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# A MULTI-WAVELET TYPE LIMITER FOR DISCONTINUOUS GALERKIN APPROXIMATIONS

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# A Multi-wavelet type limiter for discontinuous Galerkin Approximations

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**Abstract**

In this report, we present a multi-wavelet type limiter for the discontinuous Galerkin method for limiting the solution when spurious oscillations develop near a shock. This limiting leads to a loss of information in the approximation that can be detrimental to a higher order approximation ($k > 2$). The goal is therefore to retain as much information as possible in the higher order approximation. This is done by taking advantage of the evolution in time of more degrees of freedom of a DG approximation by making use of ideas from multi-resolution analysis (MRA) [3]. This differs from multi-level method [18] in that it only seeks to apply MRA ideas locally, on elements where the approximation requires limiting. This combination of techniques seems a natural pairing as it is well known that the wavelet linear approximation (*i.e.*, truncating the high frequencies) can approximate smooth functions very efficiently. Previously, the major hurdle was in devising wavelets that satisfy boundary conditions. With the discontinuous Galerkin method this is no longer an issue. Multi-wavelets can achieve arbitrary high accuracy without Gibbs' phenomena by selecting an appropriate wavelet basis, concentrating the energy to low frequencies. Standard wavelet linear approximation techniques cannot achieve similar results for functions which are not smooth, such as piecewise continuous functions with large jumps. In this paper we present results showing that the multi-wavelet idea is a promising technique for limiting solutions.

**Key Words:** limiting, discontinuous Galerkin method, hyperbolic equations, multi-wavelet methods

**AMS(MOS) subject classification:** 65M60

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# 1 Introduction

The Discontinuous Galerkin (DG) method has established its importance in an ever increasing variety of applications including neutron transport, fluid dynamics, and electromagnetic problems. In particular, the properties of this method provide a natural extension to such areas as chemistry, physics, and geophysical applications. The popularity of DG is due in part to its local nature and because it satisfies a local conservation property. Other nice properties of this method include its flexibility for adaptivity, easy implementation of boundary conditions and it is easy to parallelize. If polynomials of degree less than or equal to $k$ are used for the basis of the approximation, the approximation has $(k+\frac{1}{2})$ order of accuracy (and typically $(k+1)^{th}$ order of accuracy is observed).

This paper examines a multi-wavelet technique for improving the limiting process for the discontinuous Galerkin method for hyperbolic equations when using higher order polynomial approximations ($k > 2$). One of the challenges of implementing the discontinuous Galerkin method is that it requires the time evolution of more degrees of freedom per an element and that for applications which contain shocks there is no consensus on a proper limiting procedure. While evolving more degrees of freedom may be a disadvantage in some applications, we seek to take advantage of this through the use of multi-wavelet transforms. Initially, the ideas will be implemented locally, in regions where limiting is necessary instead of applying the usual total variation bounded (TVB) limiter. The discontinuous Galerkin method is a promising tool for many as yet unexplored applications but finding a proper limiting procedure is challenging. Further, a more robust limiter that also accounts for higher order approximations will aid in the expansion of the applicability of the discontinuous Galerkin method.

## 1.1 Limiting for DG

The discontinuous Galerkin method, without further modification, can compute solutions which are either smooth or have weak shocks and other discontinuities. However if the discontinuities are strong, the scheme will generate significant oscillations and even nonlinear instability. For this purpose, typically a slope limiter, a technique borrowed from the finite volume methodology, is used after each Runge-Kutta inner stage to control the numerical solution. There are many such limiters that exist in literature, e.g., minmod type limiters [14], [7], WENO based limiters [25], and generalize minmod moment based limiting [21] which are employed to control these oscillations. The slope limiter used in the DG methods involve a parameter, by means of which the limiting does not destroy accuracy at critical points. This parameter is problem dependent and often chosen on a trial and error basis. In one dimensional scalar conservation laws, this parameter is nothing but an upper bound of the second order derivative of the solution at critical points. Moment limiting [21] seeks to maintain as high order approximation while reducing oscillations inherent to the numerical method. Active research is focused on finding a problem independent limiter that can control the numerical oscillations and does not destroy the accuracy of the solution at critical points [22]. We seek to accomplish this through ideas used in multi-wavelet techniques.

# 2 Background

## 2.1 The discontinuous Galerkin method

The discontinuous Galerkin method has shown to be advantageous because of its ability to handle complicated geometries, allowance for simple treatment of boundary conditions, high-order accuracy, and because it is highly parallelizable. There are several places in the literature that contain a more in depth discussion [12, 11, 10, 9, 13, 14].

The discontinuous Galerkin method for the one-dimensional conservation law

$$u_t + f(u)_x = 0 \tag{1}$$

is defined as follows: Begin by defining a mesh given by mesh size $\triangle x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ and cell center $x_i$, where $I_i = (x_i - \frac{\triangle x_i}{2}, x_i + \frac{\triangle x_i}{2}) = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}),\;\; i = 1, \cdots, N$. Then an approximation space is chosen to consist of piecewise polynomials of degree less than or equal to $k$, where $k+1$ is the order of accuracy of the approximation, that is, $V_h = \{v | v \in \mathscr{P}^k \text{ for } x \in I_i\}$. The discontinuous Galerkin method is found by multiplying equation (1) by a test function $v \in V_h$ and integrating by parts to obtain the variational formulation:

Find $u_h(x,t) \in V_h$ such that

$$\int_{I_i} u_t v dx = \int_{I_i} f(u) v_x dx - f(u_{i+1/2}) v_{i+1/2} + f(u_{i-1/2}) v_{i-1/2} \qquad \forall v \in V_h.$$

The numerical scheme is then given by:

$$\int_{I_i} (u_h)_t v dx = \int_{I_i} f(u_h) v_x dx - \hat{f}_{i+1/2} v^-_{i+1/2} + \hat{f}_{i-1/2} v^+_{i-1/2} \tag{2}$$

for all test functions $v \in V_h$. The numerical flux, $\hat{f}_{i+1/2} = \hat{f}(u^-_{i+1/2}, u^+_{i+1/2})$, is chosen to be an upwind monotone flux, i.e. it is a non-decreasing function of the first argument $u^-$ and a non-increasing function of the second argument $u^+$. The test function $v$ is taken from inside the cell. The numerical integration of (2) is done by implementing the third-order SSP Runge-Kutta method (see e.g. [19], [20], and [26]) and taking the appropriate time step so that spatial errors dominate.

## 2.2 Multi-Wavelet Methods

Wavelet analysis is now an established tool in many areas of science and engineering. It provides a systematic way of representing and analyzing multiscale phenomena and has applications in diverse areas including signal and image processing, data compression, solution of partial differential equations and statistics.

The notion of multiresolution analysis (MRA) was introduced in [23] and [24]. Since then, there appeared many new constructions of orthogonal and non-orthogonal bases with controllable localization in the time-frequency domain.

The multi-resolution analysis technique we seek to implement is due to Alpert and collaborators [1, 2]. The idea is as follows: The scaling functions $\phi_0, \ldots, \phi_{k-1}$ were chosen to be $\phi_j(x) = \sqrt{(j+1/2)} P_j(x),\; j = 0, \ldots, k-1$, where $P_j$ are the Legendre polynomials. These functions form an orthonormal basis for the space of polynomials of degree less than $k$ on the interval $[-1,1]$. Alpert, Beylkin, Gines, and Vozovoi [3] introduced an alternative basis for this space, using interpolating polynomials. Given nodes $x_0,\dots,x_{k-1}$ which are roots of $P_k(x)$, and the associated Gauss-Legendre quadrature weights, $w_0,\dots,w_{k-1}$, the functions $R_j(x) = \frac{1}{\sqrt{w_j}} l_j(x), \quad j = 0,\dots,k-1$ where

$$l_j(x) = \Pi_{i=0,i\neq j}^{k-1} \frac{(x-x_i)}{(x_j-x_i)}$$

form an orthonormal basis on $[-1,1]$ such that for any polynomial $f$ of degree less than $k$ can be represented by the expansion

$$f(x) = \sum_{j=0}^{k-1} d_j R_j(x)$$

where the coefficients are given by $d_j = \sqrt{w_j} f(x_j)$, $j = 0,\dots,k-1$. We have two cases:

1. $\{\phi_{l,k}^j(x) = 2^{j/2}\phi_l(2^j x - k)\}$ scaling function basis
2. $\{\phi_{l,0}^0\ ,\ \psi_{l,k}^j(x) = 2^{j/2}\psi_l(2^j x - k)\}$ wavelet basis

as presented in [3]. We may use the basis in the case (2) for a Galerkin solution of a partial differential equation, then any function $f \in L^2[-1,1]$ in terms of its wavelet coefficients (plus its coefficients on the coarsest scale $V_0$) as

$$f(x) = \sum_{l=0}^{m} s_{l,0}^0 \phi_{l,0}^0(x) + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} \sum_{l=0}^{m} d_{l,k}^j \psi_{l,k}^j(x),$$

where $s_{l,k}^j = (f, \phi_{l,k}^j)$ and $d_{l,k}^j = (f, \psi_{l,k}^j)$. If we use the scaling function basis on some scale $V_j$ given in the case (1), then this discretization corresponds to a typical discontinuous Galerkin approximation.

# 3 Multi-Wavelet limiter

The incorporation of multi-resolution analysis into the discontinuous Galerkin method as a limiting technique is an exciting undertaking. One of the main difficulties is devising wavelets satisfying boundary conditions, but with the use of DG, this is no longer an issue. This differs from multi-level method [18] in that it only seeks to apply MRA ideas locally, on elements where the approximation requires limiting.

The idea behind using MRA for limiting is the following: The DG coefficients are decomposed and the information is represented at various scales. This leads to separation of the information into low frequency components and high frequency components. We can now remove unnecessary components using a pre-defined threshold and reconstruct the resulting coefficients back to the original scale to obtain the limited solution.

Let $s_{jl}^m$ be the Legendre expansion coefficient of the given function at level $m$. The relations between the coefficients $s_{jl}^n$ on two consecutive levels $n = m$ and $n = m+1$ are given by decomposition and reconstruction steps using the quadrature mirror filter coefficients as matrices $H^{(0)}, H^{(1)}, G^{(0)}, \text{and} G^{(1)}$ [3]. The matrix coefficients $h_{ij}^{(0)}, h_{ij}^{(1)}, g_{ij}^{(0)}$ and $g_{ij}^{(1)}$ allow us to change representation between two consecutive levels say for ex., $n = m$ and $n = m+1$.

We can use this information to examine the coefficients obtained in the discontinuous Galerkin approximation by decomposing the information, ridding the extraneous oscillations, and reconstructing the approximation.

## 3.1 Multi-wavelet decomposition

To construct the multiwavelet decomposition for Legendre polynomials, the coefficients $d_{jl}^m$ on scale $m$ may be computed using the coefficients $s_{jl}^{m+1}$ from the nearest finer level $m+1$. The relations between the coefficients on two consecutive levels $m$ and $m+1$ are

$$\begin{aligned} s_{il}^m &= \sum_{j=0}^{k-1} \left( h_{ij}^{(0)} s_{j,2l}^{m+1} + h_{ij}^{(1)} s_{j,2l+1}^{m+1} \right), \\ d_{il}^m &= \sum_{j=0}^{k-1} \left( g_{ij}^{(0)} s_{j,2l}^{m+1} + g_{ij}^{(1)} s_{j,2l+1}^{m+1} \right) \end{aligned}$$

Thus, starting with $2^n k$ values $s_{il}^n$, we apply repeatedly the decomposition procedure given above to compute the coefficients on coarser levels, $m = n-1, n-2, \ldots, 0$.

## 3.2 Multi-wavelet reconstruction

For multi-wavelet reconstruction, the coefficients $s_{jl}^n$ can be computed from the multi-wavelet coefficients $s_{j0}^0, d_{jl}^m, m = 0, \ldots, n$ using recursively the reconstruction step,

$$\begin{aligned} s_{i,2l}^{m+1} &= \sum_{j=0}^{k-1} \left( h_{ji}^{(0)} s_{jl}^m + g_{ji}^{(0)} d_{jl}^m \right) \\ s_{i,2l+1}^{m+1} &= \sum_{j=0}^{k-1} \left( h_{ji}^{(1)} s_{jl}^m + g_{jl}^{(1)} d_{jl}^m \right) \end{aligned}$$

## 3.3 Nonlinear Wavelet Transform

It is well known that wavelet linear approximation (*i.e.*, truncating the high frequencies) can approximate smooth functions very efficiently. It can achieve arbitrary high accuracy without Gibbs' phenomena by selecting an appropriate wavelet basis, concentrating the energy to low frequencies. Standard wavelet linear approximation techniques cannot achieve similar results for functions which are not smooth, such as piecewise continuous functions with large jumps. Many problems arise near discontinuities, caused primarily by the well known Gibbs' phenomenon.

Several approaches have been proposed to overcome these problems. Nonlinear and data dependent methods are often used to overcome this problem. Within the wavelet pyramidal filtering framework, non-linear data dependent approximations are often used, *e.g.*, Donoho's hard and soft thresholding techniques [17]. A more fundamental approach is to modify the wavelet transforms so as not to generate large wavelet coefficients near jumps. Claypool *et al.* use an adaptive lifting scheme which lowers the order of approximation near jumps [8], thus minimizing the Gibbs' effect. All these approaches have their limitations, and some residual Gibbs' phenomenon still exists. Another approach, due to Donoho, is to construct an orthonormal basis such as wedgelets and rigdelets [16] to represent the discontinuities. Avudainayagam and Vani developed a data dependent Haar wavelet transform and later extended to Daubechies wavelet transform [5]. This work proposes to modify this algorithm to apply to the coefficients of the discontinuous Galerkin approximation and apply it as a limiter, if needed.

This work is similar to the multi-level method of Gopalakrishnan and Kanschat [18]. In [18], the authors combined an interior penalty method with a DG scheme and applied it to an advection-diffusion equation with an arbitrarily small diffusion term. This scheme reduces to the standard DG method for advection problems when the diffusion term is zero. Their method is stable and accurate in diffusion dominated as well as convection dominated regime. This research also uses a multilevel method, but in this work we first determine the elements where a limiter is necessary and then apply MRA locally. We emphasize that this paper only concentrates on the limiting aspect and not the shock detection. However, a natural extension is to use the MRA as a shock detector as well.

## 3.4 Implementation of the multi-wavelet limiter

In implementing the multi-wavelet limiter, we first choose the element on which limiting is required. This is possible through using multi-wavelets itself, or the traditional minmod limiter to pick up the element where unphysical oscillations crop up due to Gibbs' phenomena. Another possibility is to use the local edge detection methods of Archibald, Gelb, and Yun [4]. Once we have determined the elements on which the approximation needs to be limited, we then use the information obtained from the multiwavelet decomposition to remove these oscillations. This is done by first decomposing the given coefficients. The decomposition represents the transformation of the information which was originally given at one scale to multi-scales. The given information at one scale is decomposed into lower and higher frequency components at various scales using the multi-wavelet transformation of Beylkin et al. [3]. We then start by looking at the higher frequencies and limit these coefficients by removing those components which are smaller than a particular tolerance thus removing the oscillations and then reconstruct the modified values back to original scale. This is an effective method since application of wavelet transform on a given scaling function coefficients leads to averages (low frequency components) and differences (high frequency components).

The criteria for truncation of wavelet coefficients is the following: We first define a threshold value $\varepsilon$ based on the order of the method. Given a function $f$, and its approximation $f^n$, (on scale $n$), we set to zero the difference coefficients $d_l^n$ (in interval $l$) whenever $|d_l^n| \leq \varepsilon$ where $\varepsilon$ is the desired accuracy of the approximation. After we have determined this, we can use this criteria at each step of decomposition till we reach the last stage where no further decomposition is required. We then reconstruct the low frequency and truncated high frequency components back to the function values on the original scale. Due to this truncation process, the unphysical oscillations which arise in high-order methods can be removed.

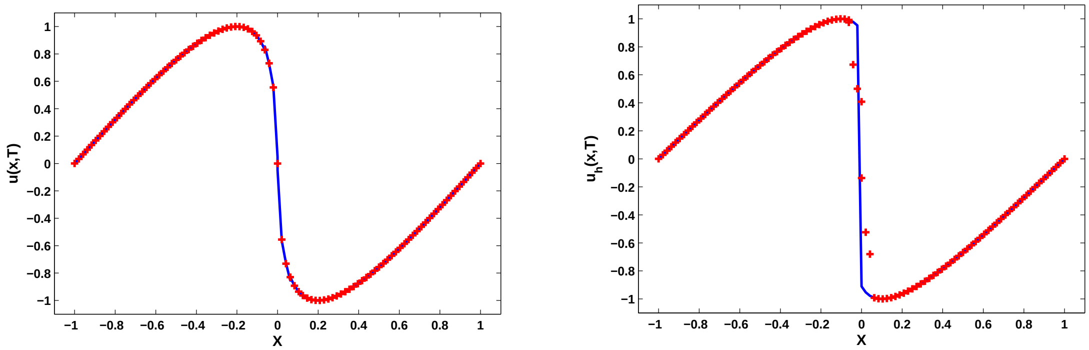


Figure 1: $k = 3, N = 32$ plot of the solution to Burgers equation. Left figure: $T = 0.4$ the multiwavelet limiter does not affect the approximation. Right figure: $T = 2$ the multiwavelet limiter helps to control spurious oscillations. Where $-$=exact and $*$=approximation.

# 4 Numerical Examples

We demonstrate the effectiveness of the multiwavelet limiter on a few preliminary test cases. First, we test it on the nonlinear scalar Burgers equation with sine initial condition. Secondly we test it on the standard Euler equations with Sod initial conditions. The cases we consider use the Legendre basis for the discontinuous Galerkin approximation with polynomial expansion of degree three and MRA tolerance of $\varepsilon = 10^{-4}$.

## 4.1 Burgers

The first case we consider is Burgers equation with sine initial condition.

$$\begin{aligned} &u_t + uu_x = 0, \\ &u(x,0) = \sin(\pi x), \qquad x \in (-1,1). \end{aligned} \tag{3}$$

The final solution is calculated before and after the shock at $t = 0.3$ and $t = 2$.

Figure 1 shows the exact and approximate solutions scaled to the interval $(-1,1)$ for $k = 3$ and $N = 32$. On the left is the solution before the shock occurs, at $T = 0.4$. It is desirable that the multiwavelet limiter not affect the solution unnecessarily, which is the case in this instance. Secondly, the figure on the right plots the solution after the shock has occurred, at $T = 2$. In this case, we can see that indeed the approximation does not contain spurious oscillations and remains stable even after the shock develops.

## 4.2 Euler Equation with Riemann Initial Condition

Next, we investigate the effectiveness of the multiwavelet limiter for the Euler equation of gas dynamics for a polytropic gas,

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, \qquad x \in (-5,5) \tag{4}$$

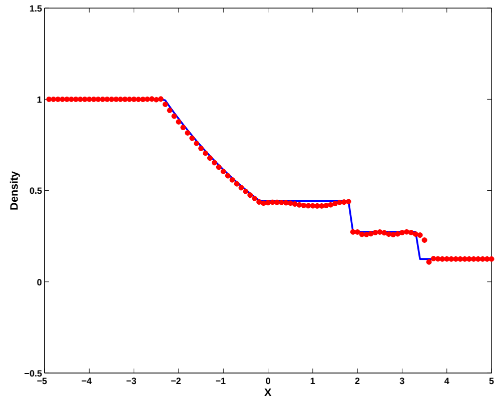


Figure 2: The density obtained by the DG approximation to the Euler equations with Sod initial conditions using the multiwavelet limiter for $k = 3,\ N = 100,\ T = 2$. The multi-wavelet limiter allows for retaining higher order information as much as possible. Where $-$=exact and $*$=approximation.

with Riemann initial conditions

$$\mathbf{u}(x,0) = \begin{cases} \mathbf{u}_L, & x < 0, \\ \mathbf{u}_R, & x > 0. \end{cases}$$

Recall that

$$\mathbf{u} = (\rho, \rho v, E)^T, \qquad \mathbf{f}(\mathbf{u}) = v\mathbf{u} + (0, v, vp)^T$$

with $E = \frac{p}{\gamma-1} + \frac{1}{2}\rho v^2$, and $\gamma = 1.4$. In this test case, we specifically consider Sod's shock tube problem [27]

$$(\rho_L, v_L, p_L) = (1, 0, 1), \quad (\rho_R, v_R, p_R) = (0.125, 0, 0.10),$$

calculated at final time $T = 2$ with $N = 100$ elements. In Figure 2, we plot the density for $k = 3$. The multiwavelet limiter allows for retaining as much higher order information as much as possible which does allow for oscillations to occur between shocks. The DG solution remains stable and maintains a high resolution approximation.

# 5 Concluding Remarks

The multi-wavelet idea is a promising technique for limiting. This technique allows for limiting the discontinuous Galerkin approximation without having to reconstruct the approximation. This is done through application of ideas in multi-resolution analysis. That is, we can decompose coefficients obtained from the discontinuous Galerkin approximation into differences and averages using a multiwavelet decomposition, examine where the spurious oscillations occur in the higher modes, and then reconstruct the coefficients. It is similar to the minmod technique, but does not require a evaluation of the approximation itself, the limiting can be accomplished in the multi-wavelet basis. The numerical examples presented demonstrate the possibility of this re-expansion. Future work is in applying to more challenging examples, examining the modification of the solution, as well as expansion to higher dimensions.

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