (2), (3) the choice of damping parameter α is quite sensitive and requires numerical testing. There exist however alternative highly efficient pricing methods, also based on the availability of the CF, that do not require a damping parameter, like the CONV method [22], or the COS method [12] based on Fourier cosine expansions.

2.2 The Hull-White model

Here, as a start, we consider the Hull-White, single-factor, no-arbitrage yield curve model in which the short-term interest rate is driven by an extended Ornstein-Uhlenbeck (OU) mean reverting process,

$$dr_t = \lambda\left(\theta_t - r_t\right)dt + \eta dW_t^r \tag{7}$$

where $\theta_t > 0, t \in \mathbb{R}^+$ is a time-dependent drift term, included to fit the theoretical bond prices to the yield curve observed in the market. Parameter η determines the overall level of volatility and the reversion rate parameter λ determines the relative volatilities. A high value of λ causes short-term rate movements to damp out quickly, so that the long-term volatility is reduced.

In the first part of our analysis we present the derivation for the CF of the interest rate process. Integrating equation (7), we obtain, for $t \geq 0$,

$$r_t = r_0 e^{-\lambda t} + \lambda \int_0^t e^{-\lambda(t-s)}\theta_s ds + \eta \int_0^t e^{-\lambda(t-s)} dW_s^{\mathbb{Q}}.$$

It is easy to show that r_t is normally distributed with

$$\mathbb{E}^{\mathbb{Q}}\left(r_t \middle| \mathcal{F}_0\right) = r_0 e^{-\lambda t} + \int_0^t \lambda e^{-\lambda(t-s)}\theta_s ds,$$

and

$$Var^{\mathbb{Q}}\left(r_t \middle| \mathcal{F}_0\right) = \frac{\eta^2}{2\lambda}\left(1 - e^{-2\lambda(t-s)}\right).$$

It is known that

$$\lim_{t\to\infty} \mathbb{E}^{\mathbb{Q}}\left(r_t | \mathcal{F}_0\right) = \theta_t,$$

i.e., for large t the first moment of the process convergences to the time-dependent drift.

In order to simplify the derivations to follow we use the following proposition (see Arnold [3], Oksendal [27]).

Proposition 2.1 (Hull-White decomposition)**.** *The Hull-White stochastic interest rate process (7) can be decomposed into* $r_t = \widetilde{r}_t + \psi_t$*, where*

$$\psi_t = e^{-\lambda t} r_0 + \lambda \int_0^t e^{-\lambda(t-s)}\theta_s ds,$$

and

$$d\widetilde{r}_t = -\lambda \widetilde{r}_t dt + \eta dW_t^{\mathbb{Q}}, \textit{ with } \widetilde{r}_0 = 0.$$

Proof. We have $dr_t = d\widetilde{r}_t + d\psi_t$ with $d\psi_t = -\lambda\psi_t dt + \lambda\theta_t dt,\;\; \psi_0 = r_0$. Thus,

$$\begin{aligned} dr_t &= -\lambda\widetilde{r}_t dt + \eta dW_t^{\mathbb{Q}} - \lambda\psi_t dt + \lambda\theta_t dt \\ &= -\lambda(r_t - \psi_t)dt + \eta dW_t^{\mathbb{Q}} - \lambda\psi_t dt + \lambda\theta_t dt = \lambda(\theta_t - r_t)dt + \eta dW_t^{\mathbb{Q}} \end{aligned}$$

□

The advantage of this transformation is that the stochastic process $\widetilde{r}_t$ is now a basic OU mean reverting process, determined only by λ and η, independent of function ψ_t. It is easier to analyze than the original Hull and White model [17].

We investigate the discounted conditional characteristic function (CF) of spot interest rate r_t,

$$\begin{aligned} \phi_{HW}(u, r_t, t, T) &= \mathbb{E}_t^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds + iur_T}|\mathcal{F}_t\right) \\ &= \mathbb{E}_t^{\mathbb{Q}}\left(e^{-\int_t^T \psi_s ds + iu\psi_T} e^{-\int_t^T \widetilde{r}_s ds + iu\widetilde{r}_T}|\mathcal{F}_t\right) \\ &= e^{-\int_t^T \psi_s ds + iu\psi_T}\phi_{HW}(u, \widetilde{r}_t, t, T), \end{aligned}$$

and see that process $\widetilde{r}_t$ is affine. Hence according to [11] the discounted CF for the affine interest rate model for $u \in \mathbb{C}$ is of the following form:

$$\phi_{HW}(u, \widetilde{r}_t, \tau) = \mathbb{E}_t^{Q}\left(e^{-\int_t^T \widetilde{r}_s ds + iu\widetilde{r}_T}|\mathcal{F}_t\right) = e^{A(u,\tau)+B(u,\tau)\widetilde{r}_t}, \tag{8}$$

with $\tau = T - t$. The necessary boundary condition accompanying (8) is $\phi_{HW}(u, \widetilde{r}_t, 0) = e^{iu\widetilde{r}_t}$, so that, $A(u, 0) = 0$ and $B(u, 0) = iu$. The solutions for $A(u, \tau)$ and $B(u, \tau)$ are provided by the following lemma:

Lemma 2.2 (Coefficients for discounted CF for the Hull-White model)**.** *The functions $A(u, \tau)$ and $B(u, \tau)$ in (8) are given by:*

$$\begin{aligned} A(u, \tau) &= \frac{\eta^2}{2\lambda^3}\left(\lambda\tau - 2\left(1 - e^{-\lambda\tau}\right) + \frac{1}{2}\left(1 - e^{-2\lambda\tau}\right)\right) - iu\frac{\eta^2}{2\lambda^2}\left(1 - e^{-\lambda\tau}\right)^2 \\ &\quad -\frac{1}{2}u^2\frac{\eta^2}{2\lambda}\left(1 - e^{-2\lambda\tau}\right), \\ B(u, \tau) &= iue^{-\lambda\tau} - \frac{1}{\lambda}\left(1 - e^{-\lambda\tau}\right). \end{aligned}$$

Proof. The proof can be found in Appendix A.1. □

By simply taking $u = 0$, we obtain the risk-free pricing formula for a zero coupon bond $P(t, T)$:

$$\phi_{HW}(0, r_t, \tau) = \mathbb{E}_t^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds} \cdot 1|\mathcal{F}_t\right) = P(t, T) = e^{-\int_t^T \psi_s ds + A(0,\tau) + B(0,\tau)\widetilde{r}_t}.$$

Moreover, we see that a zero coupon bond can be written as the product of a deterministic factor and the bond price in an ordinary Vasiček model with zero mean, under the risk neutral measure $\mathbb{Q}$. We recall that process $\widetilde{r}_t$ at time $t = 0$ is equal to 0, so

$$P(0, T) = \exp\left(-\int_0^T \psi_s ds + A(0, T)\right),$$

which gives

$$\psi_T = -\frac{\partial}{\partial T}\log P(0, T) + \frac{\partial}{\partial T}A(0, T) = f(0, T) + \frac{\eta^2}{2\lambda^2}\left(1 - e^{-\lambda T}\right)^2,$$

where $f(t, T)$ is an instantaneous forward rate.

This result shows that ψ_T can be obtained from the initial forward curve, $f(0, T)$. The other time-invariant parameters, λ and η, have to be estimated using market prices of, in particular, interest rate caps. Now from Proposition 2.1 we have $\theta_t = \frac{1}{\lambda}\frac{\partial}{\partial t}\psi_t + \psi_t$ which reads,

$$\theta_t = f(0, t) + \frac{1}{\lambda}\frac{\partial}{\partial t}f(0, t) + \frac{\eta^2}{2\lambda^2}\left(1 - e^{-2\lambda t}\right).$$

Moreover, the CF, $\phi_{HW}(u, r_t, \tau)$, for the Hull-White model can be simply obtained by integration of ψ_s over the interval $[t, T]$.

2.3 Schöbel-Zhu-Hull-White hybrid model

In this section we derive an analytic pricing formula in (semi-)closed form for European call options under the Schöbel-Zhu-Hull-White (SZHW) asset pricing model with a full matrix of correlations, defined by (4). For the state vector $\mathbf{X}_t = [S_t, r_t, \sigma_t]^T$ let us fix a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathcal{F}_n = \{\mathcal{F}_t : t \geq 0\}$, which satisfies the usual conditions. Furthermore, $\mathbf{X}_t$ is assumed to be Markovian relative to $\mathcal{F}_t$. The Schöbel-Zhu-Hull-White hybrid model can be expressed by the following 3D system of SDEs

$$\begin{cases} dS_t &=& r_t S_t dt + \sigma_t S_t dW_t^S, \\ dr_t &=& \lambda\left(\theta_t - r_t\right)dt + \eta dW_t^r, \\ d\sigma_t &=& -\kappa(\sigma_t - \overline{\sigma})dt + \gamma dW_t^\sigma, \end{cases} \tag{9}$$

with the parameters as in equations (1), (2), (3), and $p = 1$.

By extending the space vector (as in [8]) with another stochastic process, defined by $v_t := \sigma_t^2$, and choosing $x_t = \log S_t$, we obtain the following 4D system of SDEs,

$$\begin{cases} dx_t &=& \left(\widetilde{r}_t + \psi_t - \frac{1}{2}v_t\right)dt + \sigma_t dW_t^S \\ d\widetilde{r}_t &=& -\lambda \widetilde{r}_t dt + \eta dW_t^r \\ dv_t &=& \left(-2v_t\kappa + 2\kappa\sigma_t\overline{\sigma} + \gamma^2\right)dt + 2\sigma_t\gamma dW_t^\sigma \\ d\sigma_t &=& -\kappa(\sigma_t - \overline{\sigma})dt + \gamma dW_t^\sigma, \end{cases} \tag{10}$$

where we also used $r_t = \widetilde{r}_t + \psi_t$, as in Subsection 2.2. Note that θ_t is now included in ψ_t. We see that model (10) is indeed affine in the state vector $\mathbf{X}_t = [x_t, \widetilde{r}_t, v_t, \sigma_t]^T$. By the extension of the vector space we have obtained an affine model which enables us to apply the results from [11]. In order to simplify the calculations, we introduce a variable $x_t := \widetilde{x}_t + \Psi_t$ where $\Psi_t = \int_0^t \psi_s ds$ and

$$d\widetilde{x}_t = (\widetilde{r}_t - \frac{1}{2}v_t)dt + \sigma_t dW_t^S.$$

According [11] the discounted CF for $\mathbf{u} \in \mathbb{C}^4$ is of the following form,

$$\begin{aligned} \phi_{SZHW}(\mathbf{u}, \mathbf{X}_t, t, T) &= \mathbb{E}_t^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds} e^{i\mathbf{u}^T \mathbf{X}_T} | \mathcal{F}_t\right) && (11) \\ &= e^{-\int_t^T \psi_s ds + i\mathbf{u}^T[\Psi_T, \psi_T, 0, 0]^T} \mathbb{E}_t^{\mathbb{Q}}\left(e^{-\int_t^T \widetilde{r}_s ds + i\mathbf{u}^T \mathbf{X}_T^*} | \mathcal{F}_t\right) && (12) \\ &= e^{-\int_t^T \psi_s ds + i\mathbf{u}^T[\Psi_T, \psi_T, 0, 0]^T} e^{A(\mathbf{u},\tau) + \mathbf{B}^T(\mathbf{u},\tau)\mathbf{X}_t^*} && (13) \end{aligned}$$

where $\mathbf{X}_t^* = [\widetilde{x}_t, \widetilde{r}_t, v_t, \sigma_t]^T$ and $\mathbf{B}(\mathbf{u}, \tau) = [B_x(\mathbf{u}, \tau), B_r(\mathbf{u}, \tau), B_v(\mathbf{u}, \tau), B_\sigma(\mathbf{u}, \tau)]^T$. Now we set $\mathbf{u} = [u, 0, 0, 0]^T$, so that at time T we obtain the obvious boundary condition:

$$\phi_{SZHW}(\mathbf{u}, \mathbf{X}_T^*, T, T) = \mathbb{E}_T^{\mathbb{Q}}\left(e^{i\mathbf{u}^T \mathbf{X}^*(T)} | \mathcal{F}_T\right) = e^{i\mathbf{u}^T \mathbf{X}_T^*} = e^{iu\widetilde{x}_T},$$

(as the price at time T is known deterministically). This boundary condition for $\tau = 0$ gives $B_x(u, 0) = iu$, $A(u, 0) = 0$, $B_r(u, 0) = 0$, $B_\sigma(u, 0) = 0$, $B_v(u, 0) = 0$. The following lemmas define the ODEs, from (5) and (6), and detail their solution.

Lemma 2.3 (Schöbel-Zhu-Hull-White ODEs)**.** *The functions $A(u, \tau)$, $B_x(u, \tau)$, $B_\sigma(u, \tau)$, $B_v(u, \tau)$, $B_r(u, \tau)$, $u \in \mathbb{R}$, in (13) satisfy the following system of ODEs:*

$$\begin{aligned} \frac{d}{d\tau}B_x &= 0, && (14) \\ \frac{d}{d\tau}B_r &= 1 + B_x - \lambda B_r, && (15) \\ \frac{d}{d\tau}B_v &= -\frac{1}{2}B_x - 2\kappa B_v + \frac{1}{2}\left(B_x^2 + 4\gamma\rho_{x,v}B_x B_v + 4\gamma^2 B_v^2\right), && (16) \\ \frac{d}{d\tau}B_\sigma &= 2\kappa\overline{\sigma}B_v - \kappa B_\sigma + \frac{1}{2}\left(2\eta\rho_{x,r}B_x B_r + 2\gamma\rho_{x,\sigma}B_x B_\sigma + 4\eta\gamma\rho_{r,v}B_r B_v + 4\gamma^2 B_v B_\sigma\right), && (17) \\ \frac{d}{d\tau}A &= B_v\gamma^2 + B_\sigma\kappa\overline{\sigma} + \frac{1}{2}B_r^2\eta^2 + \frac{1}{2}B_\sigma^2\gamma^2 + B_\sigma B_r\eta\gamma\rho_{r,\sigma}. && (18) \end{aligned}$$

Proof. The proof can be found in Appendix A.2. □

Lemma 2.4. *The solution to the system of ODEs, specified by (14), (15), (16), (17) and (18) is given by:*

$$\begin{aligned} B_x(u,\tau) &= iu, \\ B_r(u,\tau) &= (1+iu)\lambda^{-1}\left(1-e^{-\lambda\tau}\right), \\ B_v(u,\tau) &= \frac{\beta-D}{2\theta}\left(\frac{1-e^{-\tau D}}{1-e^{-\tau D}G}\right), \end{aligned}$$

where $\beta=(\kappa-\rho_{x,v}\gamma ui)$, $D=\sqrt{\beta^2-4\alpha\gamma}$, $\theta=2\gamma^2$ *with* $\alpha=-\frac{1}{2}u(i+u)$, *and* $G=\frac{\beta-D}{\beta+D}$. *Furthermore,*

$$B_\sigma(u,\tau) = \left(\frac{e^{\frac{D}{2}\tau}}{e^{\tau D}-G}\right)\left(\frac{16\kappa\bar{\sigma}b\sinh^2\left(\frac{\tau D}{4}\right)}{D}+\frac{\eta\rho_{x,r}iu(1+iu)}{\lambda}F_1(u,\tau)+\frac{2\eta\gamma\rho_{r,v}(1+iu)b}{\lambda}F_2(u,\tau)\right),$$

with

$$\begin{aligned} F_1(u,\tau) &= \frac{2}{D}(e^{\frac{\tau D}{2}}-1)+\frac{2G}{D}(e^{-\frac{\tau D}{2}}-1)-\frac{2\left(e^{\frac{\tau}{2}(D-2\lambda)}-1\right)}{D-2\lambda}+\frac{2G\left(1-e^{-\frac{\tau}{2}(D+2\lambda)}\right)}{D+2\lambda}, \\ F_2(u,\tau) &= -\frac{4}{D}+\frac{2}{D-2\lambda}+\frac{2}{D+2\lambda}+\left(e^{-\frac{1}{2}\tau(D+2\lambda)}\right)\left(\frac{2e^{\tau\lambda}(1+e^{D\tau})}{D}-\frac{2e^{D\tau}}{D-2\lambda}-\frac{2}{D+2\lambda}\right), \end{aligned}$$

and

$$A(u,\tau) = \frac{(\beta-D)s-2\log\left(\frac{Ge^{-Ds}-1}{G-1}\right)}{4\gamma^2}-\frac{(iu+1)^2(3+e^{-2\tau\lambda}-4e^{-\tau\lambda}-2\tau\lambda)}{2\lambda^3}+F_3(u,\tau),$$

with

$$F_3(u,\tau) = \int_0^\tau B_\sigma(u,s)\left(\kappa\bar{\sigma}+\frac{1}{2}\gamma^2B_\sigma(u,s)+\eta\rho_{r,\sigma}\gamma B_r(u,s)\right)ds. \tag{19}$$

Proof. The proof is presented in Appendix A.3.

A closed form expression for the CF of the SZHW model with zero correlation between the equity and interest rate and between the interest rate and volatility processes is presented in Appendix A.4. □

Now, since we have found expressions for the coefficients $A(u,\tau)$ and $\mathbf{B}^T(u,\tau)$ we return to equation (11) and derive a representation in which the term structure is included. It is known that the price of a zero coupon bond can be obtained from the characteristic function by taking $\mathbf{u}=[0,0,0,0]^T$. So,

$$\begin{aligned} \phi_{SZHW}(0,\mathbf{X}_t,\tau) &= \exp\left(-\int_t^T\psi_s ds\right)\phi_{SZHW}(0,\mathbf{X}_t^*,\tau) \\ &= \exp\left(-\int_t^T\psi_s ds\right)\exp\left(A(0,\tau)+B_x(0,\tau)x_t+B_r(0,\tau)\widetilde{r}_t+B_v(0,\tau)v_t+B_\sigma(0,\tau)\sigma_t\right). \end{aligned}$$

Since $\widetilde{r}_0=0$ we find,

$$P(0,T)=\exp\left(-\int_0^T\psi_s ds\right)\exp\left(A(0,\tau)+B_x(0,\tau)x_0+B_v(0,\tau)v_0+B_\sigma(0,\tau)\sigma_0\right),$$

with boundary conditions $B_x(0,T)=0, B_v(0,T)=0, B_\sigma(0,T)=0$ and

$$A(0,T)=\frac{1}{2}\eta^2\int_0^T B_r(0,s)^2ds=\frac{\eta^2}{2\lambda^3}\left(\frac{1}{2}+T\lambda+2e^{-\lambda T}-\frac{1}{2}e^{-2\lambda T}\right).$$

We thus find,

$$P(0,T)=\exp\left(-\int_0^T\psi_s ds+A(0,T)\right).$$

By combining the results from the previous lemmas, we can prove the following lemma.

Lemma 2.5. *In the Schöbel-Zhu-Hull-White model, the discounted characteristic function,* $\phi_{SZHW}(u, \mathbf{X}_t, t, T)$ *for* $\log S_T$*, is given by*

$$\phi_{SZHW}(u, \mathbf{X}_t, t, T) = \exp\left(\widetilde{A}(u,\tau) + B_x(u,\tau)x_t + B_r(u,\tau)r_t + B_v(u,\tau)v_t + B_\sigma(u,\tau)\sigma_t\right),$$

where B_x*,* B_r*,* B_v*,* B_σ *are given in Lemma 2.4, and*

$$\widetilde{A}(u,\tau) = -\int_t^T \psi_s ds + iu\int_t^T \psi_s ds + A(u,\tau) = \Theta(u,\tau) + A(u,\tau), \tag{20}$$

with

$$\Theta(u,\tau) = (1-iu)\log\left(\frac{P(0,T)}{P(0,t)}\right) + (1-iu)\frac{\eta^2}{2\lambda^2}\left((T-t) + \frac{2}{\lambda}\left(e^{-\lambda T} - e^{-\lambda t}\right) - \frac{1}{2\lambda}\left(e^{-2\lambda T} - e^{-2\lambda t}\right)\right). \tag{21}$$

Proof. The proof is straightforward from the definition of the discounted CF. □

2.3.1 Numerical Test

Lemma 2.4 shows that most of the terms for the SZHW process can be obtained analytically, except the F_3-term (19), which requires numerical integration of the hyper-geometric function ${}_2F_1$ [25]. There are several ways to solve this integral. The simplest being the application of a basic integration routine. In Figure 1 (left-side picture) the numerical solutions obtained with the composite trapezoidal rule for $N = 100$ and $N = 1000$ points are compared. The parameters chosen are:

$$S_0 = 1, \lambda = 0.04, \kappa = 0.5, \eta = 0.04, \gamma = 0.01, \rho_{x,v} = -0.1, \rho_{x,r} = -0.2, \rho_{r,v} = 0.4, r_0 = 0.05, v_0 = 0.05. \tag{22}$$

It is shown that the use of approximately 100 integration points gives a fully satisfactory accuracy for our purposes. Figure 1 (right-side picture) shows that for higher values of τ in (19) the error increases. However the relative error remains acceptable.

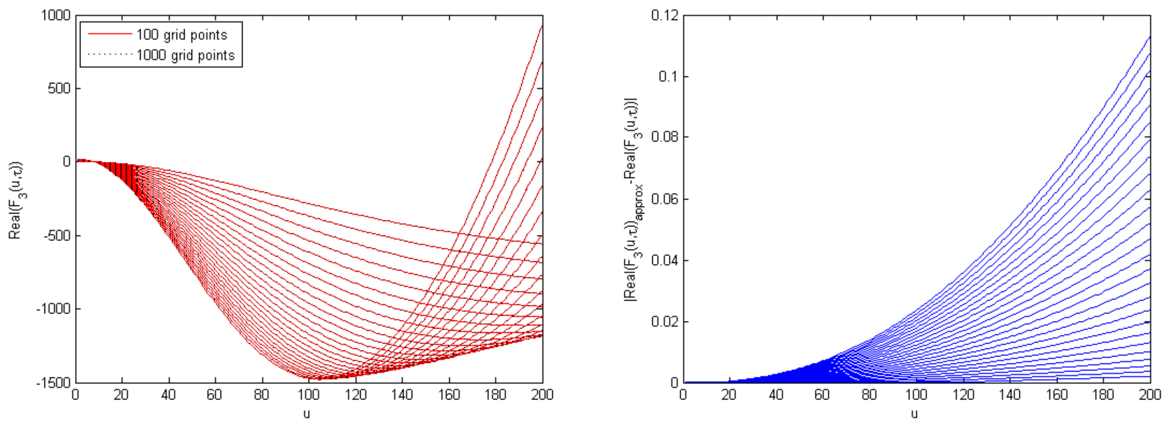

Figure 1: Left: Numerical solution of $\Re(F_3(u,\tau))$ with the trapezoidal rule, comparing $N = 100$ with $N = 1000$ points. The lines are varying with respect to parameter τ in (19) $\in \{1,\ldots,30\}$, other parameters as in (22). Right: The absolute difference between $\Re(F_3(u,\tau))$ and the "exact" solution.

Furthermore, for the same parameter set and fixing $\tau = 20$, Figure 2 compares for a different number of integration points, N, the convergence of the real part of $F_3(u,\tau)$ to the "exact" solution (with 2^{20} points). It is shown that 32 points already give an accurate numerical approximation. Table 1 compares, for $\tau = 10$, the accuracy for different values of N when evaluating integral (19) for a fixed u-vector consisting of 200 points, with the CPU time needed to compute the integral included. So, for each

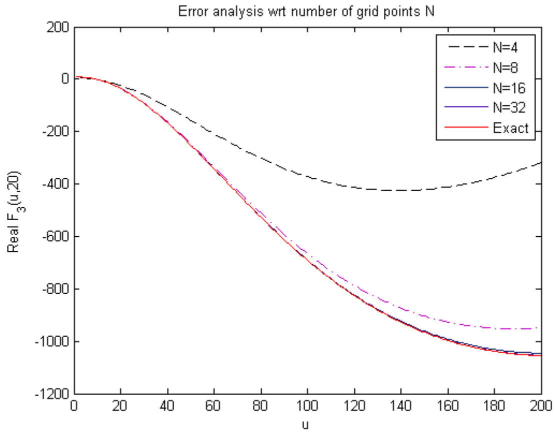

Figure 2: The influence of the number of integration points on the real part of the integral $F_3(u,\tau)$ for $\tau = 20$ years.

element of u the numerical integration is performed. The squared sum error (SSE) in the Table 1 is defined as follows

$$SSE = \sum_u \left\{ \Re\left(F_3(u,\tau)\right) - \Re\left(\widetilde{F}_3(u,\tau)\right)\right\}^2,$$

where $F_3(u,\tau)$ represents the solution obtained with $N = 2^{20}$ grid points, and $\widetilde{F}_3(u,\tau)$ is the approximation. A linear computational complexity is observed for the numerical integration, as expected.

Table 1: Trapezoidal integration errors of $F_3(u,\tau)$ with $\tau = 10$ for a whole set of u-values, consisting of 200 points.

n $(N = 2^n)$	4	5	6	7	8	9
SSE	2.2×10^{1}	7.8×10^{-1}	4.1×10^{-2}	2.5×10^{-3}	1.5×10^{-4}	9.2×10^{-6}
CPU time $[s]$	0.07	0.15	0.25	0.40	0.66	1.12

2.4 Heston-Hull-White hybrid model

It was stated, see, for example, [26], that it is not possible to formulate a so-called Heston-Hull-White hybrid process, which belongs to the AJD class, with a full matrix of correlations. For this, restrictions regarding the parameters or the correlation structure have to be introduced. One possible restriction is to assume that the interest rate process, r_t, evolves independently of the stock price, S_t, and the volatility process, σ_t, while the other correlation is not equal to zero, i.e., $dW_t^S dW_t^r = 0$, $dW_t^\sigma dW_t^r = 0$ and $dW_t^S dW_t^\sigma = \rho dt$. A second option is to solve the problem under the assumption that $dW_t^\sigma dW_t^r = 0$ and additionally that $\gamma^2/4 = \kappa\overline{\sigma}$ [26]. It may, however, be difficult to apply this latter model in practice, as the economical meaning of the parameter relationship is difficult to interpret. We therefore compare the SZHW model with a HHW model in which two of the correlations are set to zero.

Lemma 2.6 (HHW model with zero correlation)**.** *The Heston-Hull-White model defined by (1) with $p = \frac{1}{2}$, interest rate process (2), stochastic volatility (3), and correlations defined by (4) with $\rho_{r,s} = 0$ and $\rho_{r,\sigma} = 0$ has the following discounted characteristic function:*

$$\phi_{HHW}(u, \mathbf{X}_t, t, T) = \phi_H(u, \mathbf{X}_t, t, T) \exp\left(B_r(u,\tau) r_t + A_{HW}(u,\tau)\right),$$

where

$$\begin{aligned}
\phi_H(u,\mathbf{X}_t,t,T) &= \exp\left(A_H(u,\tau)+iux_t+B_\sigma(u,\tau)\sigma_t\right), && (23)\\
A_H(u,\tau) &= \frac{\kappa\theta}{\gamma^2}\left((\beta-D)\tau-2\log\left(\frac{1-Ge^{-D\tau}}{1-G}\right)\right), && (24)\\
B_\sigma(u,\tau) &= \frac{\beta-D}{\gamma^2}\left(\frac{1-e^{-\tau D}}{1-Ge^{-\tau D}}\right), && (25)\\
B_r(u,\tau) &= (1+iu)\lambda^{-1}\left(1-e^{-\lambda\tau}\right), && (26)\\
A_{HW}(u,\tau) &= -\frac{1}{2}\left(\frac{\eta^2\tau}{\lambda^2}(1+iu)-\eta^2 B_r(u,\tau)\right)+\Theta(u,\tau), && (27)
\end{aligned}$$

with $G=\frac{\beta-D}{\beta+D}$, $\beta=(\kappa-\rho_{x,\sigma}\gamma ui)$, $D=\sqrt{\beta^2-4\alpha\gamma}$, $\alpha=-\frac{1}{2}u(i+u)$, *and* $\Theta(u,\tau)$ *defined in (21).*

Proof. The proof is analogous to the proof of Lemma 2.4. □

The next lemma describes an analytic relation between the Heston model and the Schöbel-Zhu-Hull-White hybrid model.

Lemma 2.7. *The relation between the Schöbel-Zhu-Hull-White model as defined in (9) and the plain Heston Model can be expressed as follows,*

$$\begin{aligned}
\phi_{SZHW}(u,\mathbf{X}_0,\tau) &= \exp\left(B_\sigma(u,\tau)\sigma_0+B_r(u,\tau)r_0+\widetilde{A}(u,\tau)-\frac{(\beta-D)\tau-2\log\left(\frac{Ge^{-D\tau}-1}{G-1}\right)}{4\gamma^2}\right)G(u,\tau)\\
&= \phi_H(u,[x_0,\sigma_0^2]^T,\kappa^H,\gamma^H,\overline{\sigma}^H,\tau)G(u,\tau),
\end{aligned}$$

with $\widetilde{A}$ *from (20),* B_σ, B_r *as in Lemma 2.4, and* $\kappa^H=2\kappa$, $\gamma^H=2\gamma$, $\overline{\sigma}^H=\gamma^2/2\kappa$ *and*

$$G(u,\tau)=\exp\left(B_x(u,\tau)x_0+B_v(u,\tau)v_0+\frac{(\beta-D)\tau-2\log\left(\frac{Ge^{-D\tau}-1}{G-1}\right)}{4\gamma^2}\right),$$

with B_x *and* B_v *from Lemma 2.4.*

Proof. The proof is straightforward by the definition of the characteristic functions. □

3 Calibration and pricing under the hybrid model

For exotic financial products that involve more than one asset class, the pricing engine would preferably be based on a stochastic model which takes into account the interactions between the asset classes. It is interesting to evaluate price differences between the classical models and the SZHW hybrid model presented here. For this purpose we have introduced several hybrid products, treated in subsequent subsections. The pricing is done using a Monte Carlo method.

Before we can price these products, however, we need to calibrate the models, i.e., to find the model parameters so that the models recover the market prices of plain vanilla options well. This calibration procedure relies heavily on the characteristic function derived in the previous section and the appendices.

Remark (Monte Carlo simulation and negative variance)*.* For the pricing of financial derivatives, Monte Carlo methods are commonly used tools, especially for products like hybrid derivatives for which a closed-form pricing formula is not available.

Because of discretization techniques like the Euler-Maruyama or Milstein schemes (see, for example, [31]) a Monte Carlo technique may sometimes give a negative or imaginary variance in the SV models. This is not acceptable. The basic Euler-Maruyama discretization of the general SV model (3) for a given time step Δ_t reads,

$$\sigma_{t+\Delta_t}=(1-\kappa\Delta_t)\sigma_t+\kappa\overline{\sigma}\Delta_t+\gamma\sigma_t^{1-p}\sqrt{\Delta_t}Z.$$

with Z being a standard normal random variable with cumulative distribution function Φ. For a given $\sigma_t > 0$, the probability of $\sigma_{t+\Delta_t}$ being negative equals to

$$Prob\left(\sigma_{t+\Delta_t} < 0\right) = \Phi\left(\frac{(\kappa\Delta_t - 1)\sigma_t - \kappa\overline{\sigma}\Delta_t}{\gamma\sigma_t^{1-p}\sqrt{\Delta_t}}\right). \tag{28}$$

Formula (28) shows that for $\Delta_t \to 0$ this probability goes to zero, but even a small time step may cause the variance to become negative. In the literature, improved techniques to perform a simulation of the AJD processes have been developed, see [6], [2]. An analysis of the possible ways to overcome the negative variance problem can be found in [24]. We have chosen the so-called absorption scheme, from [24], where at each iteration step $\max(\sigma_{t+\Delta_t}, 0)$ is taken.

3.1 Calibration of the models

In this section we examine the extended stochastic volatility models and compare their performance to the Heston model. We use financial market data to estimate the model parameters and discuss the effect of different models on the implied market volatility. For this purpose we have chosen the CAC40 call option implied volatilities of 17.10.2007. Implied volatilities were calculated from settlement prices using BS option pricing formula for the Heston model, the HHW model (as in Lemma 2.6) and the SZHW model.

We perform the calibration of the models in two stages. Firstly, we calibrate the parameters for the interest rate process by using caplets and swaptions. Secondly, the remaining parameters, for the underlying asset, the volatility and the correlations, are calibrated to the plain vanilla option market prices. All the models are calibrated with an initial variance $v_0 = 0.02702$ obtained from the calibration of the Heston model. Tables 2 and 3 present the estimated parameters and associated squared sum errors (SSE) defines as,

$$SSE = \sum_{i=1}^{n}\sum_{j=1}^{m}\left(C\left(T_i, K_j\right) - \hat{C}\left(T_i, T_j\right)\right)^2,$$

where $C(T_i, K_i)$ and $\hat{C}\left(T_i, T_j\right)$ are the market and the model prices, respectively, T_i is the ith time to maturity and K_j is the jth strike.

The tables show that all the models are calibrated reasonably well. The calibration error is the largest for the HHW model. We cannot observe any similarities in the parameters obtained for the different models.

Table 2: Parameters estimated from the market data (model of Hull-White), r_0 is assumed to be the earliest forward rate.

parameters	r_0	λ	η	SSE
Hull-White	0.01733	1.12	0.001	0.001

Table 3: Parameters in the stochastic volatility models (1), (2) and (3), estimated from the market data.

parameters	r	κ	$\overline{\sigma}$	γ	$\rho_{x,r}$	$\rho_{x,\sigma}$	$\rho_{r,\sigma}$	SSE
Heston	0.0327	0.0125	0.5999	0.1248	-	-0.6461	-	1.2×10^{-6}
Heston-H-W	-	0.7463	0.0311	0.0811	0	-0.1247	0	5.0×10^{-2}
Schöbel-Zhu-H-W	-	0.0063	0.6446	0.0124	-0.3864	-0.6714	0.177	9.0×10^{-4}

Figure 3 indicates how well the different models replicate the market smiles and skews. The pattern of implied volatilities shows that for short maturities the fit of the SZHW model outperforms the Heston model, however, for long maturities the latter one gives a slightly better fit. The results also show that the HHW model gives a higher implied volatility for short maturities and high strikes, whereas it is lower for long maturities than the implied volatilities generated by the other models.

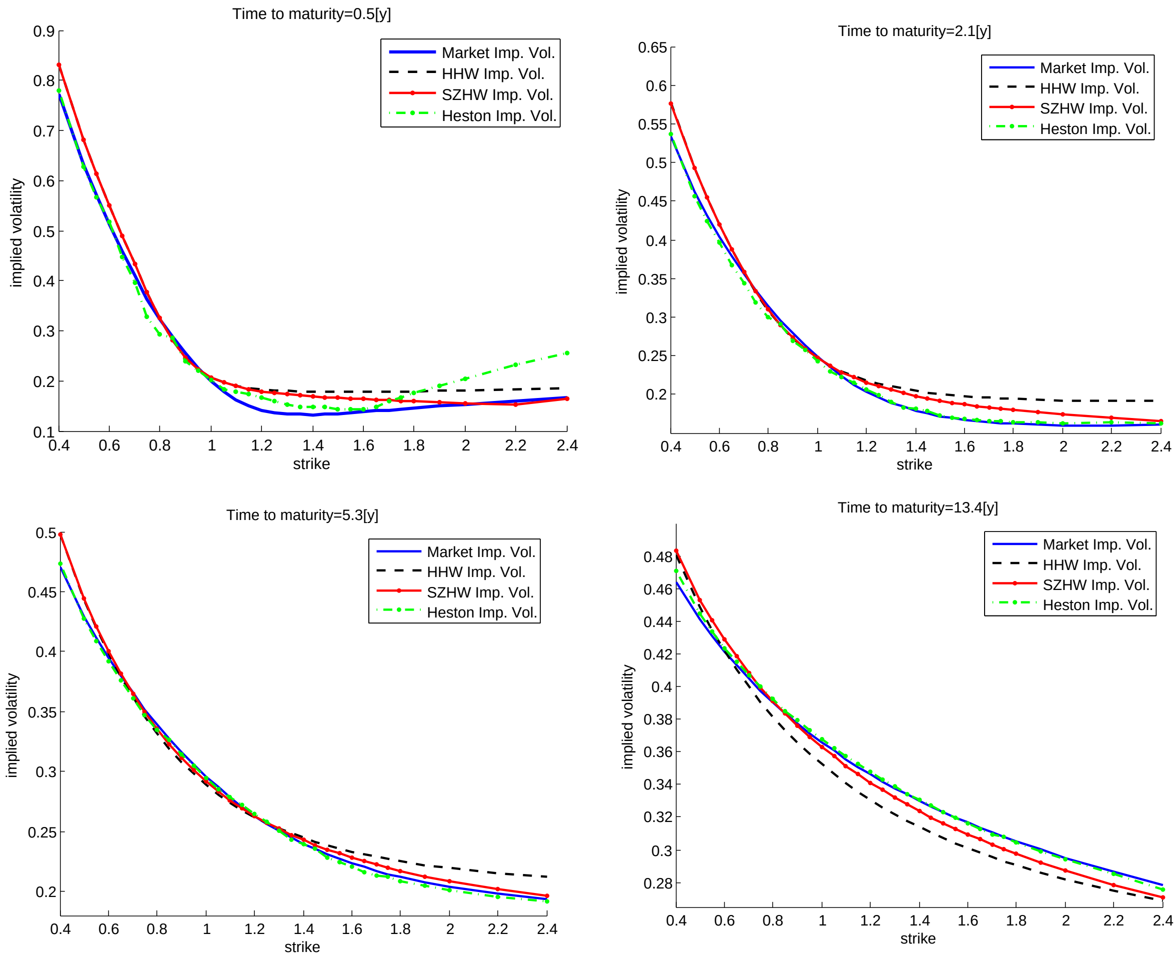

Figure 3: Implied market volatilities compared from settlement prices for the Heston's, the HHW and the SZHW model.

3.2 Variance swaps and cliquet options

Cliquet options are very popular in the world of equity derivatives [37]. The contracts are constructed to give a protection against downside risk combined with a significant upside potential. A cliquet option can be interpreted as a series of forward-starting European options, for which the total premium is determined in advance. The payout on each option can either be paid at the final maturity date, or at the end of a *reset* period. One of the cliquet type structures is a Globally Floored Cliquet with the following payoff:

$$\Pi(t_0=0,T)=\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds}\max\left(\sum_{i=1}^{M}\min\left(A_{t_i},\text{LocalCap}\right),\text{MinCoupon}\right)|\mathcal{F}_0\right). \tag{29}$$

Here $A_{t_i}=\max\left(\text{LocalFloor},\frac{S_{t_i}}{S_{t_{i-1}}}-1\right)$, $t_i=i\frac{T}{M}$, with maturity T. M indicates the number of reset periods. We notice that the term A_{t_i} can be recognized as an ATM forward starting option, which is driven by a forward skew. It has been shown in [15] that the cliquet structures are significantly underpriced under a local volatility model for which forward skews are basically too flat.

Since the forward prices are not known a-priori, we derive the values from the so-called forward characteristic function. If we define $\mathbf{X}_T$ as a state vector at time T then the forward characteristic function can be found as

$$\begin{aligned}\phi_F(\mathbf{u},\mathbf{X}_T,t^*,T) &= \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds} e^{i\mathbf{u}^T(\mathbf{X}_T-\mathbf{X}_{t^*})}|\mathcal{F}_0\right)\\ &= \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^{t^*} r_s ds - i\mathbf{u}^T\mathbf{X}_{t^*}}\phi\left(\mathbf{u},\mathbf{X}_T,t^*,T\right)|\mathcal{F}_0\right)\\ &= e^{A(\mathbf{u},t^*,T)}\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^{t^*} r_s ds - i\mathbf{u}^T\mathbf{X}_{t^*}+\mathbf{B}^T(\mathbf{u},t^*,T)\mathbf{X}_{t^*}}|\mathcal{F}_0\right).\end{aligned}$$

In the case of the plain Heston model, for $\mathbf{u}=[u,0]^T$, the forward characteristic function reads:

$$\phi_{FH}(u,\mathbf{X}_T,t^*,T) = e^{A(u,\tau^*)}\mathbb{E}^{\mathbb{Q}}\left(e^{B_\sigma(u,\tau^*)v_{t^*}}|\mathcal{F}_0\right), \qquad (30)$$

where $\tau^*=T-t^*$ and $A_H(u,\tau^*)$, $B_\sigma(u,\tau^*)$ are the Heston functions from (24), (25). The expectation under the risk neutral measure in (30) can be recognized as the Laplace transform of the transitional probability density function of a Cox-Ingersoll-Ross model [9], which is given by the following lemma:

Lemma 3.1 (Laplace transform of for Heston volatility process)**.** *The Laplace transform of the equation given by (30) for Heston stochastic volatility process has the following form*

$$\mathbb{E}^{\mathbb{Q}}\left(e^{B(u,t^*,T)v_{t^*}}|\mathcal{F}_0\right) = \left(\frac{1}{1-\frac{\eta^2}{2\kappa}\left(1-e^{-\kappa\tau}\right)B(u,t^*,T)}\right)^{\frac{2\kappa\theta}{\eta^2}}\exp\left(\frac{e^{\kappa\tau}B(u,t^*,T)\sigma_0}{1-\frac{\eta^2}{2\kappa}\left(1-e^{-\kappa\tau}\right)B(u,t^*,T)}\right).$$

Proof. A detailed proof can be found in [32] or [1]. □

Figure 4 shows the performance of all three models applied to the pricing of the cliquet option defined in (29). We choose here $T=3$, LocalCap $=0.01$, LocalFloor $=-0.01$ and $M=36$ (the contract measures the monthly performance). For large values of the MinCoupon the values of the hybrid under the three models are identical, which is expected since a large MinCoupon dominates the max operator in (29) and the expectation becomes simply the price of a zero coupon bond at time $t=0$ multiplied by the deterministic MinCoupon. We also see that the Heston model generates lower prices for small values of the MinCoupon.

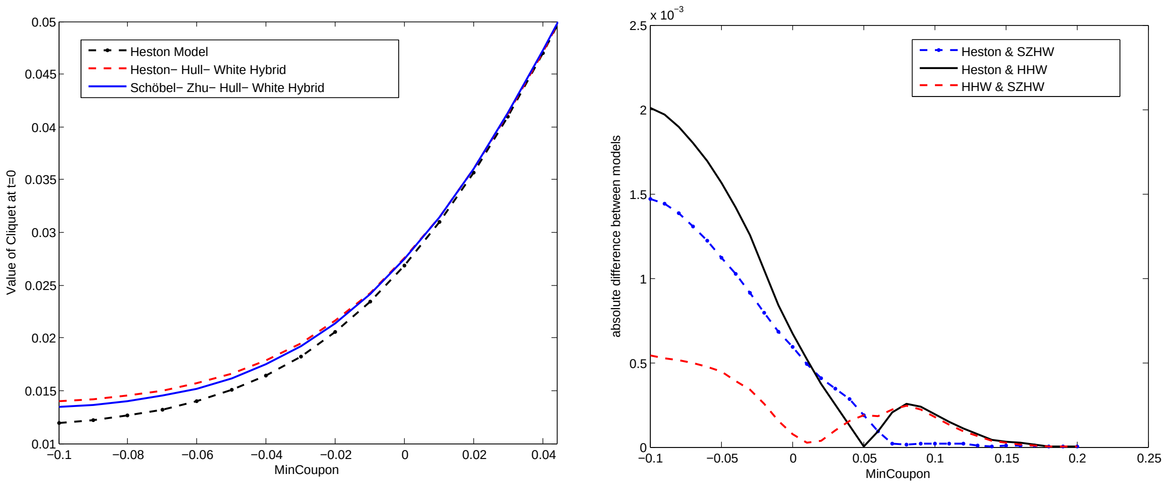

Figure 4: Pricing a cliquet product under different models with the underlying index CAC40. Left: the price of a globally floored cliquet as a function of MinCoupon given by (29) for $T=3$ years and $M=36$. Right: The absolute difference between two models decreases with MinCoupon.

3.3 A diversification product

Other hybrid products that an investor may use in strategic trading are so-called diversification products. These are based on sets of assets with different expected returns and risk levels. Proper construction of such products may give reduced risk compared to any single asset, and an expected return that is greater than that of the least risky asset [19]. A simple example is a portfolio with two assets: a stock with a high risk and high return and a bond with a low risk and low return. If one introduces an equity component in a pure bond portfolio the expected return will increase. However, because of a non-perfect correlation between these two assets also a risk reduction is expected. If the percentage of the equity in the portfolio is increased, it eventually starts to dominate the structure and the risk may increase with a higher impact for a low or negative correlation [19]. An example is a financial product, defined in the following way:

$$\Pi(t_0 = 0, T) = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds} \max\left(0, \ell \cdot \frac{S_T}{S_0} + (1-\ell) \cdot \frac{B_T}{B_0}\right) | \mathcal{F}_0\right)$$

where S_T is the underlying asset at time T, B_T is a bond, ℓ represents a percentage ratio. Figure 5 shows the pricing results for the models discussed. The product pricing is performed with the Monte Carlo method and the parameters calibrated from the market data. For $\ell \in [0,1]$ the *max* disappears from the payoff and only a sum of discounted expectations remains. The figure shows that the Heston model generates a significantly higher price, whereas the HHW and SZHW prices are relatively close. The absolute difference between the models increases with percentage ℓ.

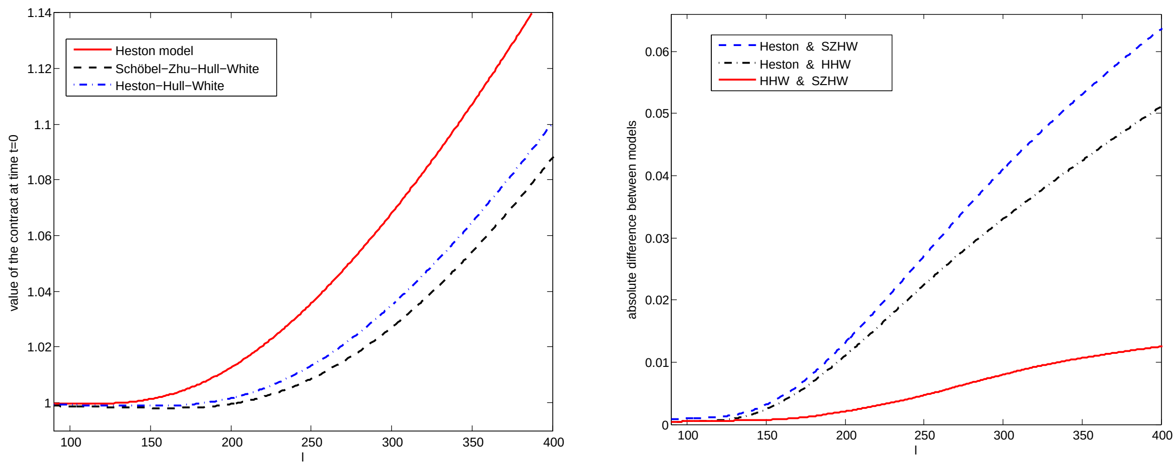

Figure 5: Pricing of a diversification hybrid product under different models. Left: The Heston model generates a significantly higher price for increasing $l = \ell \cdot 100\%$, , whereas the HHW and SZHW prices are relatively close. Right: The absolute difference between two models increases with percentage l.

3.4 Strategic investment hybrid

Suppose that an investor believes that if the price of an asset, S_t^1, like oil, goes up, then the equity markets under-perform relative to the interest rate yields, whereas, if S_t^1 drops down, the equity markets over-perform relative to the interest rate [19]. If the prices of S_t^1 are high, the market may expect an increase of the inflation and hence of the interest rates and low S_t^1 prices could have the opposite effect. In order to include such a feature in a hybrid product we define a contract in which an investor is allowed to buy a weighted performance coupon depending on the performance of another underlying. Such a product can be defined as follows,

$$\Pi(t_0 = 0, T) \quad = \quad \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^T r_s ds} V_T | \mathcal{F}_0\right) \text{ with} \tag{31}$$

$$V_T = \max\left(0, \ell \cdot \frac{L_0}{L_T} + (1-\ell)\frac{S_T}{S_0}\right)\mathbf{1}_{S_T^1 > S_0^1} + \max\left(0, (1-\ell)\frac{L_0}{L_T} + \ell \cdot \frac{S_T}{S_0}\right)\mathbf{1}_{S_T^1 < S_0^1},$$

where $0 \leq \ell$ is a weighting factor related to a percentage, $L_T = \sum_{i=1}^{M} P(T, t_i)$ with $t_1 = T$ is the T-value of projected liabilities for certain time t_M, with $\ell > 1 - \ell$.

Figure 6 shows prices obtained from Monte Carlo simulation of the contract at time $t_0 = 0$ for maturity $T = t_1 = 3$ and time horizon $t_M = 12$ with one year spacing. Since we did not model the second underlying process, S_T^1, we assume that $S_T^1 > S_0^1$. We see that for $\ell \in [0, 1]$ the *max* over the sum of performances disappears and the hybrid can be relatively easily priced, i.e., separately for both underlyings (L_0/L_T and S_T/S_0). The difference between the stochastic models becomes pronounced for $\ell > 1$ since then the correlation plays a more important role. The absolute difference between the different models for $\ell < 3$ remains below 5×10^{-3}, whereas it more significant for high ℓ. The right-side figure shows that for large ℓ the prices of HHW and SZHW are relatively close, whereas the Heston model gives higher prices.

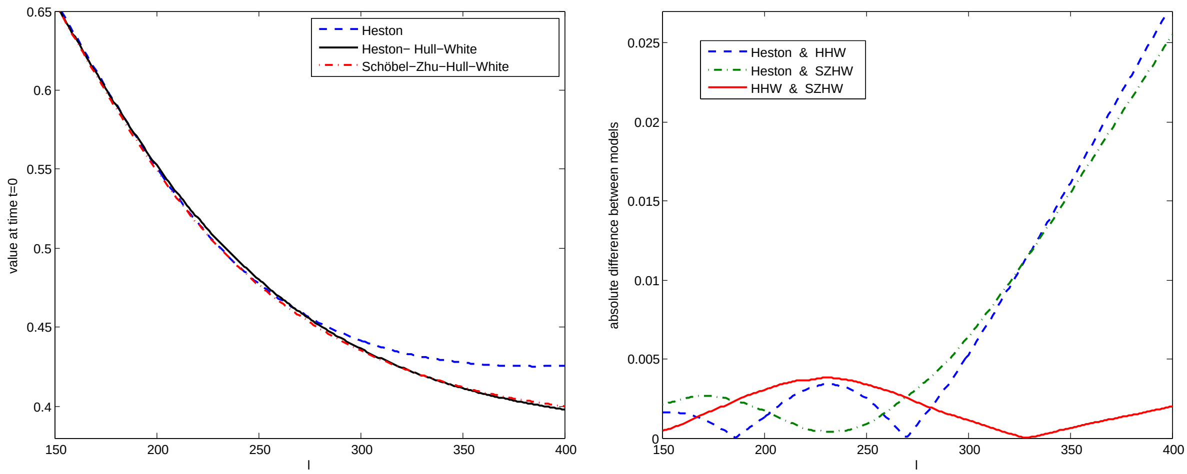

Figure 6: Left: Discounted payoffs of the strategic investment hybrid with the Heston, HHW and SZHW models in dependence of $l = \ell \cdot 100\%$. Right: Price difference between two different models.

4 Conclusions

In this paper we have presented an extension of the Schöbel-Zhu stochastic volatility model with a Hull-White interest rate process and evaluated it by means of pricing structured hybrid derivative products. The stochastic differential equations are driven by mean-reverting processes. The aim was to define a hybrid stochastic process which belongs to the class of affine jump-diffusion models, as this may lead to efficient calibration of the model. We have shown that the so-called Schöbel-Zhu-Hull-White model belongs to the category of AJD processes. No restrictions regarding the choice of correlation structure between the different Wiener processes appearing need to be made. Due to the resulting semi-analytic characteristic function we were able to calibrate the model in an efficient way by means of the Carr-Madan pricing technique.

It has been shown by numerical experiments for different hybrid products that under the same plain vanilla prices the extended SV models give different prices than the Heston model.

The present hybrid model cannot model a skew in the interest rates, which will be part of our future work.

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A Proofs of various lemmas

In this appendix we have placed the proofs of the various lemmas.

A.1 Proof of Lemma 2.2

Proof.
We depart from $\phi_{HW}(u, x_t, \tau) = \exp\left(A(u, \tau) + B(u, \tau)x_t\right)$ (with B a scalar function) and find $\phi_{HW}(u, x_t, 0) = \exp\left(iux_t\right)$, since $A(u, 0) = 0$ and $B(u, 0) = iu$. According to (5) we have to solve the following system of ODEs

$$\begin{cases} \dfrac{d}{d\tau}A = -r_0 + B^T a_0 + \dfrac{1}{2}B^T c_0 B, \\ \dfrac{d}{d\tau}B = -r_1 + a_1^T B + \dfrac{1}{2}B^T c_1 B, \end{cases}$$

with $dx_t = -\lambda x_t dt + \eta dW_t^{\mathbb{Q}}$, $r_t = x_t$, and initial condition $x_0 = 0$. Now, we recognize that $r_0 = 0$, $r_1 = 1$, $a_0 = 0$, $a_1 = -\lambda$, $\sigma(x_t)\sigma(x_t)^T = \eta^2$, $c_0 = \eta^2$ and $c_1 = 0$. Therefore, the system of ODEs reads

$$\begin{cases} \frac{d}{d\tau}A = \frac{1}{2}B^2\eta^2 \\ \frac{d}{d\tau}B = -\lambda B - 1 \end{cases}$$

where the second equation equals $d\left(Be^{\lambda\tau}\right) = -e^{\lambda\tau}d\tau$, with solution:

$$B(u,\tau) = iue^{-\lambda\tau} - \frac{1}{\lambda}\left(1 - e^{-\lambda\tau}\right).$$

The first equation gives,

$$dA = \left(-\frac{\eta^2 u^2}{2}e^{-2\lambda\tau} - 2iu\frac{\eta^2}{2\lambda}\left(e^{-\lambda\tau} - e^{-2\lambda\tau}\right) + \frac{\eta^2}{2\lambda^2}\left(1 - e^{-\lambda\tau}\right)^2\right)d\tau,$$

with solution,

$$\begin{aligned} A(u,\tau) &= \frac{\eta^2}{2\lambda^3}\left(\lambda\tau - 2\left(1 - e^{-\lambda\tau}\right) + \frac{1}{2}\left(1 - e^{-2\lambda\tau}\right)\right) - iu\frac{\eta^2}{2\lambda^2}\left(1 - e^{-\lambda\tau}\right)^2 \\ &\quad -\frac{1}{2}u^2\frac{\eta^2}{2\lambda}\left(1 - e^{-2\lambda\tau}\right). \end{aligned}$$

This concludes the proof. □

A.2 Proof of Lemma 2.3

Proof.
We need to find the solution of:

$$\begin{aligned} \frac{d}{d\tau}A(u,\tau) &= -r_0 + \mathbf{B}^T a_0 + \frac{1}{2}\mathbf{B}^T c_0 \mathbf{B}, \\ \frac{d}{d\tau}\mathbf{B}(u,\tau) &= -r_1 + a_1^T\mathbf{B} + \frac{1}{2}\mathbf{B}^T c_1 \mathbf{B}. \end{aligned}$$

For the space vector $\mathbf{X}_t^* = [\widetilde{x}_t, \widetilde{r}_t, v_t, \sigma_t]^T$ we have

$$a_0 = [0, 0, \gamma^2, \kappa\bar{\sigma}]^T, \quad a_1 = \begin{bmatrix} 0 & 1 & -\frac{1}{2} & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -2\kappa & 2\kappa\bar{\sigma} \\ 0 & 0 & 0 & -\kappa \end{bmatrix}, \quad r_0 = 0, \quad r_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

and

$$\Sigma := \sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^T = \begin{bmatrix} v & \sigma\eta\rho_{x,r} & 2v\gamma\rho_{x,v} & \sigma\gamma\rho_{x,\sigma} \\ & \eta^2 & 2\eta\sigma\gamma\rho_{r,v} & \eta\gamma\rho_{r,\sigma} \\ & & 4v\gamma^2 & 2\sigma\gamma^2 \\ & & & \gamma^2 \end{bmatrix}.$$

This leads to

$$c_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & \eta^2 & 0 & \eta\gamma\rho_{r,\sigma} \\ & & 0 & 0 \\ & & & \gamma^2 \end{bmatrix}, \quad c_1 = \begin{bmatrix} (0,0,1,0) & (0,0,0,\eta\rho_{x,r}) & (0,0,2\gamma\rho_{x,v}) & (0,0,0,\gamma\rho_{x,\sigma}) \\ & (0,0,0,0) & (0,0,0,2\eta\gamma\rho_{r,v}) & (0,0,0,0) \\ & & (0,0,4\gamma^2,0) & (0,0,0,2\gamma^2) \\ & & & (0,0,0,0) \end{bmatrix}.$$

With

$$\frac{1}{2}\mathbf{B}^T c_1 \mathbf{B} = \frac{1}{2}\begin{bmatrix} \sum_{i=1}^4\sum_{j=1}^4 B_i[s_1(1)]_{i,j}B_j \\ \sum_{i=1}^4\sum_{j=1}^4 B_i[s_1(2)]_{i,j}B_j \\ \sum_{i=1}^4\sum_{j=1}^4 B_i[s_1(3)]_{i,j}B_j \\ \sum_{i=1}^4\sum_{j=1}^4 B_i[s_1(4)]_{i,j}B_j \end{bmatrix},$$

(with $i = 1, \ldots, 4$ representing x, v, r, σ) we obtain the following system

$$\frac{dA}{d\tau} = [B_x, B_r, B_v, B_\sigma] \begin{bmatrix} 0 \\ 0 \\ \gamma^2 \\ \kappa\overline{\sigma} \end{bmatrix} + \frac{1}{2}[B_x, B_r, B_v, B_\sigma] \begin{bmatrix} 0 & 0 & 0 & 0 \\ & \eta^2 & 0 & \eta\gamma\rho_{r,\sigma} \\ & & 0 & 0 \\ & & & \gamma^2 \end{bmatrix} \begin{bmatrix} B_x \\ B_r \\ B_v \\ B_\sigma \end{bmatrix},$$

$$\frac{d\mathbf{B}}{d\tau} = \begin{bmatrix} \frac{dB_x}{d\tau} \\ \frac{dB_r}{d\tau} \\ \frac{dB_v}{d\tau} \\ \frac{dB_\sigma}{d\tau} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ -\frac{1}{2} & 0 & -2\kappa & 0 \\ 0 & 0 & 2\kappa\overline{\sigma} & -\kappa \end{bmatrix} \begin{bmatrix} B_x \\ B_r \\ B_v \\ B_\sigma \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ S_1 \\ S_2 \end{bmatrix},$$

where

$$\begin{aligned} S_1 &= B_x^2 + 4\gamma\rho_{x,v}B_xB_v + 4\gamma^2B_v^2, \\ S_2 &= 2\eta\rho_{x,r}B_xB_r + 2\gamma\rho_{x,\sigma}B_xB_\sigma + 4\eta\gamma\rho_{r,v}B_rB_v + 4\gamma^2B_vB_\sigma. \end{aligned}$$

So, we find the following system:

$$\begin{aligned} \frac{d}{d\tau}B_x &= 0, \\ \frac{d}{d\tau}B_r &= 1 + B_x - \lambda B_r, \\ \frac{d}{d\tau}B_v &= -\frac{1}{2}B_x - 2\kappa B_v + \frac{1}{2}S_1, \\ \frac{d}{d\tau}B_\sigma &= 2\kappa\overline{\sigma}B_v - \kappa B_\sigma + \frac{1}{2}S_2, \\ \frac{d}{d\tau}A &= B_v\gamma^2 + B_\sigma\kappa\overline{\sigma} + \frac{1}{2}B_r^2\eta^2 + \frac{1}{2}B_\sigma^2\gamma^2 + B_\sigma B_r\eta\gamma\rho_{r,\sigma}. \end{aligned}$$

□

A.3 Proof of Lemma 2.4

Proof.
In the 1D case, i.e., $\mathbf{u} = [u, 0, 0, 0]^T$ we start by solving the ODE for dB_r,

$$\frac{d}{d\tau}B_r + \lambda B_r = iu + 1.$$

Standard calculations give

$$\int_0^\tau d\left(e^{\lambda s}B_r(u,s)\right) = (1 + iu)\int_0^\tau e^{\lambda s}ds, \text{ i.e.,}$$

$$e^{\lambda\tau}B_r(u,\tau) - e^0B_r(u,0) = (1 + iu)\left(\frac{1}{\lambda}e^{\lambda\tau} - \frac{1}{\lambda}\right).$$

Using the boundary condition, $B_r(u, 0) = 0$, gives, $B_r(u,\tau) = (1 + iu)\lambda^{-1}\left(1 - e^{-\lambda\tau}\right)$.
The ODE for B_v now reads (using $B_x = iu$):

$$\frac{d}{d\tau}B_v = -\frac{1}{2}u(i + u) + 2\gamma^2B_v^2 - 2B_v(\kappa - \gamma\rho_{x,v}iu).$$

In order to simplify this equation we introduce the variables $\alpha = -\frac{1}{2}u(i + u)$, $\beta = 2(\kappa - \gamma\rho_{x,v}iu)$ and $\theta = 2\gamma^2$. The ODE can then be presented in the following form:

$$\frac{d}{d\tau}B_v = \alpha - \beta B_v + \theta B_v^2. \tag{32}$$

Following the calculations for the Heston model the solution of (32) reads,

$$B_v(u,\tau) = \frac{\beta - D}{2\theta}\left(\frac{1-e^{-\tau D}}{1-e^{-\tau D}\left(\frac{b}{a}\right)}\right),$$

where $a = \beta + D/2\theta$, $b = \beta - D/2\theta$, and $D = \sqrt{\beta^2 - 4\alpha\theta}$. This solution can be simplified to

$$B_v(u,\tau) = b\left(\frac{1-e^{-\tau D}}{1-e^{-\tau D}G}\right),$$

with $G = (\beta - D)/(\beta + D)$.

Next, we solve the ODE for B_σ,

$$\frac{d}{d\tau}B_\sigma = 2\kappa\bar{\sigma}B_v + \eta\rho_{x,r}B_xB_r + 2\eta\gamma\rho_{r,v}B_rB_v + \left(\gamma\rho_{x,\sigma}B_x + 2\gamma^2B_v - \kappa\right)B_\sigma.$$

We introduce the following functions,

$$\begin{aligned}\zeta(\tau) &= 2\kappa\bar{\sigma}B_v + \eta\rho_{x,r}B_xB_r + 2\eta\gamma\rho_{r,v}B_rB_v,\\ \xi(\tau) &= \gamma\rho_{x,\sigma}B_x + 2\gamma^2B_v - \kappa.\end{aligned}$$

This leads to the following ODE

$$\frac{d}{d\tau}B_\sigma - \xi(\tau)B_\sigma = \zeta(\tau),$$

whose solution follows from,

$$\frac{d}{d\tau}\left(e^{-\int_0^\tau \xi(s)ds}B_\sigma\right) = \zeta(\tau)\exp\left(-\int_0^\tau \xi(s)ds\right),$$

or

$$\exp\left(-\int_0^\tau \xi(s)ds\right)B_\sigma = \int_0^\tau \zeta(s)\exp\left(-\int_0^s \xi(k)dk\right)ds.$$

So, finally, we need to calculate

$$\begin{aligned}B_\sigma(u,\tau) &= \exp\left(\int_0^\tau \xi(s)ds\right)\int_0^\tau \zeta(s)\exp\left(-\int_0^s \xi(k)dk\right)ds + \text{ Const.}\\ B_\sigma(u,0) &= 0\end{aligned}$$

For this, we start with the integral for $\xi(k)$:

$$\begin{aligned}\int_0^s \xi(k)dk &= \int_0^s \left(\gamma\rho_{x,\sigma}iu + 2\gamma^2B_v - \kappa\right)dk\\ &= \gamma\rho_{x,\sigma}ius - \kappa s + 2\gamma^2 b\int_0^s\left(\frac{1-e^{-kD}}{1-e^{-kD}G}\right)dk\\ &= \gamma\rho_{x,\sigma}ius - \kappa s + 2\gamma^2 b\left(\frac{Ds - (G-1)\log(1-G) + (G-1)\log(e^{Ds}-G)}{DG}\right)\\ &= \left(\gamma\rho_{x,\sigma}iu - \kappa\right)s + \frac{(\beta-D)Ds}{2GD} + \frac{(\beta-D)(G-1)\log(\frac{e^{sD}-G}{1-G})}{2GD}\\ &= \left(\gamma\rho_{x,\sigma}iu - \kappa + \frac{\beta-D}{2G}\right)s + \frac{(\beta-D)(G-1)}{2GD}\log(\frac{e^{sD}-G}{1-G})\\ &= C_1s + C_2\log(\frac{e^{sD}-G}{1-G})\end{aligned}$$

where $C_1 = \left(\gamma\rho_{x,\sigma}iu - \kappa + \frac{\beta-D}{2G}\right)$, $C_2 = \frac{(\beta-D)(G-1)}{2GD}$, $\beta = 2(\kappa - \gamma\rho_{x,v}iu)$, $D = \sqrt{\beta^2 - 4\alpha\theta}$ and $G = \frac{\beta-D}{\beta+D}$. After substitution of these quantities, we find that $C_1 = D/2$ and $C_2 = -1$.

Next, we need to calculate the exponent of the integral of ξ:

$$\exp\left(\int_0^s \xi(k)dk\right) = \exp\left(C_1 s + C_2\log(\frac{e^{sD}-G}{1-G})\right) = \exp\left(\frac{D}{2}s\right)\left(\frac{1-G}{e^{sD}-G}\right),$$

and we can include ζ in the integral,

$$\int_0^\tau \zeta(s)\exp\left(-\int_0^s \xi(k)dk\right)ds = \int_0^\tau (2\kappa\overline{\sigma}B_v + \eta\rho_{x,r}B_xB_r + 2\eta\gamma\rho_{r,v}B_rB_v)\exp\left(-\frac{D}{2}s\right)\left(\frac{e^{sD}-G}{1-G}\right)ds.$$

This integral is split into three parts. The first part can be solved analytically,

$$\begin{aligned}
\int_0^\tau 2\kappa\overline{\sigma}B_v e^{-\frac{D}{2}s}\left(\frac{e^{sD}-G}{1-G}\right)ds &= 2\kappa\overline{\sigma}b\int_0^\tau\left(\frac{1-e^{-sD}}{1-e^{-sD}G}\right)e^{-\frac{D}{2}s}\left(\frac{e^{sD}-G}{1-G}\right)ds\\
&= \frac{2\kappa\overline{\sigma}b}{1-G}\int_0^\tau\left(\frac{1-e^{-sD}}{1-e^{-sD}G}\right)e^{-\frac{D}{2}s}\left(e^{sD}-G\right)ds\\
&= \frac{2\kappa\overline{\sigma}b}{1-G}\int_0^\tau e^{-\frac{sD}{2}}\left(e^{sD}-1\right)ds\\
&= \frac{16\kappa\overline{\sigma}b\sinh^2\left(\frac{\tau D}{4}\right)}{(1-G)D}.
\end{aligned}$$

The second part can be solved analytically as well,

$$\begin{aligned}
\int_0^\tau \eta\rho_{x,r}B_xB_r e^{-\frac{D}{2}s}\left(\frac{e^{sD}-G}{1-G}\right)ds &= \int_0^\tau \eta\rho_{x,r}iu(1+iu)\lambda^{-1}(1-e^{-\lambda s})e^{-\frac{Ds}{2}}\left(\frac{e^{sD}-G}{1-G}\right)ds\\
&= \frac{\eta\rho_{x,r}iu(1+iu)}{(1-G)\lambda}\int_0^\tau e^{-\frac{Ds}{2}}(1-e^{-\lambda s})(e^{sD}-G)ds\\
&= \frac{\eta\rho_{x,r}iu(1+iu)}{(1-G)\lambda}A_1(u,\tau).
\end{aligned}$$

where

$$A_1(u,\tau) = \frac{2}{D}(e^{\frac{\tau D}{2}}-1) + \frac{2G}{D}(e^{-\frac{\tau D}{2}}-1) - \frac{2\left(e^{\frac{\tau}{2}(D-2\lambda)}-1\right)}{D-2\lambda} + \frac{2G\left(1-e^{-\frac{\tau}{2}(D+2\lambda)}\right)}{D+2\lambda}, \tag{33}$$

and the third part reads,

$$\begin{aligned}
\int_0^\tau 2\eta\gamma\rho_{r,v}B_rB_v e^{-\frac{D}{2}s}\left(\frac{e^{sD}-G}{1-G}\right)ds &= \frac{2\eta\gamma\rho_{r,v}}{1-G}\int_0^\tau B_rB_v e^{-\frac{D}{2}s}\left(e^{sD}-G\right)ds\\
&= \frac{2\eta\gamma\rho_{r,v}(1+iu)b}{(1-G)\lambda}\int_0^\tau (1-e^{-\lambda s})\left(\frac{1-e^{-sD}}{1-e^{-sD}G}\right)e^{-\frac{D}{2}s}\left(e^{sD}-G\right)ds\\
&= \frac{2\eta\gamma\rho_{r,v}(1+iu)b}{(1-G)\lambda}\int_0^\tau e^{-\frac{1}{2}s(D+2\lambda)}(e^{Ds}-1)(e^{s\lambda}-1)ds\\
&= \frac{2\eta\gamma\rho_{r,v}(1+iu)b}{(1-G)\lambda}(B_1(u,\tau)+B_2(u,\tau)),
\end{aligned}$$

where

$$B_1(u,\tau) = -\frac{4}{D} + \frac{2}{D-2\lambda} + \frac{2}{D+2\lambda}, \tag{34}$$

$$B_2(u,\tau) = \left(e^{-\frac{1}{2}\tau(D+2\lambda)}\right)\left(\frac{2e^{\tau\lambda}(1+e^{D\tau})}{D} - \frac{2e^{D\tau}}{D-2\lambda} - \frac{2}{D+2\lambda}\right). \tag{35}$$

So, finally, we have:

$$\begin{aligned}
B_\sigma(u,\tau) &= \exp\left(\int_0^\tau \xi(s)ds\right)\int_0^\tau \zeta(s)\exp\left(-\int_0^s \xi(k)dk\right)ds\\
&= \left(\frac{e^{\frac{D}{2}\tau}}{e^{\tau D}-G}\right)\left(\frac{16\kappa\overline{\sigma}b\sinh^2\left(\frac{\tau D}{4}\right)}{D} + \frac{\eta\rho_{x,r}iu(1+iu)}{\lambda}A_1(u,\tau) + \frac{2\eta\gamma\rho_{r,v}(1+iu)b}{\lambda}(B_1(u,\tau)+B_2(u,\tau))\right),
\end{aligned}$$

with $A_1(u,\tau)$ from (33), $B_1(u,\tau)$ from (34) and $B_2(u,\tau)$ from (35).

Now, we solve the ODE for $A(u,\tau)$:

$$\frac{d}{d\tau}A = B_v\gamma^2 + B_\sigma\kappa\bar{\sigma} + \frac{1}{2}B_r^2\eta^2 + \frac{1}{2}B_\sigma^2\gamma^2 + B_\sigma B_r\eta\gamma\rho_{r,\sigma},$$

with solution,

$$A(u,\tau) - A(u,0) = \gamma^2\int_0^\tau B_v ds + \kappa\bar{\sigma}\int_0^\tau B_\sigma ds + \frac{1}{2}\eta^2\int_0^\tau B_r^2 ds + \frac{1}{2}\gamma^2\int_0^\tau B_\sigma^2 ds + \eta\gamma\rho_{r,\sigma}\int_0^\tau B_\sigma B_r ds.$$

Or,

$$A(u,\tau) = \underbrace{\int_0^\tau \left(\gamma^2 B_v + \frac{1}{2}\eta^2 B_r^2\right) ds}_{A_2(u,\tau)} + \underbrace{\int_0^\tau B_\sigma\left(\kappa\bar{\sigma} + \frac{1}{2}\gamma^2 B_\sigma + \eta\rho_{r,\sigma}\gamma B_r\right) ds}_{A_3(u,\tau)} \quad (36)$$

In order to find $A(u,\tau)$ we have to evaluate the integrals A_2 and A_3. Integral A_3 involves a hypergeometric function (called the ${}_2F_1$ function or simply Gaussian function), which is computed numerically here. For integral A_2 we have two representations,

$$A_2(u,\tau) = \frac{(\beta - D)s - 2\log\left(\frac{Ge^{-Ds}-1}{G-1}\right)}{4\gamma^2} - A_4(u,\tau), \text{ or}$$

$$A_2(u,\tau) = \frac{(\beta + D)s - 2\log\left(\frac{e^{Ds}-G}{1-G}\right)}{4\gamma^2} - A_4(u,\tau), \text{ where}$$

$$A_4(u,\tau) = \frac{(iu+1)^2(3 + e^{-2\tau\lambda} - 4e^{-\tau\lambda} - 2\tau\lambda)}{2\lambda^3}.$$

Since in $A_2(u,\tau)$ a complex-valued logarithm appears, it should be treated with some care. It turns out that the second formulation gives rise to discontinuities which may cause inaccuracies. According to [23], an easy way to avoid any errors due to complex-valued discontinuities is to apply numerical integration.

We know that the price of a zero coupon bond can be obtained from the characteristic function, $\phi_{SZHW}(\mathbf{u}, \mathbf{X}_t, t, T)$, by setting $\mathbf{u} = [0,0,0,0]^T$. So,

$$\begin{aligned} P(t,T) &= \phi(0, \mathbf{X}_t, \tau) \\ &= \exp\left(-\int_t^T \psi_s ds\right)\exp\left(A(0,\tau) + B_x(0,\tau)x_t + B_r(0,\tau)\widetilde{r}_t + B_v(0,\tau)v_t + B_\sigma(0,\tau)\sigma_t\right). \end{aligned}$$

Since $\widetilde{r}_0 = 0$, we have $P(0,T) = \exp\left(-\int_0^T \psi_s ds\right)\exp\left(A(0,\tau) + B_x(0,\tau)x_0 + B_v(0,\tau)v_0 + B_\sigma(0,\tau)\sigma_0\right)$ and it is easy to check that $B_x(0,T) = 0,\ B_v(0,T) = 0,\ B_\sigma(0,T) = 0$, and,

$$\begin{aligned} A(0,T) &= \frac{1}{2}\eta^2\int_0^T B_r(0,s)^2 ds \\ &= \frac{\eta^2}{2\lambda^3}\left(\frac{1}{2} + T\lambda + 2e^{-\lambda T} - \frac{1}{2}e^{-2\lambda T}\right). \end{aligned}$$

Therefore, $P(0,T) = \exp\left(-\int_0^T \psi_s ds + A(0,T)\right)$, or, $\log\left(P(0,T)\right) = -\int_0^T \psi_s ds + A(0,T)$, which finally gives us:

$$\psi_T = -\frac{\partial}{\partial T}\log P(0,T) + \frac{\partial}{\partial T}A(0,T) = f(0,T) + \frac{\eta^2}{2\lambda^2}\left(1 - e^{-\lambda T}\right)^2.$$

Since $\psi_0 = f(0,0) \equiv r_0$, where r_0 is the initial value of the interest rate process r_t.

With $\mathbf{u} = [u,0,0,0]^T$, we find:

$$\phi_{SZHW}(u, \mathbf{X}_t, t, T) = \exp\left(\widetilde{A}(u,\tau) + B_x(u,\tau)x_t + B_r(u,\tau)r_t + B_v(u,\tau)v_t + B_\sigma(u,\tau)\sigma_t\right),$$

with

$$
\begin{aligned}
\widetilde{A}(u,\tau) &= -\int_t^T \psi_s ds + iu\int_t^T \psi_s ds + A(u,\tau)\\
&= (iu-1)\int_t^T \left(f(0,s)+\frac{\eta^2}{2\lambda^2}\left(1-e^{-\lambda s}\right)^2\right)ds + A(u,\tau)\\
&= (1-iu)\int_t^T d\left(\log(P(0,s))\right) + (1-iu)\frac{\eta^2}{2\lambda^2}\int_t^T \left(1-e^{-\lambda s}\right)^2 ds + A(u,\tau)\\
&= (1-iu)\log\left(\frac{P(0,T)}{P(0,t)}\right) + (1-iu)\frac{\eta^2}{2\lambda^2}\left((T-t)+\frac{2}{\lambda}\left(e^{-\lambda T}-e^{-\lambda t}\right)-\frac{1}{2\lambda}\left(e^{-2\lambda T}-e^{-2\lambda t}\right)\right) + A(u,\tau),
\end{aligned}
$$

and $A(u,\tau)$ as in (36).

The discounted CF for the Schöbel-Zhu-Hull-White hybrid process is now determined, and reads,

$$
\phi_{SZHW}(u,\mathbf{X}_0,\tau) = \exp\left(\widetilde{A}(u,\tau)+B_x(u,\tau)x_0+B_r(u,\tau)r_0+B_v(u,\tau)v_0+B_\sigma(u,\tau)\sigma_0\right),
$$

where:

$$
\begin{aligned}
B_x(u,\tau) &= iu,\\
B_r(u,\tau) &= (1+iu)\lambda^{-1}\left(1-e^{-\lambda\tau}\right),\\
B_v(u,\tau) &= \frac{\beta-D}{2\theta}\left(\frac{1-e^{-\tau D}}{1-e^{-\tau D}G}\right),\\
B_\sigma(u,\tau) &= \left(\frac{e^{\frac{D}{2}\tau}}{e^{\tau D}-G}\right)\left(\frac{16\kappa\bar{\sigma}b\sinh^2\left(\frac{\tau D}{4}\right)}{D}+\frac{\eta\rho_{x,r}iu(1+iu)}{\lambda}F_1(u,\tau)+\frac{2\eta\gamma\rho_{r,v}(1+iu)b}{\lambda}F_2(u,\tau)\right),\\
\widetilde{A}(u,\tau) &= (1-iu)\left(\log\left(\frac{P(0,T)}{P(0,t)}\right)+\frac{\eta^2}{2\lambda^2}\left((T-t)+\frac{2}{\lambda}\left(e^{-\lambda T}-e^{-\lambda t}\right)-\frac{1}{2\lambda}\left(e^{-2\lambda T}-e^{-2\lambda t}\right)\right)\right)+A(u,\tau),
\end{aligned}
$$

and,

$$
\begin{aligned}
A(u,\tau) &= \frac{(\beta-D)s-2\log\left(\frac{Ge^{-Ds}-1}{G-1}\right)}{4\gamma^2}-\frac{(iu+1)^2(3+e^{-2\tau\lambda}-4e^{-\tau\lambda}-2\tau\lambda)}{2\lambda^3}+F_3(u,\tau),\\
F_1(u,\tau) &= \frac{2}{D}(e^{\frac{\tau D}{2}}-1)+\frac{2G}{D}(e^{-\frac{\tau D}{2}}-1)-\frac{2\left(e^{\frac{\tau}{2}(D-2\lambda)}-1\right)}{D-2\lambda}+\frac{2G\left(1-e^{-\frac{\tau}{2}(D+2\lambda)}\right)}{D+2\lambda},\\
F_2(u,\tau) &= -\frac{4}{D}+\frac{2}{D-2\lambda}+\frac{2}{D+2\lambda}+\left(e^{-\frac{1}{2}\tau(D+2\lambda)}\right)\left(\frac{2e^{\tau\lambda}(1+e^{D\tau})}{D}-\frac{2e^{D\tau}}{D-2\lambda}-\frac{2}{D+2\lambda}\right),\\
F_3(u,\tau) &= \int_0^\tau B_\sigma(u,s)\left(\kappa\bar{\sigma}+\frac{1}{2}\gamma^2B_\sigma+\eta\rho_{r,\sigma}\gamma B_r\right)ds.
\end{aligned}
$$

□

A.4 Exact solution for Schöbel-Zhu-Hull-White model for $\rho_{x,r}=0$ and $\rho_{r,\sigma}=0$

Here we present the closed form solution of the SZHW model with the correlations $\rho_{x,r}=\rho_{r,\sigma}=0$. Then, coefficient B_σ reads

$$
B_\sigma(u,\tau) = \left(\frac{e^{\frac{D}{2}\tau}}{e^{\tau D}-G}\right)\left(\frac{16\kappa\bar{\sigma}b\sinh^2\left(\frac{\tau D}{4}\right)}{D}\right).
$$

The integral (19) that we have to deal with to calculate coefficient A is of the following form:

$$\begin{aligned}
F_3(u,\tau) &= \int_0^\tau B_\sigma\left(\kappa\bar{\sigma}+\frac{1}{2}\gamma^2 B_\sigma\right)ds = \kappa\overline{\sigma}\int_0^\tau B_\sigma ds + \frac{1}{2}\gamma^2\int_0^\tau B_\sigma^2 ds \\
&= \frac{-4\kappa^2\overline{\sigma}^2 b}{GD^2}\left(D\tau + 4\sqrt{G}\left(Arctanh\left(\frac{1}{\sqrt{G}}\right) - Arctanh\left(\frac{e^{\frac{\tau D}{2}}}{\sqrt{G}}\right)\right) + (1-G)\log\left(\frac{1-G}{e^{\tau D}-G}\right)\right) \\
&\quad -\frac{8\gamma^2\kappa^4\overline{\sigma}^4 b^2}{D^3G^2}\left(\left(-1-2\sqrt{G}+2G^{\frac{3}{2}}+G^2\right)\log\left(1+iG^{\frac{1}{4}}\right)+\left(-1+2\sqrt{G}-2G^{\frac{3}{2}}+G^2\right)\log\left(1+G^{\frac{1}{4}}\right)\right) \\
&\quad -\frac{8\gamma^2\kappa^4\overline{\sigma}^4 b^2}{D^3G^2}\left(\left(\sqrt{G}-1\right)^3\left(\sqrt{G}+1\right)\log\left(1-G^{\frac{1}{4}}\right)+\left(\sqrt{G}-1\right)\left(\sqrt{G}+1\right)^3\log\left(1+iG^{\frac{1}{4}}\right)+3G+G^2\right) \\
&\quad -\frac{8\gamma^2\kappa^4\overline{\sigma}^4 b^2}{D^3G^2(G-e^{D\tau})}\left(G\left(1-e^{\frac{D\tau}{2}}+6G-4e^{\frac{D\tau}{2}}G+G^2\right)-De^{D\tau}\tau+DG\tau\right) \\
&\quad -\frac{8\gamma^2\kappa^4\overline{\sigma}^4 b^2}{D^3G^2(G-e^{D\tau})}\left(e^{D\tau}-G\right)(G-1)\left(\left(\sqrt{G}-1\right)^2\log\left(e^{\frac{D\tau}{2}}-G^{\frac{1}{4}}\right)+\left(\sqrt{G}+1\right)^2\log\left(e^{\frac{D\tau}{4}}-iG^{\frac{1}{4}}\right)\right) \\
&\quad -\frac{8\gamma^2\kappa^4\overline{\sigma}^4 b^2}{D^3G^2(G-e^{D\tau})}\left(e^{D\tau}-G\right)(G-1)
\end{aligned}$$

This expression can be confirmed with the help of Mathematica, for example.