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Saddlepoint approximations for expectations

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Abstract

We derive two types of saddlepoint approximations to expectations in the form of $\mathbb{E}[(X-K)^+]$ and $\mathbb{E}[X|X \geq K]$, where X is the sum of n independent random variables and K is a known constant. We establish error convergence rates for both types of approximations in the i.i.d. case. The approximations are further extended to cover the case of lattice variables.

1 Introduction

We consider the saddlepoint approximations of $\mathbb{E}[(X-K)^+]$ and $\mathbb{E}[X|X \geq K]$, where X is the sum of n independent random variables X_i, $i = 1, \ldots, n$, and K is a known constant. These two expectations can be frequently encountered in finance and insurance. In option pricing, $\mathbb{E}[(X-K)^+]$ is the payoff of a call option (Rogers & Zane, 1999). It also plays an integral role in the pricing of the Collateralized Debt Obligations (CDO) (Yang et al., 2006; Antonov et al., 2005). In insurance, $\mathbb{E}[(X-K)^+]$ is known as the stop-loss premium. The term $\mathbb{E}[X|X \geq K]$ corresponds to the expected shortfall, also known as the tail conditional expectation, of a credit or insurance portfolio, which plays an increasingly important role in risk management in financial and insurance institutions.

We derive two types of saddlepoint expansions for the two quantities. The first type of approximation formulas for $\mathbb{E}[(X-K)^+]$ already appeared in Antonov et al. (2005). We here provide a simpler and more statistically-oriented derivation that requires no knowledge of complex analysis. The second type of approximations is obtained by two distinct approaches. The resulting formulas distinguish themselves from all existing approximation formulas by their remarkable simplicity. We also establish error convergence rates for both types of approximations in the i.i.d. case. The approximations are further extended to cover the case of lattice variables. The lattice case is largely ignored, even in applications where lattice variables are more relevant, for example, the pricing of CDOs.

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The two quantities are related as follows,

$$\mathbb{E}[X|X \geq K] = \frac{\mathbb{E}[(X-K)^+]}{\mathbb{P}(X \geq K)} + K, \tag{1}$$

$$\mathbb{E}[(X-K)^+] = \mathbb{E}\left[X1_{\{X\geq K\}}\right] - K\mathbb{P}(X \geq K), \tag{2}$$

$$\mathbb{E}[X|X \geq K] = \frac{\mathbb{E}\left[X1_{\{X\geq K\}}\right]}{\mathbb{P}(X \geq K)}. \tag{3}$$

It is also straightforward to extend our results to the functions $\mathbb{E}[(K-X)^+]$ and $\mathbb{E}[X|X < K]$. The connections are well known and we put them here only for completeness.

$$\begin{aligned}
&\mathbb{E}[(K-X)^+] = \mathbb{E}[(X-K)^+] - \mathbb{E}[X] + K, \\
&\mathbb{E}[X1_{\{X<K\}}] = \mathbb{E}[X] - \mathbb{E}[X1_{\{X\geq K\}}], \\
&\mathbb{E}[X|X < K] = \left(\mathbb{E}[X] - \mathbb{E}[X1_{\{X\geq K\}}]\right)/\mathbb{P}(X < K).
\end{aligned}$$

For simplicity of notation, we define

$$\begin{cases} C & := \mathbb{E}[(X-K)^+], \\ S & := \mathbb{E}[X|X \geq K], \\ J & := \mathbb{E}\left[X1_{\{X\geq K\}}\right]. \end{cases} \tag{4}$$

2 Densities and tail probabilities

Dating back to Esscher (1932), the saddlepoint approximation has been recognized as a valuable tool in asymptotic analysis and statistical computing. It has found a wide range of applications in finance and insurance, reliability theory, physics and biology. The saddlepoint approximation literature so far mainly focuses on the approximation of densities (Daniels, 1954) and tail probabilities (Lugannani & Rice, 1980; Daniels, 1987). For a comprehensive exposition of saddlepoint approximations, see Jensen (1995).

We start with some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $X_i, i = 1 \ldots n$ be n independent continuous random variables all defined on the given probability space and $X = \sum_{i=1}^n X_i$. Suppose that for all i, the moment generating function (MGF) of X_i is analytic and given by M_{X_i}, the MGF of the sum X is then simply the product of the MGF of X_i, i.e.,

$$M(t) = \prod_{i=1}^n M_{X_i}(t),$$

for t in some open neighborhood of zero. Let $\kappa(t) = \log M(t)$ be the Cumulant Generating Function(CGF) of X. The density and tail probability of X can be represented by the following inversion formulas

$$f_X(K) = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \exp(\kappa(t) - tK)dt, \tag{5}$$

$$\mathbb{P}(X \geq K) = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \frac{\exp(\kappa(t) - tK)}{t}dt \quad (\tau > 0). \tag{6}$$

Throughout this paper we adopt the following notation:

- $\phi(\cdot)$ and $\Phi(\cdot)$ denote, respectively, the pdf and cdf of a standard normal random variable,

- $\mu := \mathbb{E}[X]$ is the expectation of X under $\mathbb{P}$,
- T represents the saddlepoint that gives $\kappa'(T) = K$,
- $\lambda_r := \kappa^{(r)}(T)/\kappa''(T)^{r/2}$ is the standardized cumulant of order r evaluated at T,
- $Z := T\sqrt{\kappa''(T)}$,
- $W := sgn(T)\sqrt{2[KT - \kappa(T)]}$ with $sgn(T)$ being the sign of T.

The saddlepoint approximation for densities is then given by the Daniels (1954) formula

$$f_X(K) \approx \phi(W)\frac{T}{Z}\left(1 + \frac{\lambda_4}{8} - \frac{5\lambda_3^2}{24}\right) =: f_D. \tag{7}$$

For tail probabilities, two types of distinct saddlepoint expansions exist. The first type of expansion is given by

$$\mathbb{P}(X \geq K) \approx e^{-\frac{W^2}{2}+\frac{Z^2}{2}}[1 - \Phi(Z)] =: P_1, \tag{8}$$

$$\mathbb{P}(X \geq K) \approx P_1\left(1 - \frac{\lambda_3}{6}Z^3\right) + \phi(W)\frac{\lambda_3}{6}\left(Z^2 - 1\right) =: P_2, \tag{9}$$

in the case $T \geq 0$. For $T < 0$ similar formulas are available, see Daniels (1987). The second type of expansion is obtained by Lugannani & Rice (1980), with

$$\mathbb{P}(X \geq K) \approx 1 - \Phi(W) + \phi(W)\left[\frac{1}{Z} - \frac{1}{W}\right] =: P_3, \tag{10}$$

$$\mathbb{P}(X \geq K) \approx P_3 + \phi(W)\left[\frac{1}{Z}\left(\frac{\lambda_4}{8} - \frac{5\lambda_3^2}{24}\right) - \frac{\lambda_3}{2Z^2} - \frac{1}{Z^3} + \frac{1}{W^3}\right] =: P_4. \tag{11}$$

The saddlepoint approximations are asymptotic approximations but they are known to give accurate results in terms of relative error even for relatively small n. When X_i are i.i.d. random variables, the rate of convergence of f_D is n^{-2} and the rates of convergence of P_1 to P_4 are $n^{-1/2}, n^{-1}, n^{-3/2}, n^{-5/2}$, respectively. Widely known as the Lugannani-Rice formula, P_3 is most popular among the four tail probability approximations for both simplicity and accuracy. A good review of saddlepoint approximations for the tail probability is given in Daniels (1987).

3 Measure change approaches

Before we derive the formulas for $\mathbb{E}[(X-K)^+]$ and $\mathbb{E}[X|X \geq K]$, we would like to briefly review a different approach to approximating the two quantities. This usually involves a change of measure and borrows the saddlepoint expansions for densities or tail probabilities.

An inversion formula similar to those for densities and tail probabilities also exists for $\mathbb{E}[(X-K)^+]$, which is given by

$$\mathbb{E}\left[(X-K)^+\right] = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \frac{\exp(\kappa(t) - tK)}{t^2}dt \quad (\tau > 0). \tag{12}$$

Yang et al. (2006) rewrite the inversion formula to be

$$\mathbb{E}\left[(X-K)^+\right] = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \exp(\kappa(t) - 2\log|t| - tK)dt. \tag{13}$$

Take $\kappa_{\mathbb{Q}}(t) = \kappa(t) - 2\log|t|$, where subscript $\mathbb{Q}$ denotes a probability measure different from the original measure $\mathbb{P}$, the right-hand side of (13) is then in the form of (5) and the Daniels formula (7) can be used for approximation. It should be pointed out, however, that in this case always two saddlepoints exist. Moreover, the MGF of X under the new measure $\mathbb{Q}$ is problematic as $M_{\mathbb{Q}}(0) \to \infty$,which suggests that $\mathbb{Q}$ is *not a probability measure.*

Bounded random variables

Studer (2001) considers the approximation of the expected shortfall, in two models of the associated random variable.

The first case deals with bounded random variables. Without loss of generality, we only consider the case that X has a positive lower bound. Define the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ by $\mathbb{Q}(A) = \int_A X/\mu d\mathbb{P}$ for $A \in \mathcal{F}$, then

$$\begin{aligned}\mathbb{E}[X|X \geq K] &= \frac{1}{\mathbb{P}(X \geq K)} \int_{\{X \geq K\}} X d\mathbb{P} = \frac{\mu}{\mathbb{P}(X \geq K)} \int_{\{X \geq K\}} \frac{X}{\mu} d\mathbb{P} \\ &= \frac{\mu}{\mathbb{P}(X \geq K)} \mathbb{Q}(X \geq K). \end{aligned} \tag{14}$$

Hence the expected shortfall is transformed to be a multiple of the ratio of two tail probabilities. The MGF of X under probability $\mathbb{Q}$ reads

$$M_{\mathbb{Q}}(t) = \int e^{tX} \frac{X}{\mu} d\mathbb{P} = \frac{M'(t)}{\mu} = \frac{M(t)\kappa'(t)}{\mu}$$

as $\kappa'(t) = [\log M(t)]' = M'(t)/M(t)$. It follows that

$$\kappa_{\mathbb{Q}}(t) = \log M_{\mathbb{Q}}(t) = \kappa(t) + \log(\kappa'(t)) - \log(\mu). \tag{15}$$

For bounded variables in general it is only necessary to apply a linear transform on the random variable X beforehand so that the new variable has a positive lower bound and thus $\mathbb{Q}(\cdot)$ is a valid probability measure.

The saddlepoint approximation for tail probability can be applied for both probabilities $\mathbb{P}$ and $\mathbb{Q}$ in (14). A disadvantage of this approach is that two saddlepoints need to be found as the saddlepoints under the two probability measures are generally different.

Log-return model

The second case in Studer (2001) deals with $\mathbb{E}[e^X|X \geq K]$ rather than $\mathbb{E}[X|X \geq K]$. The expected shortfall $\mathbb{E}[e^X|X \geq K]$ can also be written to be a multiple of the ratio of two tail probabilities. Define the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ by $\mathbb{Q}(A) = \int_A e^X/M(1) d\mathbb{P}$ for $A \in \mathcal{F}$, then

$$\begin{aligned}\mathbb{E}[e^X|X \geq K] &= \frac{1}{\mathbb{P}(X \geq K)} \int_{\{X \geq K\}} e^X d\mathbb{P} = \frac{M(1)}{\mathbb{P}(X \geq K)} \int_{\{X \geq K\}} \frac{e^X}{M(1)} d\mathbb{P} \\ &= \frac{M(1)}{\mathbb{P}(X \geq K)} \mathbb{Q}(X \geq K). \end{aligned} \tag{16}$$

The MGF and CGF of X under probability $\mathbb{Q}$ are given by

$$\begin{aligned} M_{\mathbb{Q}}(t) &= \int e^{tX} \frac{e^X}{M(1)} d\mathbb{P} = \frac{M(t+1)}{M(1)}, \\ \kappa_{\mathbb{Q}}(t) &= \kappa(t+1) - \kappa(1). \end{aligned}$$

This also forms the basis for the approach used in Rogers & Zane (1999) for option pricing where the log-price process follows a Lévy process.

4 Classical saddlepoint approximations

In this and in the sections to follow we give, in the spirit of Daniels (1987), two types of explicit saddlepoint approximations for $\mathbb{E}[(X-K)^+]$. For each type of approximation, we give a lower order version and a higher order version. The approximations to $\mathbb{E}[X|X \geq K]$ then simply follow from (1). No measure change is required and only one saddlepoint need to be computed.

Following Jensen (1995), we call this first type of approximations the classical saddlepoint approximations. Approximation formulas for $\mathbb{E}[(X-K)^+]$ of this type already appeared in Antonov et al. (2005). They are obtained by means of routine application of the saddlepoint approximation to (12), i.e., on the basis of the Taylor expansion of $\kappa(t) - tK$ around $t = T$. Here we provide a simpler and more statistically-oriented derivation that employs Esscher tilting and the Edgeworth expansion. Rates of convergence for the approximations are readily available with our approach in the i.i.d. case.

For now we assume that the saddlepoint $t = T$ that solves $\kappa'(t) = K$ is positive. The expectation $\mathbb{E}[(X-K)^+]$ is reformulated under an exponentially tilted probability measure,

$$\mathbb{E}\left[(X-K)^+\right] = \int_K^\infty (x-K)f(x)dx = e^{-\frac{W^2}{2}} \int_K^\infty (x-K)e^{-T(x-K)}\tilde{f}(x)dx, \tag{17}$$

where $\kappa'(T) = K$ and $\tilde{f}(x) = f(x)\exp(Tx - \kappa(T))$. The MGF associated with $\tilde{f}(x)$ is given by $\tilde{M}(t) = M(T+t)/M(T)$. It immediately follows that the mean and variance of a random variable $\tilde{X}$ with density $\tilde{f}(\cdot)$ are given by $E\tilde{X} = K$ and $Var(\tilde{X}) = \kappa''(T)$. Writing $\xi = (x-K)/\sqrt{\kappa''(T)}$, $Z = T\sqrt{\kappa''(T)}$ and $\tilde{f}(x)dx = g(\xi)d\xi$, (17) reads

$$\mathbb{E}\left[(X-K)^+\right] = e^{-\frac{W^2}{2}}\sqrt{\kappa''(T)} \int_0^\infty \xi e^{-Z\xi} g(\xi)d\xi. \tag{18}$$

Suppose that $g(\xi)$ is approximated by a normal distribution $\phi(\cdot)$. The integral in (18) then becomes

$$\int_0^\infty \xi e^{-Z\xi} g(\xi)d\xi \approx \frac{\exp(\frac{Z^2}{2})}{\sqrt{2\pi}} \int_0^\infty \xi e^{-\frac{(\xi+Z)^2}{2}} d\xi = \frac{1}{\sqrt{2\pi}} - Ze^{\frac{Z^2}{2}}\left[1 - \Phi(Z)\right]. \tag{19}$$

Inserting (19) in (18) leads to the approximation denoted by C_1,

$$\mathbb{E}\left[(X-K)^+\right] \approx e^{-\frac{W^2}{2}} \left\{\sqrt{\kappa''(T)/(2\pi)} - T\kappa''(T)e^{\frac{Z^2}{2}}\left[1-\Phi(Z)\right]\right\} =: C_1. \tag{20}$$

Higher order terms enter if $g(\xi)$ is approximated by its Edgeworth expansion $g(\xi) \approx \phi(\xi)[1 + \frac{\lambda_3}{6}(\xi^3 - 3\xi)]$. Then

$$\begin{aligned}
\mathbb{E}\left[(X-K)^+\right] &\approx C_1 + e^{-\frac{W^2}{2}}\sqrt{\kappa''(T)}\frac{\lambda_3}{6}\int_0^\infty \xi e^{-Z\xi}\phi(\xi)(\xi^3 - 3\xi)d\xi \\
&= C_1 + e^{-\frac{W^2}{2}}\sqrt{\kappa''(T)}\frac{\lambda_3}{6}\frac{e^{\frac{Z^2}{2}}}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{(\xi+Z)^2}{2}}\left(-\xi^4 + 3\xi^2\right)d\xi \\
&= C_1 + e^{\frac{Z^2}{2} - \frac{W^2}{2}}\sqrt{\kappa''(T)}\frac{\lambda_3}{6}\left\{[1-\Phi(Z)](Z^4 + 3Z^2) - \phi(Z)(Z^3 + 2Z)\right\} =: C_2. \qquad (21)
\end{aligned}$$

The approximations C_1 and C_2 are in agreement with the formulas given by Antonov et al. (2005).

Generally, bounds of relative error are not available for the above approximations. However in case that X_i, $i = 1, \ldots, n$, are i.i.d. random variables, it is known that

$$\begin{aligned}
g(\xi) &= \phi(\xi)[1 + O(n^{-1/2})], \\
g(\xi) &= \phi(\xi)\left[1 + \frac{\lambda_3}{6}(\xi^3 - 3\xi) + O(n^{-1})\right].
\end{aligned}$$

So, the rates of convergence of C_1 and C_2 are of the order $n^{-1/2}$ and n^{-1}, respectively.

Negative saddlepoint

We have assumed that the saddlepoint is positive, when deriving C_1 and C_2 in (20) and (21), or, in other words, $\mu < K$. If the saddlepoint T equals 0, or equivalently, $\mu = K$, it is straightforward to see that C_1 and C_2 both reduce to the following formula,

$$\mathbb{E}[(X-\mu)^+] = \sqrt{\frac{\kappa''(0)}{2\pi}} =: C_0. \tag{22}$$

In case that $\mu > K$, we should work with $Y = -X$ and $\mathbb{E}[Y 1_{\{Y \geq -K\}}]$ instead since

$$\mathbb{E}[X 1_{\{X \geq K\}}] = \mu + \mathbb{E}[-X 1_{\{-X \geq -K\}}] = \mu + \mathbb{E}[Y 1_{\{Y \geq -K\}}].$$

The CGF of Y is given by $\kappa_Y(t) = \kappa_X(-t)$. The saddlepoint that solves $\kappa_Y(t) = -K$ is $-T > 0$ so that C_1 and C_2 can again be used. Note that

$$\kappa_Y^{(r)}(t) = (-1)^r \kappa_X^{(r)}(-t),$$

where the superscript (r) denotes the r-th derivative. Therefore, in the case of a negative saddlepoint, $\mathbb{E}[(X-K)^+]$ can be approximated by

$$C_1^- = \mu - K + e^{-\frac{W^2}{2}} \left\{ \sqrt{\kappa''(T)/(2\pi)} + T\kappa''(T) e^{\frac{Z^2}{2}} \Phi(Z) \right\}, \tag{23}$$

$$C_2^- = C_1^- - e^{\frac{Z^2}{2} - \frac{W^2}{2}} \sqrt{\kappa''(T)} \frac{\lambda_3}{6} \left\{ \Phi(Z)(Z^4 + 3Z^2) + \phi(Z)(Z^3 + 2Z) \right\}. \tag{24}$$

Log-return model revisited

It is also possible to stay with the original probability when approximating the expected shortfall in the log-return model in Studer (2001). We work with $\mathbb{E}\left[e^X 1_{\{X \geq K\}}\right]$ which equals $\mathbb{E}\left[e^X | X \geq K\right] \mathbb{P}(X \geq K)$. Replace x in (17) by e^x and make the same change of variables,

$$\mathbb{E}\left[e^X 1_{\{X \geq K\}}\right] = e^{-\frac{W^2}{2}} \int_0^\infty e^{K + \xi\sqrt{\kappa''(T)}} e^{-Z\xi} g(\xi) d\xi.$$

Approximating $g(\xi)$ by the standard normal density, we obtain

$$\begin{aligned} \mathbb{E}\left[e^X 1_{\{X \geq K\}}\right] &= e^{-\frac{W^2}{2} + K + \frac{\dot{Z}^2}{2}} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{(\xi + \dot{Z})^2}{2}} d\xi \\ &= e^{-\frac{W^2}{2} + K + \frac{\dot{Z}^2}{2}} [1 - \Phi(\dot{Z})], \end{aligned} \tag{25}$$

where $\dot{Z} = (T-1)\sqrt{\kappa''(T)}$. Eq (25) is basically $e^K P_1$, where P_1 is given by (8), with Z replaced by $\dot{Z}$. It is easy to verify that this approximation is exact when X is normally distributed. A higher order approximation would be

$$\mathbb{E}\left[e^X 1_{\{X \geq K\}}\right] = e^{-\frac{W^2}{2} + K + \frac{\dot{Z}^2}{2}} \left\{ [1 - \Phi(\dot{Z})] \left(1 - \frac{\lambda_3}{6}\dot{Z}^3\right) + \frac{\lambda_3}{6}\phi(\dot{Z})(\dot{Z}^2 - 1) \right\}.$$

5 The Lugannani-Rice type formulas

The second type of saddlepoint approximations to $\mathbb{E}[(X-K)^+]$ can be derived in a very similar way as was done in section 4 of Daniels (1987) where the Lugannani-Rice formula to tail probability was derived. As a result we shall call the obtained formulas "Lugannani-Rice type formulas".

To start, we derive the following inversion formula for $\mathbb{E}\left[X 1_{\{X \geq K\}}\right]$.

Theorem 1. *Let $\kappa(t) = \log M(t)$ be the cumulant generating function of a continuous random variable X. Then*

$$\mathbb{E}\left[X1_{\{X\geq K\}}\right] = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \kappa'(t)\frac{\exp(\kappa(t)-tK)}{t}dt \quad (\tau>0). \tag{26}$$

Proof. We start with the case that X has a positive lower bound. Employing the same change of measure as in (14), we have $E\left[X1_{\{X\geq K\}}\right] = \mu\mathbb{Q}(X\geq K)$, where

$$\mathbb{Q}(X\geq K) = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \frac{\exp(\kappa_{\mathbb{Q}}(t)-tK)}{t}dt \quad (\tau>0).$$

Plug in $\kappa_{\mathbb{Q}}(t)$, which is given by (15), we find

$$\begin{aligned}\mathbb{E}\left[X1_{\{X\geq K\}}\right] &= \mu\frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \frac{\exp\left[\kappa(t)+\log\kappa'(t)-\log\mu-tK\right]}{t}dt\\ &= \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \kappa'(t)\frac{\exp(\kappa(t)-tK)}{t}dt.\end{aligned}$$

In the case that X has a negative lower bound, $-a$, with $a>0$, we define $Y = X+a$ so that Y has a positive lower bound. Then, the CGF of Y and its first derivative are given by $\kappa_Y(t) = \kappa(t)+ta$ and $\kappa_Y'(t) = \kappa'(t)+a$, respectively. Since

$$\mathbb{E}\left[X1_{\{X\geq K\}}\right] = \mathbb{E}\left[(Y-a)1_{\{Y-a\geq K\}}\right] = \mathbb{E}\left[Y1_{\{Y-a\geq K\}}\right] - a\mathbb{P}(Y-a\geq K),$$

and

$$\mathbb{E}\left[Y1_{\{Y-a\geq K\}}\right] = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \kappa'(t)\frac{\exp(\kappa(t)-tK)}{t}dt + a\mathbb{P}(Y-a\geq K),$$

we are again led to (26). Extension to variables bounded from above is straightforward.

For unbounded X, we take $X_L = \max(X, L)$, where $L < -1/\tau$ is a constant. Since X_L is bounded from below, we have

$$\begin{aligned}\mathbb{E}\left[X_L1_{\{X_L\geq K\}}\right] &= \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \kappa_{X_L}'(t)\frac{\exp(\kappa_{X_L}(t)-tK)}{t}dt,\\ &= \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} M_{X_L}'(t)\frac{\exp(-tK)}{t}dt,\end{aligned} \tag{27}$$

where $M_{X_L}'(\tau) = M'(\tau)+\int_{-\infty}^{L}(Le^{\tau L}-xe^{\tau x})d\mathbb{P}(x)$. For $L<-1/\tau$, $M_{X_L}'(\tau)$ increases monotonically as L decreases and approaches $M'(\tau)$ as $L\to-\infty$. Note also that $\mathbb{E}\left[X1_{\{X\geq K\}}\right] = \mathbb{E}\left[X_L1_{\{X_L\geq K\}}\right]$ for all $L<K$. Now take the limit of both sides of (27) as $L\to-\infty$. Due to the monotone convergence theorem, we again obtain

$$\begin{aligned}\mathbb{E}\left[X1_{\{X\geq K\}}\right] &= \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} M'(t)\frac{\exp(-tK)}{t}dt\\ &= \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \kappa'(t)\frac{\exp(\kappa(t)-tK)}{t}dt.\end{aligned}$$

□

Now, we follow Daniels (1987) to approximate $\kappa(t)-tK$ over an interval containing both $t=0$ and $t=T$ by a quadratic function. Here, T need not to be positive any more. Recall from section

2 that $-\frac{1}{2}W^2 = \kappa(T) - TK$ with W taking the same sign as T. Let w be defined between 0 and W such that

$$\frac{1}{2}(w - W)^2 = \kappa(t) - tK - \kappa(T) + TK. \tag{28}$$

Then we have

$$\frac{1}{2}w^2 - Ww = \kappa(t) - t\kappa'(T), \tag{29}$$

and $t = 0 \Leftrightarrow w = 0$, $t = T \Leftrightarrow w = W$. Differentiate both sides of (29) once and twice to obtain

$$w\frac{dw}{dt} - W\frac{dw}{dt} = \kappa'(t) - \kappa'(T), \quad \left(\frac{dw}{dt}\right)^2 + (w - W)\frac{d^2w}{dt^2} = \kappa''(t).$$

So, in the neighborhood of $t = T$ (or, equivalently, $w = W$) we have $\frac{dw}{dt} = \sqrt{\kappa''(T)}$. Note that $\mu = \mathbb{E}[X] = \kappa'(0)$. In the neighborhood of $t = 0$ (or, equivalently, $w = 0$), we have

$$\frac{dw}{dt} = \sqrt{\kappa''(0)} \quad \text{if } T = 0, \tag{30}$$

$$\frac{dw}{dt} = \frac{\kappa'(T) - \kappa'(0)}{W} = \frac{K - \mu}{W} \quad \text{if } T \neq 0.$$

Hence, in the neighborhood of $t = 0$ we have $w \propto t$. Moreover,

$$\frac{1}{t}\frac{dt}{dw} \sim \frac{1}{w}, \quad \frac{\kappa'(t)}{t}\frac{dt}{dw} \sim \frac{\mu}{w}. \tag{31}$$

Based on Theorem 1, the inversion formula for $\mathbb{E}\left[X1_{\{X\geq K\}}\right]$ can be formulated to be

$$\begin{aligned}
\mathbb{E}\left[X1_{\{X\geq K\}}\right] &= \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \kappa'(t)e^{\frac{1}{2}w^2-Ww}\frac{1}{t}\frac{dt}{dw}dw \\
&= \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} e^{\frac{1}{2}w^2-Ww}\left[\frac{\mu}{w} + \frac{\kappa'(t)}{t}\frac{dt}{dw} - \frac{\mu}{w}\right]dw \\
&= \mu\int_{\tau-i\infty}^{\tau+i\infty} \frac{1}{2\pi i}e^{\frac{1}{2}w^2-Ww}\frac{dw}{w} + \frac{1}{2\pi i}\int_{W-i\infty}^{W+i\infty} e^{\frac{1}{2}w^2-Ww}\left[\frac{\kappa'(t)}{t}\frac{dt}{dw} - \frac{\mu}{w}\right]dw.
\end{aligned} \tag{32}$$

The first integral takes the value $1 - \Phi(W)$. The second integral has no singularity because of (31). Hence there is no problem to change the integration contour from the imaginary axis along $\tau > 0$ to that along W, as done in (32), not even if W and T are negative. The major contribution to the second integral comes from the saddlepoint. The terms in the brackets are expanded around T separately. Only taking the leading terms into account we obtain

$$\frac{\kappa'(t)}{t}\frac{dt}{dw} - \frac{\mu}{w} \approx \frac{\kappa'(t)}{t}\frac{dt}{dw}\Big|_T - \frac{\mu}{w}\Big|_W = \frac{K}{Z} - \frac{\mu}{W}.$$

Therefore we are led to

$$\mathbb{E}\left[X1_{\{X\geq K\}}\right] \approx \mu\left[1 - \Phi(W)\right] + \phi(W)\left[\frac{K}{Z} - \frac{\mu}{W}\right] =: J_3. \tag{33}$$

Subtracting $K\mathbb{P}(X \geq K)$ from J_3 with the tail probability approximated by the Lugannani-Rice formula P_3, we see immediately that

$$\mathbb{E}\left[(X - K)^+\right] \approx (\mu - K)\left[1 - \Phi(W) - \frac{\phi(W)}{W}\right] =: C_3. \tag{34}$$

This is a surprisingly neat formula requiring only knowledge of W. A more statistical approach to derive the approximation J_3 in (33) can be found in Appendix A.

Now consider the higher order approximation. Write $U := \kappa''(T)T - \kappa'(T)$. The Taylor expansion of $\kappa'(t)/t$ around T gives

$$\frac{\kappa'(t)}{t} \approx \frac{\kappa'(T)}{T} + (t-T)\frac{U}{T^2} + \frac{(t-T)^2}{2}\left[\frac{\kappa'''(T)}{T} - \frac{2U}{T^3}\right]. \tag{35}$$

By the Taylor expansion on the line $t = T + iy$, we have

$$\begin{aligned}
G :=& \frac{1}{2\pi i}\int_{T-i\infty}^{T+i\infty} e^{\kappa(t)-tK}\frac{\kappa'(t)}{t}dt \\
\approx& \frac{e^{-\frac{W^2}{2}}}{2\pi i}\int_{T-i\infty}^{T+i\infty} e^{\frac{1}{2}\kappa''(T)(t-T)^2}\left[1 + \frac{1}{6}\kappa'''(T)(t-T)^3 + \frac{1}{24}\kappa^{(4)}(T)(t-T)^4 + \right. \\
&\left. + \frac{1}{72}\kappa'''(T)^2(t-T)^6\right]\left\{\frac{\kappa'(T)}{T} + (t-T)\frac{U}{T^2} + \frac{(t-T)^2}{2}\left[\frac{\kappa'''(T)}{T} - \frac{2U}{T^3}\right]\right\}dt \\
=& \frac{e^{-\frac{W^2}{2}}}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\kappa''(T)y^2}\left[1 - \frac{1}{6}\kappa'''(T)iy^3 + \frac{1}{24}\kappa^{(4)}(T)y^4 + \right. \\
&\left. - \frac{1}{72}\kappa'''(T)^2y^6\right]\left\{\frac{\kappa'(T)}{T} + iy\frac{U}{T^2} - \frac{y^2}{2}\left[\frac{\kappa'''(T)}{T} - \frac{2U}{T^3}\right]\right\}dy \\
=& \phi(W)\left[\frac{\kappa'(T)}{Z} + \frac{\kappa'(T)}{Z}\left(\frac{\lambda_4}{8} - \frac{5}{24}\lambda_3^2\right) + \frac{U\lambda_3}{2Z^2} - \frac{\lambda_3}{2T} + \frac{U}{Z^3}\right] \\
=& \phi(W)\left[\frac{K}{Z}\left(1 + \frac{\lambda_4}{8} - \frac{5}{24}\lambda_3^2\right) + \frac{1}{TZ} - \frac{K\lambda_3}{2Z^2} - \frac{K}{Z^3}\right] =: G_1.
\end{aligned}$$

Notice that G_1 is itself a saddlepoint approximation to $E[X1_{\{X\geq K\}}]$ for $K > \mu$. However, it becomes inaccurate when T approaches zero due to the presence of a pole at zero in the integrand. Meanwhile expanding $1/w$ around W gives

$$\begin{aligned}
H :=& \frac{e^{-\frac{W^2}{2}}}{2\pi i}\int_{W-i\infty}^{W+i\infty} e^{\frac{1}{2}(w-W)^2}\frac{\mu}{w}dw \\
\approx& \frac{\mu e^{-\frac{W^2}{2}}}{2\pi i}\int_{W-i\infty}^{W+i\infty} e^{\frac{1}{2}(w-W)^2}\left[\frac{1}{W} + \frac{(w-W)}{W^2} + \frac{(w-W)^2}{W^3}\right]dw \\
=& \mu\phi(W)\left(\frac{1}{W} - \frac{1}{W^3}\right) =: H_1.
\end{aligned}$$

Finally we obtain the higher order version of the Lugannani-Rice type formulas as follows,

$$J_4 \approx \mu\left[1 - \Phi(W)\right] + G_1 - H_1, \tag{36}$$

$$C_4 \approx C_3 + \phi(W)\left[\frac{1}{TZ} + (\mu - K)\frac{1}{W^3}\right]. \tag{37}$$

To study the error convergence of C_3 and C_4 when X_i, $i = 1, \ldots, n$, are i.i.d. random variables, we look at $E[(X - nx)^+]$ for fixed x. Combining (2) and (32) we obtain

$$\mathbb{E}[(X - nx)^+] = n\mu_1\left[1 - \Phi(W)\right] + G - H - nx\mathbb{P}(X \geq nx).$$

Let $\kappa_1(t)$ be the CGF of X_1 and $\lambda_{1,r}$ be the r-th standardized cumulant of X_1. Since $\kappa(t) = n\kappa_1(t)$, the saddlepoint T that solves $\kappa(T) = nx$ is also a solution of $\kappa_1(t) = x$. Now write $\mu_1 := \mathbb{E}[X_1]$,

$Z_1 := T\sqrt{\kappa_1(T)}$ and $W_1 := sgn(T)\sqrt{2[xT - \kappa_1(T)]}$. It is obvious that $\mu = n\mu_1$, $Z = \sqrt{n}Z_1$, $W = \sqrt{n}W_1$, $\lambda_3 = \lambda_{1,3}/\sqrt{n}$ and $\lambda_4 = \lambda_{1,4}/n$.

Apparently, the remainder term $H - H_1$ is of the order $O(n^{-3/2})$. Less obvious but by term-by-term multiplication and integration we find that the remainder term $G - G_1$ is also $O(n^{-3/2})$. More precisely, we have,

$$G = \phi(W)\left[\sqrt{n}\frac{x}{Z_1} + \frac{1}{\sqrt{n}}\left(\frac{x\lambda_{1,4}}{8Z_1} - \frac{5x\lambda_{1,3}^2}{24Z_1} + \frac{1}{TZ_1} - \frac{x\lambda_{1,3}}{2Z_1^2} - \frac{x}{Z_1^3}\right) + O\left(n^{-3/2}\right)\right],$$

$$H = \mu_1\phi(W)\left[\frac{\sqrt{n}}{W_1} - \frac{1}{\sqrt{n}W_1^3} + O\left(n^{-3/2}\right)\right].$$

In addition, from Daniels (1987), we know

$$\mathbb{P}(X \geq nx) = [1 - \Phi(W)] + \phi(W)\left\{n^{-1/2}\left(\frac{1}{Z_1} - \frac{1}{W_1}\right) + n^{-3/2}\left[\frac{1}{Z_1}\left(\frac{\lambda_{1,4}}{8} - \frac{5\lambda_{1,3}^2}{24}\right) - \frac{\lambda_{1,3}}{2Z_1^2} - \frac{1}{Z_1^3} + \frac{1}{W_1^3}\right] + O\left(n^{-5/2}\right)\right\}.$$

Substitution of above three formulas gives

$$\mathbb{E}[(X - nx)^+] = (\mu_1 - x)\left\{n[1 - \Phi(W)] + \sqrt{n}\frac{\phi(W)}{W_1}\right\} + \phi(W)\left\{\frac{1}{\sqrt{n}}\left[\frac{1}{TZ_1} + \frac{\mu_1 - x}{W_1^3}\right] + O\left(n^{-3/2}\right)\right\}. \tag{38}$$

It follows that the rates of convergence of C_3 and C_4 in (34) and (37) are of order $O(n^{-1/2})$ and $O(n^{-3/2})$, respectively. Notice that in the problem considered here, the rates of convergence of C_1 and C_2, in (20) and (21), are of order $O(1)$ and $O(n^{-1/2})$, respectively, since in eq. (18) the term $\sqrt{\kappa''(T)}$ in front of the integral should be written as $\sqrt{n\kappa_1''(T)}$, which gives rise to an additional $\sqrt{n}$ term.

Remark 2. Interestingly, Martin (2006) gives an approximation formula for $\mathbb{E}[(X - K)^+]$, decomposing the expectation to one term involving the tail probability and another term involving the probability density,

$$\mathbb{E}\left[(X - K)^+\right] \approx (\mu - K)\mathbb{P}(X \geq K) + \frac{K - \mu}{T} f_X(K).$$

Martin (2006) suggests to approximate $\mathbb{P}(X \geq K)$ by the Lugannani-Rice formula P_3 in (10) and $f_X(K)$ by the Daniels formula f_D in (7). In the i.i.d. case, this leads to an approximation $C_M := (\mu - K)P_3 + (K - \mu)f_D/T$ with a rate of convergence $n^{-1/2}$ as the first term has an error of order $n^{-1/2}$ and the second term has an error of order $n^{-3/2}$. We propose to replace P_3 by its higher order version, P_4 in (11). This gives the following formula,

$$\mathbb{E}\left[(X - K)^+\right] \approx C_3 + (\mu - K)\phi(W)\left(\frac{1}{W^3} - \frac{\lambda_3}{2Z^2} - \frac{1}{Z^3}\right). \tag{39}$$

Not only eq. (39) is simpler than C_M as λ_4 is not involved, but also it has a higher rate of convergence of order $n^{-3/2}$. However compared to C_4 eq. (39) contains a term of λ_3 and is certainly more complicated to evaluate. Note further that if we neglect in C_M the terms of the higher order standard cumulants λ_3 and λ_4 in f_D we get precisely C_3 as given in (34). For these reasons, C_4 is to be preferred.

Zero saddlepoint

It is mentioned in Daniels (1987) that in case that the saddlepoint $T = 0$, or in other words, $\mu = K$, the approximations to tail probability P_1 to P_4 all reduce to

$$\mathbb{P}(X \geq K) = \frac{1}{2} - \frac{\lambda_3(0)}{6\sqrt{2\pi}}.$$

We would like to show that, under the same circumstances, C_3 and C_4 also reduce to the formula C_0 in (22). To show that $C_3 = C_0$ when $T = 0$, we point out that

$$\lim_{T\to 0} C_3 = \lim_{T\to 0} \frac{\kappa'(0) - \kappa'(T)}{T} \left[T(1 - \Phi(W)) - \phi(W)\frac{T}{W} \right].$$

Note that when $T \to 0$, $\frac{\kappa'(0)-\kappa'(T)}{T} \to -\kappa''(0)$, $T(1 - \Phi(W)) \to 0$ and $\frac{T}{W} \to [\kappa''(0)]^{-\frac{1}{2}}$ (see (30)). This implies that $\lim_{T\to 0} C_3 = C_0$. Similarly we also have $\lim_{T\to 0} C_4 = C_0$.

6 Lattice variables

So far we have only considered approximations to continuous variables. Let us now turn to the lattice case. This is largely ignored in the literature, even in applications where lattice variables are much more relevant. For example, in the pricing of CDOs, the random variable concerned is essentially the number of defaults in the pool of companies and is thus discrete.

Suppose that $\hat{X}$ only takes integer values k with nonzero probabilities $p(k)$. The inversion formula of $\mathbb{E}[(\hat{X} - K)^+]$ can then be formulated as

$$\begin{aligned}
\mathbb{E}[(\hat{X} - K)^+] &= \sum_{k=K+1}^{\infty} (k - K)p(k) = \sum_{k=K+1}^{\infty} (k - K)\frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \exp(\kappa(t) - tk)dt \\
&= \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \exp(\kappa(t) - tK)\sum_{m=1}^{\infty} me^{-tm}dt \\
&= \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \frac{\exp(\kappa(t) - tK)}{t^2}\frac{t^2e^{-t}}{(1 - e^{-t})^2}dt.
\end{aligned}$$

Expanding the two terms in the integrand separately, we find, for lattice variables, the following formulas corresponding to C_1 and C_2 in (20) and (21), respectively,

$$\hat{C}_1 = C_1\frac{T^2e^{-T}}{(1 - e^{-T})^2}, \tag{40}$$

$$\begin{aligned}
\hat{C}_2 &= C_2\frac{T^2e^{-T}}{(1 - e^{-T})^2} \\
&\quad + e^{-\frac{W^2}{2}+\frac{Z^2}{2}}\{\phi(Z) - Z[(1 - \Phi(Z)]\}\frac{Te^{-T}\left(2 - T - 2e^{-T} - Te^{-T}\right)}{\sqrt{\kappa''(T)}(1 - e^{-T})^3}.
\end{aligned} \tag{41}$$

For the approximations to $\mathbb{E}[\hat{X}|\hat{X} \geq K]$, we also need the lattice version for the tail probability

$$\mathbb{P}(\hat{X} \geq K) = e^{-\frac{W^2}{2}+\frac{Z^2}{2}}[1 - \Phi(Z)]\frac{T}{1 - e^{-T}} =: \hat{P}_1 \tag{42}$$

or its higher order version

$$\mathbb{P}(\hat{X} \geq K) = e^{-\frac{W^2}{2}+\frac{Z^2}{2}} \frac{T}{1-e^{-T}} \times \left\{ [1-\Phi(Z)] \left(2 - \frac{\lambda_3}{6} Z^3 - \frac{T}{e^T - 1} \right) \right.$$
$$\left. + \phi(Z) \left[\frac{\lambda_3}{6}(Z^2-1) + \frac{1}{Z} - \frac{T}{Z(e^T-1)} \right] \right\} =: \hat{P}_2. \quad (43)$$

Recall that the Lugannani-Rice formula for lattice variables reads

$$\mathbb{P}(\hat{X} \geq K) = 1 - \Phi(W) + \phi(W) \left[\frac{1}{\hat{Z}} - \frac{1}{W} \right] =: \hat{P}_3, \quad (44)$$

where $\hat{Z} = (1-e^{-T})\sqrt{\kappa''(T)}$. A similar lattice formula can also be obtained for J_3, which we will denote by $\hat{J}_3$. We first write down the inversion formula of the tail probability of a lattice variable,

$$\mathbb{Q}(\hat{X} \geq K) = \sum_{k=K}^{\infty} \mathbb{Q}(\hat{X} = k) = \frac{1}{2\pi i} \int_{\tau - i\infty}^{\tau + i\infty} \frac{\exp(\kappa_{\mathbb{Q}}(t) - tK)}{1-e^{-t}} dt. \quad (45)$$

Combining (45) with Theorem 1, we obtain

$$\mathbb{E}\left[\hat{X} 1_{\{\hat{X} \geq K\}}\right] = \frac{1}{2\pi i} \int_{\tau - i\infty}^{\tau + i\infty} \kappa'(t) \frac{\exp(\kappa(t) - tK)}{1-e^{-t}} dt.$$

By the same change of variables as in section 5, we have

$$\begin{aligned}\mathbb{E}\left[\hat{X} 1_{\{\hat{X} \geq K\}}\right] &= \frac{1}{2\pi i} \int_{\tau - i\infty}^{\tau + i\infty} \kappa'(t) e^{\frac{1}{2}w^2 - Ww} \frac{1}{1-e^{-t}} \frac{dt}{dw} dw \\ &= \frac{1}{2\pi i} \int_{\tau - i\infty}^{\tau + i\infty} e^{\frac{1}{2}w^2 - Ww} \left[\frac{\mu}{w} + \frac{\kappa'(t)}{1-e^{-t}} \frac{dt}{dw} - \frac{\mu}{w} \right] dw.\end{aligned}$$

Now we can proceed exactly as in section 5 as $\lim_{t \to 0} 1 - e^{-t} = t$. This leads to

$$\hat{J}_3 = \mu [1 - \Phi(W)] + \phi(W) \left[\frac{K}{\hat{Z}} - \frac{\mu}{W} \right], \quad (46)$$

$$\hat{C}_3 = (\mu - K) \left[1 - \Phi(W) - \frac{\phi(W)}{W} \right] \equiv C_3. \quad (47)$$

Including higher order terms we obtain

$$\hat{C}_4 = \hat{C}_3 + \phi(W) \left[\frac{e^{-T}}{\hat{Z}(1-e^{-T})} + (\mu - K) \frac{1}{W^3} \right]. \quad (48)$$

A higher order version of $\hat{P}_3$ can be derived similarly,

$$\mathbb{P}(\hat{X} \geq K) = 1 - \Phi(W) + \phi(W) \left[\frac{1}{\hat{Z}} \left(1 + \frac{\lambda_4}{8} - \frac{5\lambda_3^2}{24} \right) \right.$$
$$\left. - \frac{e^{-T}\lambda_3}{2\hat{Z}^2} - \frac{e^{-T}(1+e^{-T})}{2\hat{Z}^3} - \frac{1}{W} + \frac{1}{W^3} \right] =: \hat{P}_4. \quad (49)$$

This can be used to estimate $\mathbb{E}[\hat{X} | \hat{X} \geq K]$.

The rates of convergence of $\hat{C}_1$ to $\hat{C}_4$ in the i.i.d. case are identical to their non-lattice counterparts and we shall not elaborate further.

7 Numerical results

By two numerical experiments we evaluate the quality of the various approximations that are derived in the earlier sections.

In our first example $X = \sum X_i$ where X_i are i.i.d. *exponentially* distributed with density $p(x) = e^{-x}$. The CGF of X reads $\kappa(t) = -n \log(1-t)$. The saddlepoint to $\kappa'(t) = K$ is given by $T = 1 - n/K$. Moreover, we have

$$\kappa''(T) = \frac{K^2}{n}, \lambda_3 = \frac{2}{\sqrt{n}}, \lambda_4 = \frac{6}{n}.$$

Their exact values are available as $X \sim Gamma(n, 1)$. The tail probability is then given by

$$\mathbb{P}(X \geq K) = 1 - \frac{\gamma(n, K)}{\Gamma(n)},$$

and

$$\mathbb{E}[X 1_{\{X \geq K\}}] = n \left[1 - \frac{\gamma(n+1, K)}{\Gamma(n+1)}\right],$$

where Γ and γ are the gamma function and the incomplete gamma function, respectively.

In the second example we set $X = \sum X_i$ where X_i are i.i.d. *Bernoulli* variables with $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = p = 0.15$. Its CGF is given by $\kappa(t) = n \log(1 - p + pe^t)$. Here the saddlepoint to $\kappa'(t) = K$ equals $T = \log\left[\frac{K(1-p)}{(n-K)p}\right]$ and

$$\kappa''(T) = \frac{K(n-K)}{n}, \lambda_3 = \frac{n-2K}{\sqrt{nK(n-K)}}, \lambda_4 = \frac{n^2 - 6nK + 6K^2}{nK(n-K)}.$$

In this specific case, X is binomially distributed with

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k},$$

which means that C and S as defined in (4) can also be calculated exactly.

We report in Tables 1 and 2 on the approximations obtained in the exponential case and in Tables 3 and 4 approximations in the Bernoulli case. For the approximations to S we take $S_r = C_r/P_r + K$ for $r = 1, 2, 3, 4$. The saddlepoint approximations in the Bernoulli case are based on the formulas for lattice variables derived in section 6.

In general we see that all approximations work remarkably well in our experiments. The higher order Lugannani-Rice type formula, S_4, C_4 and their lattice versions, produce almost exact approximations. Particularly worth mentioning is the quality of approximations C_4 and $\hat{C}_4$, that use the same information as C_1 and $\hat{C}_1$, but show errors that are significantly smaller than C_2 and $\hat{C}_2$.

8 Conclusions

We have derived two types of saddlepoint approximations to $\mathbb{E}[(X-K)^+]$ and $\mathbb{E}[X|X \geq K]$, where X is the sum of n independent random variables and K is a known constant. For each type of approximation, we have given a lower order version and a higher order version. We have also established the error convergence rates for the approximations in the i.i.d. case. The approximations have been further extended to cover the case of lattice variables. Numerical examples show that all these approximations work remarkably well. The Lugannani-Rice type formulas to $\mathbb{E}[(X-K)^+]$ are particularly attractive because of their simplicity.

K	Exact	C_1	C_2	C_3	C_4
105	2.0331	2.0852	2.0341	2.0360	2.0331
115	3.5773e-1	3.7292e-1	3.5743e-1	3.5892e-1	3.5773e-1
125	3.7283e-2	3.8873e-2	3.7240e-2	3.7508e-2	3.7283e-2
135	2.3657e-3	2.4574e-3	2.3635e-3	2.3881e-3	2.3657e-3
145	9.5270e-5	9.8546e-5	9.5210e-5	9.6553e-5	9.5269e-5

Table 1: Exact values of $\mathbb{E}[(X-K)^+]$ and their saddlepoint approximations. $X = \sum_{i=1}^n X_i$ where X_i is exponentially distributed with density $f(x) = e^{-x}(x \geq 0)$ and $n = 100$.

K	Exact	S_1	S_2	S_3	S_4
105	111.7826	111.7313	111.7883	111.7924	111.7826
115	119.9954	120.0448	119.9937	120.0120	119.9954
125	128.9751	129.0343	128.9715	128.9990	128.9571
135	138.3421	138.3938	138.3389	138.3737	138.3421
145	147.9199	147.9626	147.9175	147.9592	147.9199

Table 2: Exact values of $\mathbb{E}[X|X \geq K]$ and their saddlepoint approximations. $X = \sum_{i=1}^n X_i$ where X_i is exponentially distributed with density $f(x) = e^{-x}(x \geq 0)$ and $n = 100$.

K	Exact	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$	$\hat{C}_4$
18	4.2046e-1	4.3660e-1	4.2330e-1	4.2579e-1	4.2045e-1
20	1.5110e-1	1.5757e-1	1.5217e-1	1.5397e-1	1.5109e-1
23	2.3355e-2	2.4313e-2	2.3529e-2	2.4075e-2	2.3353e-2
25	5.3880e-3	5.5924e-3	5.4279e-3	5.6041e-3	5.3874e-3
28	4.2976e-4	4.4395e-4	4.3281e-4	4.5375e-4	4.2969e-4

Table 3: Exact values of $\mathbb{E}[(X-K)^+]$ and their saddlepoint approximations. $X = \sum_{i=1}^n X_i$ where X_i is X_i is Bernoulli distributed with $p(X_i = 1) = 0.15$ and $n = 100$.

K	Exact	$\hat{S}_1$	$\hat{S}_2$	$\hat{S}_3$	$\hat{S}_4$
18	19.7762	19.9213	19.7874	19.7984	19.7761
20	21.4184	21.5218	21.4276	21.4448	21.4181
23	24.0548	24.1191	24.0619	24.0870	24.0547
25	25.8861	25.9340	25.8919	25.9213	25.8860
28	28.7012	28.7330	28.7053	28.7400	28.7011

Table 4: Exact values of $\mathbb{E}[X|X \geq K]$ and their saddlepoint approximations. $X = \sum_{i=1}^n X_i$ where X_i is Bernoulli distributed with $p(X_i = 1) = 0.15$ and $n = 100$.

A Alternative derivation of J_3

The approximation J_3 in (33) can also be derived by a more statistical approach. Let us replace the density of X by its saddlepoint approximation (7), we then obtain

$$\mathbb{E}\left[X1_{\{X\geq K\}}\right] \approx \frac{1}{\sqrt{2\pi}}\int_K^\infty x\frac{e^{\kappa(t)-xt}}{\sqrt{\kappa''(t)}}\left[1+\frac{\lambda_4(t)}{8}-\frac{5\lambda_3(t)^2}{24}\right]dx \tag{50}$$

where $x=\kappa'(t)$. Let again $K=\kappa'(T)$. A change of variables from x to t gives

$$\mathbb{E}\left[X1_{\{X\geq K\}}\right] \approx \frac{1}{\sqrt{2\pi}}\int_T^\infty \kappa'(t)\sqrt{\kappa''(t)}e^{\kappa(t)-\kappa'(t)t}\left[1+\frac{\lambda_4(t)}{8}-\frac{5\lambda_3(t)^2}{24}\right]dt$$

Let $w^2/2=\kappa'(t)t-\kappa(t)$ and $W^2/2=\kappa'(T)T-\kappa(T)$ so that $wdw=t\kappa''(t)dt$, $t=0\Leftrightarrow w=0$, $t=T\Leftrightarrow w=W$. A second change of variables from t to w gives

$$\mathbb{E}\left[X1_{\{X\geq K\}}\right] \approx \frac{1}{\sqrt{2\pi}}\int_W^\infty e^{-\frac{w^2}{2}}\frac{w\kappa'(t)}{t\sqrt{\kappa''(t)}}\left[1+\frac{\lambda_4(t)}{8}-\frac{5\lambda_3(t)^2}{24}\right]dw,$$

which is precisely in the form of eq. (3.2.1) in Jensen (1995). According to Theorem 3.2.1 therein, one finds

$$\begin{aligned}\mathbb{E}\left[X1_{\{X\geq K\}}\right] =& [1-\Phi(W)]\left\{q(0)\left[1+\frac{\lambda_4(0)}{8}-\frac{5\lambda_3(0)^2}{24}\right]+\frac{q''(0)}{2}\right\}\\ &+\phi(W)\frac{q(W)-q(0)}{W},\end{aligned} \tag{51}$$

where $q(w)=\frac{w\kappa'(t)}{t\sqrt{\kappa''(t)}}$. Let $\tilde{q}(w)=\frac{w}{t\sqrt{\kappa''(t)}}$, then $q(w)=\kappa'(t)\tilde{q}(w)$.

Lemma 1.

$$\begin{aligned}&\tilde{q}(w)=1-\frac{1}{6}\lambda_3(0)+\left[\frac{5}{24}\lambda_3(0)^2-\frac{1}{8}\lambda_4(0)\right]w^2+O(|w|^3),\\ &\tilde{q}(0)=1, \qquad \tilde{q}''(0)=-2\left[\frac{\lambda_4(0)}{8}-\frac{5\lambda_3^2(0)}{24}\right],\\ &t=\frac{1}{\sqrt{\kappa''(0)}}\left[w-\frac{1}{3}\lambda_3(0)w^2+O(|w|^3)\right].\end{aligned}$$

Proof. See Jensen(1995) Lemma 3.3.1. □

According to Lemma 1, we have

$$q(0)=\mu, \quad q(W)=\frac{W\kappa'(T)}{T\sqrt{\kappa''(T)}}, \tag{52}$$

$$q''(w)=\tilde{q}''(w)\kappa'(t)+2\tilde{q}'(w)\kappa''(t)\frac{dt}{dw}+\tilde{q}(w)\left[\kappa'''(t)\left(\frac{dt}{dw}\right)^2+\kappa''(t)\frac{d^2t}{dw^2}\right],$$

where $\frac{dt}{dw}=\frac{1}{\sqrt{\kappa''(0)}}\left[1-\frac{2}{3}\lambda_3(0)w\right]$, $\frac{d^2t}{dw^2}=\frac{-2\lambda_3(0)}{3\sqrt{\kappa''(0)}}$. When $w=0$ we find

$$\begin{aligned}q''(0)=&-2\left[\frac{\lambda_4(0)}{8}-\frac{5\lambda_3^2(0)}{24}\right]\mu+2\left[-\frac{\lambda_3(0)}{6}\right]\frac{\kappa''(0)}{\sqrt{\kappa''(0)}}+\frac{\kappa'''(0)}{\kappa''(0)}+\kappa''(0)\frac{-2\lambda_3(0)}{3\sqrt{\kappa''(0)}}\\ =&-2\left[\frac{\lambda_4(0)}{8}-\frac{5\lambda_3^2(0)}{24}\right]\mu.\end{aligned} \tag{53}$$

Plugging (52) and (53) in (51) we again get

$$\mathbb{E}\left[X1_{\{X\geq K\}}\right] = \mu\left[1-\Phi(W)\right] + \phi(W)\left[\frac{\kappa'(T)}{T\sqrt{\kappa''(T)}} - \frac{\mu}{W}\right] \equiv J_3. \tag{54}$$

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