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THAT'S WHY, *sort of ...*
Classical Mechanics derived from Self-evident Axioms

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Summary

Classical point-mechanics is derived from three principles —called axioms— that are based on observations of simple kinematical phenomena. Predefined concepts of 'force' and 'mass' are not required. The concept 'mass' and corresponding concepts of momentum and energy follow from the first and second axiom. Together with the third axiom, a basic way for constructing equations of motion is derived, more or less equivalent to Gauss' principle of least constraint.

1 Introduction

Although the topic 'Classical Mechanics' is completely established since the beginning of the 20th century, still a lot is written nowadays on the subject. Also critical remarks have been made, by authors of outstanding reputation, like C. Lanczos [5], and C.A. Truesdell [7, 8]. Critisism concentrates on, among other things, ambiguities in the definition of force, Newton's diffuse definition of mass, the status of the 'action and reaction' law, and the status of the 'conservation of angular momentum' [7]. Critical remarks have been made (f.i. in Lanczos [5]) on the almost religious status of what is usually called *Newton's laws of mechanics*. All this however doesn't affect the reliability of these laws, at least as far relativistic and quantum effects may be neglected.
In the opinion of many scientists and engineers (classical) mechanics is *mathematics*, and indeed, studying and applying mechanics requires a lot of mathematics, where the physical base of the theory is present in the form of axioms. According to E.A. Desloge [2] there are two possibilities for founding mechanics:

A. The conservation of linear momentum, requiring the pre-assumption of *mass* (related to the work of Huygens and Descartes);

B. Newton's laws, requiring the pre-assumption of *force* (and mass).

Truesdell [7] 'advises' to accept the conservation of linear momentum *and* angular momentum as two necessary and sufficient 'axioms', which, together of course with proper definitions of mass, inertial systems, and all of geometry, enables us to build the whole theory.
The idea of establishing a small set of axioms as the basis for a theory probably stems from Euclid. His theory of geometry is usually considered as a piece of (pure) mathematics. But it was developed as a model for describing and understanding (a part of) the 'real world'. He choosed his axioms not only to be *the simplest* statements, but also the most *self evident* ones. His axioms were meant to be *convincing*, whereas nowadays in pure mathematics an axiomatic system only must obey rules like mutual independence, consistency, and completeness.
Like Euclid's geometry, classical mechanics is a model for description and understanding a part of the real world. In fact it is the extension of Euclidean geometry with the concepts of time, matter and 'how it works'. Concerning classical mechanics, the author has never seen a set of basic principles that are equally self-evident and convincing as Euclid's axioms for geometry are. H. Hertz [4] and P. Appell [1] present alternatives for Newton's approach, based on Gauss' principle of least constraint [3], which is based on the principle of d'Alembert — part of the 'classical' theory. Gauss' principle is not very self evident, nor are Appel's and Hertz' principles. Nevertheless, Gauss' principle has an attractive simplicity.
The present study is about founding classical mechanics on first principles, with kinematics as starting point. There will not be used a predefined concept

of *mass*, nor of *force*.
The principle of relativity, stating the equivalence of all inertial systems, is chosen as first axiom.
The two new axioms are based on thought experiments [1]. They describe the experienced results of two elementary experiments with colliding balls and with forced motion of a material point respectively.
All axioms are expressed in terms of *velocity* and *acceleration* only. Understanding the axioms requires only a notion of these *kinematical* concepts. The development of the theory on the basis of these axioms require knowledge of Euclidian Geometry, linear algebra, and the concepts of time, motion and inertial systems. And, of course, a little calculus.
The concepts of *mass*, *momentum* and *energy* arise as mathematical consequences of the first two axioms. The mathematical analysis leading to this result is a mere excercise in elementary linear algebra.
The third axiom gives rise to a new derivation of Gauss' *principle of least constraint* [3], from which the general equations of motion for so-called flexible constructions are derived. No use is made of 'physical forces', like elastic, electrical and gravitational forces.
Finally, as a demonstration of the usability of the theory, the second and third laws of Newtonian mechanics are derived, and also Eulers equations for the motion of a rigid body.

1.1 Notation and basic linear algebra topics.

Since there are differences in notational conventions between physicists, civil engineers, mathematicians etc, we (loosely) describe the notation used in this paper.

$\mathbb{R}^n$ denotes the linear space of real n-dimensional vectors, represented by boldface lowercase letters: $\boldsymbol{a}, \boldsymbol{b}, \ldots$. A vector $\boldsymbol{a}$ in $\mathbb{R}^n$ has n real entries (or elements) $a_1, a_2, \ldots, a_n$, usually displayed as a column of n numbers. The (Euclidean) norm $\|\boldsymbol{a}\|$ of a vector $\boldsymbol{a}$, is defined by $\|\boldsymbol{a}\| = \sqrt{\sum_k a_k^2}$.

$\mathbb{R}^{m\times n}$ is the space of real matrices with m rows and n columns ($m \times n$ matrices), denoted by boldface capitals: $\boldsymbol{A}, \boldsymbol{B}, \ldots$. The matrix element on the k-th row and the l-th column of a matrix $\boldsymbol{A}$ is represented by $a_{k,l}$. The unit matrix $\boldsymbol{I}^{(n)}$ for $\mathbb{R}^n$ is the $n \times n$ matrix with ones of the main diagonal, and zeros everywhere else: $i_{k,k} = 1$, $i_{k,l} = 0$ whenever $k \neq l$. If the size n of the space follows from the context, the superscript (n) is omitted.
An $m \times n$ matrix represents a linear mapping form $\mathbb{R}^n$ into $\mathbb{R}^m$, such that if $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, the components y_k are given by $y_k = \sum_l a_{k,l}x_l$. The product $\boldsymbol{C}$ of an $m \times r$ matrix $\boldsymbol{A}$ and an $r \times n$ matrix $\boldsymbol{B}$ is defined by $\boldsymbol{C}\boldsymbol{x} = \boldsymbol{A}(\boldsymbol{B}\boldsymbol{x})$, which definition implies the associativity of matrix multiplication. Of course $\boldsymbol{I}^{(m)}\boldsymbol{A} = \boldsymbol{A}$ for every $\boldsymbol{A} \in \mathbb{R}^{m\times n}$. The set of all images $\boldsymbol{y}$ of the mapping $\boldsymbol{A}$ is a linear subspace

[1] meaning that the author have not done them actually, but many others will have at least *experienced* the outcome.

of $\mathbb{R}^m$, the column space of $\boldsymbol{A}$, and is denoted by $\mathcal{R}(\boldsymbol{A})$. The set of all vectors $\boldsymbol{x} \in \mathbb{R}^n$ that are mapped to the nullvector in $\mathbb{R}^m$ is called the null space of $\boldsymbol{A}$, and denoted by $\mathcal{N}(\boldsymbol{A})$.
The transpose of a matrix, denoted by $\boldsymbol{A}^T$, is defined by the rule: if $\boldsymbol{B} = \boldsymbol{A}^T$, then $b_{k,l} = a_{l,k}$. The transpose of a product satisfies $(\boldsymbol{AB})^T = \boldsymbol{A}^T\boldsymbol{B}^T$. Since vectors $\boldsymbol{a} \in \mathbb{R}^n$ can be considered as matrices of one column and n rows, they also have a transpose $\boldsymbol{a}^T$, a matrix with one row and n columns.

The scalar product or 'dot product' of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, notation $\boldsymbol{a}\cdot\boldsymbol{b}$, is defined as $\boldsymbol{a}^T\boldsymbol{b}$. Geometrical interpretation: $\boldsymbol{a}\cdot\boldsymbol{b} = \|\boldsymbol{a}\| \cdot \|\boldsymbol{b}\| \cos(\theta)$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal (or perpendicular) if $\boldsymbol{a}\cdot\boldsymbol{b} = 0$. Notation: $\boldsymbol{a} \perp \boldsymbol{b}$. The Euclidean norm of $\boldsymbol{a}$ satisfies $\|\boldsymbol{a}\|^2 = \boldsymbol{a}\cdot\boldsymbol{a} = \boldsymbol{a}^T\boldsymbol{a}$.
In $\mathbb{R}^3$, the vector product or 'cross product' $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ of two vectors is defined by

$$\begin{aligned} c_1 &= a_2b_3 - a_3b_2 \\ c_2 &= a_3b_1 - a_1b_3 \\ c_3 &= a_1b_2 - a_2b_1 \end{aligned} \tag{1}$$

If $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$, then $\boldsymbol{c} \perp \boldsymbol{a}$, $\boldsymbol{c} \perp \boldsymbol{b}$. Geometrical interpretation: $\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\| \cdot \|\boldsymbol{b}\| \cdot \sin(\theta)$, in which θ is again the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. The vector product $\boldsymbol{a} \times \boldsymbol{b}$ can also be interpreted as a skew symmetric matrix $\boldsymbol{A}$ acting on $\boldsymbol{b}$:

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{Ab}, \text{ where } \boldsymbol{A} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{2}$$

The following identities are valid:

$$\begin{aligned} &\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a}, \ \boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}, \ \boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0} \\ &\boldsymbol{a}\cdot(\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c}\cdot(\boldsymbol{a} \times \boldsymbol{b}), \ \text{(scalar triple product)} \\ &\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}, \ \text{(vector triple product)} \\ &\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a}^T\boldsymbol{c}\boldsymbol{I} - \boldsymbol{c}\boldsymbol{a}^T)\boldsymbol{b}, \ \text{(matrix interpretation)} \end{aligned}$$

For the time derivatives of time dependent quantities, we use the 'dot-notation':

$$\dot{\boldsymbol{x}} = \frac{d\boldsymbol{x}}{dt}, \ \ddot{\boldsymbol{x}} = \frac{d^2\boldsymbol{x}}{dt^2}, \ \text{etc}$$

Finally, for partial derivatives of a function with respect to the components of a vector $\boldsymbol{x}$ in $\mathbb{R}^n$ we use the following shorthand expressions:

$$\frac{\partial F}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{x}} F = \left[\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \ldots, \frac{\partial F}{\partial x_n}\right]$$

2 Basic properties and definitions

2.1 Axiom 1: Principle of relativity

Homogeneity and isotropy of space. In Euclidean geometry space is considered as isotropic and homogeneous. That is: geometrical theorems are valid independent from the orientations and locations in space. Therefore, an elementary mechanical law must be invariant with respect to translation-, rotation-, and reflection transformations. In practice this is contradicted continuously by the fact that in real life, motions in vertical direction certainly differ from motions in the two horizontal directions. However, we assume that in outer space, far from heavy matter, space 'behaves' isotropically.

Homogeneity of time. Experiments done tomorrow under 'equal conditions' will proceed identically to the same experiments done today. So mechanical formulas are invariant for time shifts.

Isotropy of Space-Time. For a description of this property, we need the concept of *inertial systems*. These are systems of observation that are moving freely in space, and in which Newton's first law is valid: *every free moving point describes a straight line at constant speed*. Such motion is called *uniform motion*. Formally:

Definition 1 (Inertial system) *An inertial system is any reference system in which each free material point moves uniformly.*

Now experience learns that not only the behaviour of free material points, but also of other physical phenomena, is independent from the inertial system in which they are observed. This is expressed in the following axiom:

Axiom 1 (Principle of relativity) *All inertial systems are equivalent.*

This principle is explicitly used by Christiaan Huygens, in his analysis of the behaviour of colliding balls, although not stated as a 'principle. Apparently, Huygens considered it as common knowledge. A strong argument for accepting this principle is the fact that it is clearly impossible to determine whether an inertial system is moving or not.
We may interprete the principle of relativity as 'isotropy of space-time'.
Since Newton's first axiom — the law of inertia — plays a key role in the present analysis, it will be stated as a lemma:

Lemma 1 (Law of inertia) *In an inertial system, every free material point moves uniformly.*

Proof: This is trivially implied by definition of inertial systems □

Galilei transformations. Any reference system that moves uniformly with respect to an inertial system, is itself an inertial system. Also, it can be derived that all inertial systems are related by linear transformations, called *Galilei transformations*. Denote a basic inertial frame by $\mathcal{G}$, with coordinates $\boldsymbol{x}$, and another by $\widetilde{\mathcal{G}}$, with coordinates $\widetilde{\boldsymbol{x}}$. Then the corresponding transformation can be written as

$$\widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{x}}_0 + \alpha \boldsymbol{C}(\boldsymbol{x} - \boldsymbol{x}_0 - \boldsymbol{u}t) \tag{3}$$

In this expression, $\boldsymbol{u}$ is the velocity of the frame $\widetilde{\mathcal{G}}$ with respect to the frame $\mathcal{G}$, α is a positive scale factor, and $\boldsymbol{C}$ is a real unitary matrix (an 'orthogonal matrix'). We'll often use a special subset of these transforms in which no shifts, no scaling and no rotation occur:

$$\widetilde{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{u}t \tag{4}$$

The transformations in this restricted class have the properties
– The coordinate systems $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ use the same units of length and time,
– they have the same orientation 'in space',
– the origin of $\widetilde{\mathcal{G}}$ moves at velocity $\boldsymbol{u}$ along the line $\boldsymbol{x}_{\text{origin}} = \boldsymbol{u}t$.

Quantities that are invariant under Galilei transformations are called *Galilei-invariant*.
This family of transformations, and their rotated, shifted and rescaled variants, are based on the assumption that *time* is a universal scalar parameter. In fact we have also the equation $\widetilde{t} = t$. This is not the case if Einstein's special relativity plays a role.

2.2 Material balls.

We initially study the behaviour of rigid material balls.

Definition 2 (Material Ball) *A 'material ball', or 'ball' is a rigid piece of matter with spherical shape. Its geometric properties are completely determined by the location $\boldsymbol{x}_c$ of its centre, and its radius R. By the term 'location of the ball', we mean the location of its centre.*
A ball is **isotropic** *in the sense that its mechanical behaviour is independent from its orientation in space.*
Two balls are **identical**, *if they behave exactly the same way under the same circumstances.*
An **ideal ball** *is a material ball with a surface so perfectly smooth, that it cannot be brought into rotation. Therefore the motion of an ideal ball is completely determined by the motion of its central point. An ideal ball has only three degrees of freedom.*
The **velocity** *and the* **acceleration** *of an ideal ball are the velocity and acceleration of its centre respectively. By a* **material point** *is meant an ideal ball of which the radius is irrelevantly small compared to the size of the events in which it plays a role. The mechanical behaviour of a material point is described completely by the path of its 'centre'. A material point cannot have other kinds of motion than translatory motions.*

2.3 Collisions.

A free material ball, observed in an inertial frame of reference, will move uniformly, unless it collides with an obstacle. In case of a collision *between* moving balls, both balls will change their velocity abruptly, since parts of different rigid objects cannot fill the same space simultaneously. After a collision, the balls move on uniformly with new velocities. We call these events *transactions* between material balls.
Consider a collision between two balls B_1 and B_2, with initial velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ respectively. We call these velocities *primary velocities*. After the collision, the velocities have changed into $\boldsymbol{v}'_1$ and $\boldsymbol{v}'_2$, the *secondary* velocities. At collision time, the balls touch each other in a common point P on the surfaces of both balls. Let $\boldsymbol{n}$ be the exterior unit vector normal to the surface of B_1 in P, the *collision normal*. The collision can only take place if the velocity difference $\boldsymbol{v}_1 - \boldsymbol{v}_2$ has a positive component in the direction $\boldsymbol{n}$, since otherwise the balls are *moving away* from each other.
The primary velocities, and the collision normal together determine the collision completely. Now observe the collision in the inertial frame $\mathcal{G}(\boldsymbol{u})$ with rotated axes:

$$\boldsymbol{Q}\widetilde{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{u}t$$

where $\boldsymbol{Q}$ is a real unitary matrix. Then $\boldsymbol{n} = \boldsymbol{Q}\widetilde{\boldsymbol{n}}$, and for $k = 1, 2$ we have

$$\begin{aligned} \boldsymbol{v}_k &= \boldsymbol{Q}\widetilde{\boldsymbol{v}}_k + \boldsymbol{u} \\ \boldsymbol{v}'_k &= \boldsymbol{Q}\widetilde{\boldsymbol{v}}'_k + \boldsymbol{u} \end{aligned}$$

where the tilded quantities refer to $\mathcal{G}(\boldsymbol{u})$
The velocity jumps are '$\boldsymbol{Q}$-rotated Galilei-invariant':

$$\delta\boldsymbol{v}_k = \boldsymbol{v}'_k - \boldsymbol{v}_k = \boldsymbol{Q}(\widetilde{\boldsymbol{v}}'_k - \widetilde{\boldsymbol{v}}_k) = \boldsymbol{Q}\delta\widetilde{\boldsymbol{v}}_k$$

Now choose $\boldsymbol{u} = \boldsymbol{v}_2$, then the second ball is initially at rest in $\mathcal{G}(\boldsymbol{v}_2)$: $\widetilde{\boldsymbol{v}}_2 = \mathbf{0}$. Choose the orthogonal matrix $\boldsymbol{Q}$ such that $\widetilde{\boldsymbol{n}} = \widetilde{\boldsymbol{e}}_1$, $\widetilde{\boldsymbol{v}}_1 = \widetilde{v}_{1,1}\widetilde{\boldsymbol{e}}_1 + \widetilde{v}_{2,1}\widetilde{\boldsymbol{e}}_2$, meaning that the collision normal is along the $\widetilde{x}$-axis, and the velocity difference is in the $\widetilde{x}, \widetilde{y}$-plane. We call this coordinate system the collision's *own system*. This means that in fact the collision result depends on the scalars $\widetilde{v}_{1,1}$ and $\widetilde{v}_{2,1}$. These scalars satisfy

$$\begin{bmatrix} \widetilde{v}_{1,1} \\ \widetilde{v}_{2,1} \end{bmatrix} = \|\widetilde{\boldsymbol{v}}_1\| \cdot \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} = \|\boldsymbol{v}_1 - \boldsymbol{v}_2\| \cdot \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

where θ is the angle between $\boldsymbol{v}_1 - \boldsymbol{v}_2$ and $\boldsymbol{n}$. So in fact the collision is determined by $\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|$, and θ.

Definition 3 (Collision parameters) *For a collision between two balls, with velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, and hitting each other with a collision normal $\boldsymbol{n}$, the collision parameters are*

1. *The impact speed $\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|$*
2. *The collision angle $\theta \in (0, \frac{\pi}{2})$, satisfying*

$$\cos(\theta) = \frac{(\boldsymbol{v}_1 - \boldsymbol{v}_2)\boldsymbol{\cdot}\boldsymbol{n}}{\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|}$$

3 Mechanics of transactions.

3.1 Axiom 2: Law of decrease.

Over the ages, people have attempted to build machinery for doing work at zero cost. These perpetuum mobile builders can be compared with the alchemists, who tried to find an elixir for eternal life, or to transfer lead into gold. There have been very skilled people amongst them.
But they all share the following experience: **It doesn't work**. However: many of them **almost succeeded**, a most irritating fact indeed.
Philosophically, this experience is quite satisfactory: One cannot have anything for free, one has to pay for everything, only sunshine is for free, etc.
It is rather difficult to analyse why a particular perpetuum mobile fails, since these machines are often very complicated. So usually we say: "It is contradicting the basic laws of mechanics. Period".

Now let us consider an extremely simple example of a mechanical event: the collision between two moving balls. Suppose the balls have, before they collide, velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ respectively. After their collision, the velocities are $\boldsymbol{v}_1'$ and $\boldsymbol{v}_2'$. Nearly everyone knows the example of a central collision between two identical balls, obtaining (nearly) each others velocities: $\boldsymbol{v}_1' = \boldsymbol{v}_2$, and $\boldsymbol{v}_2' = \boldsymbol{v}_1$. With non-identical balls, colliding in an arbitrary way, the result of the collision is not so easy to predict. But never people will observe a collision where $\|\boldsymbol{v}_1'\| > \|\boldsymbol{v}_1\|$, *and* $\|\boldsymbol{v}_2'\| > \|\boldsymbol{v}_2\|$, meaning an increase of *both* velocities simultaneously.
How can we be so sure of this statement? Because if collisions contradicting this observation were possible, some ingeneous craftsman would have constructed a perpetuum mobile based on this kind of event. How? For instance by bringing two arbitrary springs S_1 and S_2 in appropriate states, launching two balls with these springs, making them collide in the right way, and catching the balls with two springs identical to S_1 and S_2 respectively. Then in the end, the similar springs are excited more than the original springs: we have

won 'energy'.
By the same argument, also collisions for which $\|\boldsymbol{v}'_1\| = \|\boldsymbol{v}_1\|$ and $\|\boldsymbol{v}'_2\| > \|\boldsymbol{v}_2\|$ are impossible.
This simple example can easily be extended to more complicated transaction events. We therefore formulate the following axiom

Axiom 2 (Law of decrease) *Let N material balls have initial velocities $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_N$ at time t. Assume the balls are involved in collisions with each other, and some other material objects. Assume at some moment in time $t' > t$ the other material objects are in exactly the same state as before, and the balls have velocities $\boldsymbol{v}'_j$.*
Then if for some i $\|\boldsymbol{v}'_i\| > \|\boldsymbol{v}_i\|$, then for some $j \neq i$, $\|\boldsymbol{v}'_j\| < \|\boldsymbol{v}_j\|$.

This statement is equivalent to

$$\|\boldsymbol{v}'_i\| \geq \|\boldsymbol{v}_i\| \text{ for all } i \implies \|\boldsymbol{v}'_i\| = \|\boldsymbol{v}_i\| \text{ for all } i \tag{5}$$

Strictly spoken, the 'law of non-increase' would be a more correct terminology, but since in practice (nearly) always losses occur, the term 'decrease' comes closer to the *practical manifestations* of this axiom.

3.2 Collision experiments.

The law of decrease, combined with the principle of relativity, has surprisingly strong consequences, as the following lemmas will show.

Lemma 2 (Dependency lemma) *Let $S = \{B_1, B_2, \ldots, B_n\}$ be a system of ideal balls that are moving freely, apart from some mutual collisions. Let the velocities of B_k at time t and t' be $\boldsymbol{v}_k$ and $\boldsymbol{v}'_k = \boldsymbol{v}_k + \delta\boldsymbol{v}_k$ respectively.*
Then the set of vectors $\delta\boldsymbol{v}_k$ is linearly dependent.

Proof: Observe the development of the system from an arbitrary inertial frame $\mathcal{G}(\boldsymbol{u})$, moving at velocity $\boldsymbol{u}$ with respect to the basic frame. Denoting the velocities and velocity changes as observed in $\mathcal{G}(\boldsymbol{u})$ by tildered symbols, we have $\widetilde{\boldsymbol{v}}_k = \boldsymbol{v}_k - \boldsymbol{u}$, and therefore $\delta\widetilde{\boldsymbol{v}}_k = \delta\boldsymbol{v}_k$ for all balls: the velocity changes are Galilei invariant.
The changes in the squared absolute velocities, observed in $\mathcal{G}(\boldsymbol{u})$, satisfy

$$\delta\|\widetilde{\boldsymbol{v}}_k\|^2 = \delta\|\boldsymbol{v}_k - \boldsymbol{u}\|^2 = \delta\|\boldsymbol{v}_k\|^2 - 2\delta\boldsymbol{v}_k \boldsymbol{\cdot} \boldsymbol{u}$$

Suppose we are looking for an inertial system $\mathcal{G}(\boldsymbol{u})$ in which $\delta\|\widetilde{\boldsymbol{v}}_k\|^2 = b_k$, then $\boldsymbol{u}$ must satisfy

$$2\delta\boldsymbol{v}_k \boldsymbol{\cdot} \boldsymbol{u} = \delta\|\boldsymbol{v}_k\|^2 - b_k, \quad k = 1, 2, \ldots, n$$

Now the law of decrease prevents this system to have a solution $\boldsymbol{u}$ if $b_k > 0$ for all k. Hence the rows of its matrix, i.e. the vectors $\delta\boldsymbol{v}_k$, must be linearly dependent. □

This lemma is not too impressive, since n vectors in $\mathbb{R}^3$ are linearly dependent anyway if $n \geq 4$. Only the cases $n = 2, 3$ may provide us with some new information.

Lemma 3 *Let two ideal balls B_1 and B_2 collide, with primary velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, and secondary velocities $\boldsymbol{v}'_1$ and $\boldsymbol{v}'_2$ respectively. Then a positive constant β, the* **collision ratio** *and a non negative number Δ, the* **collision defect** *exist such that in an arbitrary inertial frame $\mathcal{G}(\boldsymbol{u})$*

$$\beta\widetilde{\boldsymbol{v}}'_1 + \widetilde{\boldsymbol{v}}'_2 = \beta\widetilde{\boldsymbol{v}}_1 + \widetilde{\boldsymbol{v}}_2 \quad (6)$$

$$\beta\|\widetilde{\boldsymbol{v}}'_1\|^2 + \|\widetilde{\boldsymbol{v}}'_2\|^2 = \beta\|\widetilde{\boldsymbol{v}}_1\|^2 + \|\widetilde{\boldsymbol{v}}_2\|^2 - \Delta \quad (7)$$

where $\widetilde{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{u}$ for all velocities.

Proof: According to the dependency lemma, the velocity jumps must be linearly dependent: $\beta\delta\boldsymbol{v}_1 + \delta\boldsymbol{v}_2 = \mathbf{0}$ for some nonzero scalar β. This is part of (6). Now write for $k = 1, 2$

$$\delta\|\widetilde{\boldsymbol{v}}_k\|^2 = \delta\|\boldsymbol{v}_k\|^2 - 2\boldsymbol{v}_k \boldsymbol{\cdot} \boldsymbol{u} \quad (8)$$

Now for all $\boldsymbol{u}$

$$\beta\delta\|\widetilde{\boldsymbol{v}}_1\|^2 + \delta\|\widetilde{\boldsymbol{v}}_2\|^2 = \beta\delta\|\boldsymbol{v}_1\|^2 + \delta\|\boldsymbol{v}_2\|^2 - 2(\beta\delta\boldsymbol{v}_1 + \delta\boldsymbol{v}_2)\boldsymbol{\cdot}\boldsymbol{u} = \beta\delta\|\boldsymbol{v}_1\|^2 + \delta\|\boldsymbol{v}_2\|^2 = C \quad (9)$$

with C a constant, independent from $\boldsymbol{u}$.
In (8), we can choose $\boldsymbol{u}$ to give $\delta\|\widetilde{\boldsymbol{v}}_k\|^2$ any prescribed value. If we choose $\boldsymbol{u}$ such that $\delta\|\widetilde{\boldsymbol{v}}_1\|^2 = 0$, then according to the law of decrease we must have $\delta\|\widetilde{\boldsymbol{v}}_2\|^2 \leq 0$. Therefore

$$C = \beta\delta\|\widetilde{\boldsymbol{v}}_1\|^2 + \delta\|\widetilde{\boldsymbol{v}}_2\|^2 = \delta\|\widetilde{\boldsymbol{v}}_2\|^2 \leq 0$$

which proves (7) with $\Delta = -C$.
Next choose $\boldsymbol{u}$ such that $\delta\|\widetilde{\boldsymbol{v}}_2\|^2 > 0$, then

$$\beta\delta\|\widetilde{\boldsymbol{v}}_1\|^2 = -\Delta - \delta\|\widetilde{\boldsymbol{v}}_2\|^2 < 0$$

Because $\delta\|\widetilde{\boldsymbol{v}}_1\|^2 < 0$ by the law of decrease, it follows $\beta > 0$, completing the proof. □

The non-negativity of the collision defect is a first step in the quantification of an energy concept. It also enables us to define the concept of an ideal collision: a collision with zero collision defect.

The collision ratio can be regarded as a ratio of inertia. If $\beta \gg 1$, then $\boldsymbol{v}_1$ will be much less influenced by the collision than $\boldsymbol{v}_2$. This ratio depends not only on the properties of the individual balls, but *on the collision conditions as well*. For different collisions between balls B_1 and B_2, we write $\beta_{k,l}$, $\beta'_{k,l}$ $\beta''_{k,l}$ etc
The same collision event could have been described with swapped indices.

$$\delta\boldsymbol{v}_1 + \beta_{2,1}\delta\boldsymbol{v}_2 = \mathbf{0}, \quad \text{with } \beta_{2,1} = \frac{1}{\beta_{1,2}}$$

Lemma 4 *Each collision between two identical ideal balls has ratio $\beta = 1$*

Proof: Consider two identical balls of radius R, with primary orbits $\boldsymbol{x}_{1,2}(t) = \pm(\boldsymbol{x}_c + \boldsymbol{v}(t - t_c))$, that collide at $t = t_c$. The origin is chosen such that it coincides with the contact point of the collision. So $R = \|\boldsymbol{x}_c\|$.
The two balls have a completely symmetric history. If we rotate the Carthesian coordinate frame over π radians in the plane containing $\boldsymbol{v}$ and $\boldsymbol{x}_c$, then the balls simply have exchanged positions.
So the balls meet each other under equal conditions. Identical balls under identical circumstances react identically. Therefore the secondary velocities satisfy $\boldsymbol{v}_2' = -\boldsymbol{v}_1'$, Hence $\delta\boldsymbol{v}_1 = -\delta\boldsymbol{v}_2$, and $\beta_{1,2} = 1$. □

Next theorem, about the case $N = 3$ of the dependency lemma, is based on two thought experiments with colliding balls.

Theorem 1 (Mass-momentum-theorem) *To each material ball is associated a positive number m, it's* mass, *such that for each collision between two balls B_1 and B_2:*

$$m_1\delta\boldsymbol{v}_1 + m_2\delta\boldsymbol{v}_2 = \boldsymbol{0} \tag{10}$$

$$m_1\delta\|\boldsymbol{v}_1\|^2 + m_2\delta\|\boldsymbol{v}_2\|^2 \leq 0 \tag{11}$$

Proof: Let B_1 , B_2, and B_3 be free ideal balls, moving such way that the following sequence of collisions occurs:

$$\begin{array}{llll} B_1 \oplus B_2 : & \delta_1\boldsymbol{v}_1 = \boldsymbol{a}_1, & \delta_1\boldsymbol{v}_2 = -\beta_{1,2}\boldsymbol{a}_1, & \delta_1\boldsymbol{v}_3 = \boldsymbol{0} \\ B_2 \oplus B_3 : & \delta_2\boldsymbol{v}_2 = \boldsymbol{a}_2, & \delta_2\boldsymbol{v}_3 = -\beta_{2,3}\boldsymbol{a}_2, & \delta_2\boldsymbol{v}_1 = \boldsymbol{0} \\ B_2 \oplus B_1 : & \delta_3\boldsymbol{v}_2 = \boldsymbol{a}_3, & \delta_3\boldsymbol{v}_1 = -\beta_{2,1}'\boldsymbol{a}_3, & \delta_3\boldsymbol{v}_3 = \boldsymbol{0} \end{array}$$

The experiment can be set up in such way that $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ is a linearly independent set of vectors, and that the first and second collisions have arbitrary, but prescribed collision parameters.
The total velocity change $\delta\boldsymbol{v}_k$ of B_k can be written as $\sum_{j=1}^{3} \delta_j\boldsymbol{v}_k$.
This leads to

$$\begin{aligned} \delta\boldsymbol{v}_1 &= \boldsymbol{a}_1 - \beta_{2,1}'\boldsymbol{a}_3 \\ \delta\boldsymbol{v}_2 &= -\beta_{1,2}\boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_3 \\ \delta\boldsymbol{v}_3 &= -\beta_{2,3}\boldsymbol{a}_2 \end{aligned}$$

According to the dependency lemma, these velocity jumps are linearly dependent,

$$\sum_{k=1}^{3} \lambda_k \delta\boldsymbol{v}_k = \boldsymbol{0}$$

for λ's not all zero. Working out, this can be written as

$$(\lambda_1 - \beta_{1,2}\lambda_2)\boldsymbol{a}_1 + (\lambda_2 - \lambda_3\beta_{2,3})\boldsymbol{a}_2 + (-\lambda_1\beta_{2,1}' + \lambda_2)\boldsymbol{a}_3 = \boldsymbol{0}$$

Since the vectors $\boldsymbol{a}_k$ for $k = 1, 2, 3$ are linearly independent, this implies $\boldsymbol{M\lambda} = \boldsymbol{0}$, with

$$\boldsymbol{M} = \begin{bmatrix} 1 & -\beta_{1,2} & 0 \\ 0 & 1 & -\beta_{2,3} \\ -\beta_{2,1}' & 1 & 0 \end{bmatrix}$$

So $\boldsymbol{M}$ is singular, and therefore $\det(\boldsymbol{M}) = \beta_{2,3} \cdot (1 - \beta_{1,2}\beta'_{2,1}) = 0$. Hence $\beta_{1,2}\beta'_{2,1} = 1$, or equivalently

$$\beta'_{1,2} = \beta_{1,2}$$

It follows that the collision ratio is *independent of the collision parameters.*

Next let again B_1 , B_2, and B_3 be free moving ideal balls, and consider the following sequence of collisions:

$$\begin{array}{llll} B_1 \oplus B_2: & \delta_1 \boldsymbol{v}_1 = \boldsymbol{a}_1, & \delta_1 \boldsymbol{v}_2 = -\beta_{1,2}\boldsymbol{a}_1, & \delta_1 \boldsymbol{v}_3 = \boldsymbol{0} \\ B_2 \oplus B_3: & \delta_2 \boldsymbol{v}_2 = \boldsymbol{a}_2, & \delta_2 \boldsymbol{v}_3 = -\beta_{2,3}\boldsymbol{a}_2, & \delta_2 \boldsymbol{v}_1 = \boldsymbol{0} \\ B_3 \oplus B_1: & \delta_3 \boldsymbol{v}_3 = \boldsymbol{a}_3, & \delta_3 \boldsymbol{v}_1 = -\beta_{3,1}\boldsymbol{a}_3, & \delta_3 \boldsymbol{v}_2 = \boldsymbol{0} \end{array}$$

The differences with the first experiment are
(1) There are no restrictions to the collision parameters,
(2) The third collision is now between B_3 and B_1. In this case we have

$$\begin{aligned} \delta \boldsymbol{v}_1 &= \boldsymbol{a}_1 - \beta_{3,1}\boldsymbol{a}_3 \\ \delta \boldsymbol{v}_2 &= -\beta_{1,2}\boldsymbol{a}_1 + \boldsymbol{a}_2 \\ \delta \boldsymbol{v}_3 &= -\beta_{2,3}\boldsymbol{a}_2 + \boldsymbol{a}_3 \end{aligned}$$

Similarly, the linear dependence of the velocity jumps, and the linear independence of the vectors $\boldsymbol{a}_k$, lead to $\widetilde{\boldsymbol{M}}\boldsymbol{\lambda} = \boldsymbol{0}$, for some nonzero vector $\boldsymbol{\lambda}$, where $\widetilde{\boldsymbol{M}}$ is defined by

$$\widetilde{\boldsymbol{M}} = \begin{bmatrix} 1 & -\beta_{1,2} & 0 \\ 0 & 1 & -\beta_{2,3} \\ -\beta_{3,1} & 0 & 1 \end{bmatrix}$$

So $\widetilde{\boldsymbol{M}}$ is singular, and $\det(\widetilde{\boldsymbol{M}}) = 1 - \beta_{3,1}\beta_{1,2}\beta_{2,3} = 0$. Therefore the following relation holds for the collision ratios:

$$\beta_{2,3} = \frac{1}{\beta_{1,2}\beta_{3,1}} = \frac{\beta_{2,1}}{\beta_{3,1}}$$

Define the mass m_k of ball B_k by

$$m_k = \beta_{k,1}$$

in which B_1 is considered as having unit mass. Then $\beta_{2,3} = m_2/m_3$, from which (10) and (11) follow. □

3.3 Mass, momentum and energy.

We now can describe the behaviour of systems of material balls, free moving except for mutual collisions that may take place [2]. For such systems we define:

[2] Simple (one-atomic) gasses under moderate physical circumstances are examples of these systems.

Definition 4 (Energy and momentum) *Let S be any system of N free ideal balls, with masses m_k, and velocities $\boldsymbol{v}_k$, for $k = 1, 2, \ldots, N$. The system's* **momentum** *is defined by*

$$\boldsymbol{p} = \sum_{k=1}^{N} m_k \boldsymbol{v}_k \tag{12}$$

The system's **energy** *W is defined by*

$$W = \sum_{k=1}^{N} \frac{1}{2} m_k \|\boldsymbol{v}_k\|^2 \tag{13}$$

If the same system is observed in $\mathcal{G}(\boldsymbol{u})$, then momentum and energy read

$$\widetilde{\boldsymbol{p}} = \sum_{k=1}^{N} m_k(\boldsymbol{v}_k - \boldsymbol{u}) = \boldsymbol{p} - M\boldsymbol{u} \tag{14}$$

$$\widetilde{W} = \sum_{k=1}^{N} \frac{1}{2} m_k \|\boldsymbol{v}_k - \boldsymbol{u}\|^2 = W - \boldsymbol{p}\boldsymbol{\cdot}\boldsymbol{u} + \frac{1}{2} M \|\boldsymbol{u}\|^2 \tag{15}$$

with $M = \sum m_k$ is the total mass in the system. The direct consequence of theorem 1 is that in a system of free material balls, not interacting with anything but each other, (a so-called *closed system*)

1. the total momentum is a constant vector (is 'conserved').
2. the total energy cannot increase

We next consider what happens with momentum and energy if masses collide with other objects, such as rigid bodies of arbitrary shape. We select a family of objects with relatively simple properties.

Definition 5 (State of rest of an object) *A material object is in a state of rest if all points of the object are at rest, and stay at rest.*

Definition 6 (Admissible objects) *A material object is admissible if*
1. it can be in a state of rest,
2. it can always be brought into a state of rest by a sequence of collisions with material balls.
3. it has zero momentum in a state of rest.

In this definition 'momentum' must be interpreted as: *if nothing material moves, there is zero momentum.*
Rigid bodies, mechanisms of rigid bodies, as well as classical mass-spring systems are admissible objects, which can be verified by inspection of the definition. We restrict the analysis to admissible objects.

Lemma 5 *Let a system of N free material balls interact with each other and with an admissible object, and let at $t = t_e$ this object be in exactly the same state as before all interactions. Then the total energy of the balls has not increased:*

$$\delta W = \sum_{k=1}^{N} \frac{1}{2} m_k \left(\|\boldsymbol{v}_k(t_e)\|^2 - \|\boldsymbol{v}_k(t_0)\|^2 \right) \leq 0 \tag{16}$$

Proof: Assume (16) doesn't hold, so $W' > W$. We prove that this violates the law of decrease.
After the transactions, the balls have velocities $\boldsymbol{v}'_k$. For some balls $\|\boldsymbol{v}'_k\| > \|\boldsymbol{v}_k\|$, for others $\|\boldsymbol{v}'_k\| \leq \|\boldsymbol{v}_k\|$, so we cannot verify or falsify the law of decrease directly. Now arrange extra *ideal* collisions between the balls and if necessary suitable stand-ins [3], such that in the end $\|\boldsymbol{v}''_k\| = \|\boldsymbol{v}_k\|$, $k = 1, 2, \ldots N' \leq N$, with N' is as large as possible. Since all collisions in this process are ideal, the total energy doesn't change, and therefore $W'' = W' > W$, which implies $\|\boldsymbol{v}''_k\| > \|\boldsymbol{v}_k\|$ for $k = N' + 1, \ldots, N$. This contradicts the law of decrease, and therefore proves (16). □

With lemma 5, we can extend the energy and momentum properties to admissible objects.

Theorem 2 (Energy and momentum of admissible objects)
i. An admissible object carries energy W_{obj} and momentum $\boldsymbol{p}_{\text{obj}}$.
ii. In interactions with a system of free mass-points with total momentum $\boldsymbol{p}$, the following relation holds

$$\delta \boldsymbol{p}_{\text{obj}} + \delta \boldsymbol{p} = \mathbf{0} \tag{17}$$

iii. A composition of a finite number of separate admissible objects is an admissible object, of which the total energy and momentum equals the sum of energies and momentums of the components.

Proof:
(i) If an admissible object is not at rest, then at least one of it's mass-points has nonzero velocity. If such mass collides with a free mass with zero velocity, the free mass will get a nonzero secondary velocity $\boldsymbol{v}'$, and therefore nonzero energy. Since no other objects are

[3] identical balls, with the same velocities, but a more suitable orbit.

involved, this energy ‘comes from’ the object. So apparently the object carries positive energy if not in a state of rest.
(ii) Let an admissible object $\mathcal{O}$ be initially in some state S. Let it collide with a number of free mass-points, and let $\delta_1 \boldsymbol{p}$ and $\delta_1 W$ be the total change of momentum and energy of these mass-points. Now continue with hitting the object until it is in the state S again. Let the total change in momentum and energy of the free mass-points due to this process be $\delta_2 \boldsymbol{p}$ and $\delta_2 W$ respectively. Since the state of the object $\mathcal{O}$ has not changed after these transactions, the energy of the free masses may not have increased:

$$\delta W = \delta_1 W + \delta_2 W \leq 0$$

If the transactions are observed from within the inertial system $\mathcal{G}(\boldsymbol{u})$, with arbitrary $\boldsymbol{u}$, then also in $\mathcal{G}(\boldsymbol{u})$ the object $\mathcal{O}$ has returned to its original state, so in $\mathcal{G}(\boldsymbol{u})$ the energy of the free masses may not have increased as well. Applying (15), with $\delta \boldsymbol{p} = \delta_1 \boldsymbol{p} + \delta_2 \boldsymbol{p}$, we have

$$\delta \widetilde{W} = \delta W - \boldsymbol{u} \cdot \delta \boldsymbol{p} \leq 0, \text{ for all } \boldsymbol{u}$$

It follows

$$\delta \boldsymbol{p} = \delta_1 \boldsymbol{p} + \delta_2 \boldsymbol{p} = \boldsymbol{0}$$

The free masses lost $-\delta_1 \boldsymbol{p}$ momentum in the first transaction, and received $\delta_2 \boldsymbol{p} = -\delta_1 \boldsymbol{p}$ after the next transactions. In between, the lost momentum was apparently held by the object $\mathcal{O}$. Since $\mathcal{O}$ has zero momentum if at rest, we can define its momentum $\boldsymbol{p}_{\text{obj}}$ as the total momentum *received* by a pool of free masses in bringing the object back to a state of rest:

$$\boldsymbol{p}_{\text{obj}} = \delta_2 \boldsymbol{p} \tag{18}$$

Obviously, the vector $\boldsymbol{p}_{\text{obj}}$ satisfies (17).
(iii) For each component of the compositon, momentum and energy can be transfered to free masses, together counting for the total energy and momentum, according to definition 4. □

3.4 Semi-rigid mass-point constructions.

In theorem 2 the *existence* of energy and momentum were proved for admissible objects. For a specific class of admissible objects, we can quantify these properties.

Definition 7 (semi-rigid mass-point construction)
1. An object consisting of N mass-points is a semi-rigid mass-point construction if some or all of the mutual distances are constant in time:

$$\|\boldsymbol{x}_k - \boldsymbol{x}_l\| = d_{k,l} \text{ is constant for some } k, l \text{ combinations.}$$

2. If all k, l combinations have constant distances, then the construction is called rigid.

The common terminology for this kind of construction in the field of mechanisms is *flexible*. We choose the term *semi-rigid*, because the definition must also cover a rigid body, and it is contra-intuitive to call a rigid body 'flexible'.

A semi-rigid mass-point construction is at rest if all masses are at rest. Since in that case the joining bars are not active, the masses are effectively free, and stay at rest. Therefore a semi-rigid mass-point construction is an admissible object.

In the definition nothing is mentioned about how the distances are kept fixed. We'll use suggestive terminology like 'connections' and 'links', but all assumed properties of these concepts may not be used in the analysis. We only use the requirements $\|\boldsymbol{x}_k - \boldsymbol{x}_l\|$ is constant in time, regardless the way this is achieved. For practical implementations of semi-rigid mass-point constructions one can imagine material pieces like bars and pivots, to keep distances constant. The pivots must have spherical freedom, such that two free bars, only connected by such a pivot, can take any relative orientations freely. But we don't want bars and pivots to play a role in the *dynamics* of the semi-rigid mass-point construction . Therefore we imagine the use of extremely lightweight material for these purposes, and in fact we assume these parts to have 'zero mass'. We call such bars and pivots *ideal bars* and *ideal pivots*.

Energy and momentum of a semi-rigid mass-point construction must be functions of the individual velocities of the masses in the construction. The obvious quantification appears to be $\boldsymbol{p} = \sum m_k \boldsymbol{v}_k$ and $W = \sum \frac{1}{2} m_k \|\boldsymbol{v}_k\|^2$, according to (12) and (13). However, we do not know whether these quantities behave like energy and momentum if collisions with mass-points are involved. Therefore we call them 'formal energy' and 'formal momentum'

Definition 8 *The formal energy W^{form} and formal momentum $\boldsymbol{p}^{\text{form}}$ of a N-point semi-rigid mass-point construction are given by*

$$W^{\text{form}} = \sum_{k=1}^{N} \frac{1}{2} m_k \|\dot{\boldsymbol{x}}_k\|^2, \quad \boldsymbol{p}^{\text{form}} = \sum_{k=1}^{N} m_k \dot{\boldsymbol{x}}_k \tag{19}$$

We start the analysis with the simplest non-trivial semi-rigid mass-point construction , the *barbell*.

Definition 9 (Barbell) *A barbell is a semi-rigid mass-point construction consisting of two mass-points m_1 and m_2, connected by an ideal bar of fixed length L. L is called the length of the barbell.*

Lemma 6 *A free moving barbell has energy and momentum equal to the formal energy and momentum respectively.*

Proof: Consider a barbell with masses m_1 and m_2 located in $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ respectively. If the barbell is moving uniformly, that is if it is at rest in some inertial system $\mathcal{G}(\boldsymbol{u})$, the connecting bar is not active, and could as well be removed. So a uniform moving barbell is equivalent to a system of free moving masses, hence it's momentum and energy satisfy

$$\boldsymbol{p} = (m_1 + m_2)\boldsymbol{u},\ W = \frac{1}{2}(m_1 + m_2)\|\boldsymbol{u}\|^2$$

which correspond with the formal versions.
Consider a free mass m_e with velocity $\boldsymbol{v}_e = \boldsymbol{u} + \widetilde{\boldsymbol{v}}_e$, where $\widetilde{\boldsymbol{v}}_e \perp \boldsymbol{x}_2 - \boldsymbol{x}_1$. Let this mass collide *centrally* with m_1, then the connecting bar is not a constraint for this collision so the collision is practically free. Assume for simplicity $m_e = m_1$, then in $\mathcal{G}(\boldsymbol{u})$ the two masses m_1 and m_e 'exchange velocities', whereas the motion of m_2 is not affected:

$$\widetilde{\boldsymbol{v}}_1' = \widetilde{\boldsymbol{v}}_e,\ \widetilde{\boldsymbol{v}}_e' = \boldsymbol{0},\ \widetilde{\boldsymbol{v}}_2' = \widetilde{\boldsymbol{v}}_2 = \boldsymbol{0}$$

Momentum transfer satisfies $\delta\boldsymbol{p}_e = m_e\delta\boldsymbol{v}_e$, and according to conservation of momentum, we must have for the barbell:

$$\delta\boldsymbol{p} = -\delta\boldsymbol{p}_e = m_1\delta\boldsymbol{v}_1$$

Therefore the new momentum of the barbell reads

$$\boldsymbol{p}' = (m_1 + m_2)\boldsymbol{u} + m_1\delta\boldsymbol{v}_1 = m_1\boldsymbol{v}_1' + m_2\boldsymbol{v}_2'$$

Similarly for the energy:

$$W' = \frac{1}{2}(m_1 + m_2)\|\boldsymbol{u}\|^2 + \frac{1}{2}m_1\left(\|\boldsymbol{v}_1'\|^2 - \|\boldsymbol{u}\|^2\right) = \frac{1}{2}\left(m_1\|\boldsymbol{v}_1'\|^2 + m_2\|\boldsymbol{v}_2'\|^2\right)$$

So energy and momentum equal the formal versions.
Now let the barbel be in any state, with velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Because of the rigidity, we must have

$$\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|^2 \text{ is constant } \Longrightarrow (\boldsymbol{x}_2 - \boldsymbol{x}_1)\bullet(\boldsymbol{v}_2 - \boldsymbol{v}_1) = 0$$

so $\boldsymbol{v}_2 - \boldsymbol{v}_1 \perp \boldsymbol{x}_2 - \boldsymbol{x}_1$. Now choose the inertial system $\mathcal{G}(\boldsymbol{u})$ with $\boldsymbol{u} = \boldsymbol{v}_2$. Then $\widetilde{\boldsymbol{v}}_2 = \boldsymbol{0}$, and $\widetilde{\boldsymbol{v}}_1 = \boldsymbol{v}_1 - \boldsymbol{v}_2 \perp \boldsymbol{x}_2 - \boldsymbol{x}_1$. The barbell can be brought into this state by a suitable collision like described before, from a state of rest in $\mathcal{G}(\boldsymbol{v}_2)$. Therefore in any state of motion, the barbell's momentum and energy equal the formal momentum and formal energy respectively. □

For a general semi-rigid mass-point construction , we cannot execute 'smart' collisions, for which the affected mass can be considered as momentarily free. So we need an extra tool for proving a 'formal = actual' statement for arbitrary semi-rigid mass-point constructions
Suppose we break the link between the masses of a barbell at time t, by applying a clipping device in a point $\widehat{\boldsymbol{x}} = \theta\boldsymbol{x}_1 + (1-\theta)\boldsymbol{x}_2$. In general, after the clipping, the masses will move on with velocities $\boldsymbol{v}_1' \neq \boldsymbol{v}_1(t)$ and $\boldsymbol{v}_2' \neq \boldsymbol{v}_2(t)$, where these velocity jumps and the corresponding momentum and energy jumps are due to the clipping. To which extent this happens depends on the velocity difference between the clipping device and the point $\widehat{\boldsymbol{x}}$ at the moment of clipping.

At zero velocity difference, the clipping procedure will not act as a constraint, and hence *no momentum transfer will take place to either part of the clipped bar*. So if the clipping is done carefully, we may expect $\boldsymbol{v}_1' = \boldsymbol{v}_1(t)$ and $\boldsymbol{v}_2' = \boldsymbol{v}_2(t)$, and both masses continue in a uniform motion at the same velocities they had immediately before the clipping. We describe this procedure formally:

Definition 10 *An ideal clipper is an instrument with which an ideal bar can be 'clipped' in two parts, without transfer of energy or momentum to either part of the bar.*

Theorem 3 *A semi-rigid construction of N mass-points $m_1, m_2, \ldots, m_N$, with velocities $\boldsymbol{v}_k(t)$, has energy $W = W^{\text{form}}$, and momentum $\boldsymbol{p} = \boldsymbol{p}^{\text{form}}$*

Proof: Denote the actual momentum and energy of construction with b bars by $\boldsymbol{p}_b$, and W_b. In the formal momentum and energy, the number of bars doesn't play a role.
A semi-rigid mass-point construction that contains only one bar consists of a barbell and $N-2$ free masses. For the barbell the hypothesis has been proved in lemma 6, and according to theorem 2 (iii), the theorem holds for $b = 1$.
Assumption: Suppose there are semi-rigid mass-point constructions not satisfying the theorem, then there is one with a minimal number $\widehat{b}$ of bars, with $\widehat{b} > 1$.
For this minimal construction we must have $W_{\widehat{b}} \neq W^{\text{form}}$ or $\boldsymbol{p}_{\widehat{b}} \neq \boldsymbol{p}^{\text{form}}$. After removing one bar with an ideal clipper, energy and momentum haven't changed, so $W_{\widehat{b}-1} = W_{\widehat{b}}$, and $\boldsymbol{p}_{\widehat{b}-1} = \boldsymbol{p}_{\widehat{b}}$. On the other hand, $W_{\widehat{b}-1} = W^{\text{form}}$, and $\boldsymbol{p}_{\widehat{b}-1} = \boldsymbol{p}^{\text{form}}$. It follows

$$\begin{aligned} \boldsymbol{p}_b &= \boldsymbol{p}^{\text{form}} \\ W_b &= W^{\text{form}} \end{aligned}$$

contradicting the assumption, which proves the theorem. □

4 Mechanics of interactions.

4.1 Axiom 3: Law of least frustration.

The law of inertia can also be frustrated in more regular ways, like throwing a ball, or starting a car by dragging it. We model this kind of actions by *dragging* operations. If we consider a mass-point, pulled by a dragging point with a prescribed motion, then we can see the following behaviour:

Observation 1 *The acceleration of the mass-point* is always directed to the dragging point, *regardless the dragging point's own motion* [4].

[4]Simple explanation in classical mechanics: *Assume the mass-point is connected to the dragging point by an ultra lightweight, infinitely flexible cord (a so called ideal cord). This cord can only transfer forces in its own direction, that is the direction from mass-point to dragging point. And, by Newton's second law, acceleration is proportional to the force. Not usable explanation because we do not yet have forces, and certainly not Newton's second law.*

The observation can be extended to the case of two or more dragging points acting on one mass-point:

Observation 2 *With two or more dragging points, in one plane with the mass-point, the acceleration vector of the mass-point is in this plane.*

For the case that three or more dragging points are not in one plane with the mass, the acceleration of the mass is kinematically determined (or overdetermined).

We analyse the thought-experiments in an elementary way. Imagine a mass-point located at $\boldsymbol{x}$, dragged by N dragging points $\boldsymbol{x}_k(t)$, $k = 1, 2, \dots, N$, such that $\|\boldsymbol{x}_k(t) - \boldsymbol{x}(t)\| = L_k$. Write $\boldsymbol{r}_k = \boldsymbol{x}_k - \boldsymbol{x}$, then $\boldsymbol{r}_k$ has constant length, and we have for $k = 1, 2, \dots, N$

$$\|\boldsymbol{r}_k\|^2 = L_k^2 \Longrightarrow \dot{\boldsymbol{r}}_k \boldsymbol{\cdot} \boldsymbol{r}_k = 0 \Longrightarrow \ddot{\boldsymbol{r}}_k \boldsymbol{\cdot} \boldsymbol{r}_k + \|\dot{\boldsymbol{r}}_k\|^2 = 0 \tag{20}$$

The mass-point's acceleration $\ddot{\boldsymbol{x}}$ must satisfy the N relations

$$\ddot{\boldsymbol{x}} \boldsymbol{\cdot} \boldsymbol{r}_k = \ddot{\boldsymbol{x}}_k \boldsymbol{\cdot} \boldsymbol{r}_k + \|\dot{\boldsymbol{r}}_k\|^2 = b_k \tag{21}$$

The righthand sides of these equalities are given quantities, since all positions, velocities and the accelerations of the dragging points are known.
The equations can be inconsistent. If the vectors $\boldsymbol{r}_k$ are linearly dependent (which is certainly the case for $N > 3$), then combinations exist for which $\sum \lambda_k \boldsymbol{r}_k = \boldsymbol{0}$, with not all λ_k zero. Then also $\sum \lambda_k b_k = 0$ must hold. If this is true for all possible null-combinations, then the configuration is *kinematically consistent*.
The outcome of the dragging experiments can be described as

$$\ddot{\boldsymbol{x}} = \sum_{k=1}^{N} \xi_k \boldsymbol{r}_k$$

Let $\boldsymbol{R} = [\boldsymbol{r}_1\, \boldsymbol{r}_2\, \cdots\, \boldsymbol{r}_N]$, the $3 \times N$ matrix of which $\boldsymbol{r}_k$ are the columns, and let $b_k = \ddot{\boldsymbol{x}}_k \boldsymbol{\cdot} \boldsymbol{r}_k + \|\dot{\boldsymbol{r}}_k\|^2$, for $k = 1, 2, \dots, N$. We can describe the experiment as follows

$$\ddot{\boldsymbol{x}} = \boldsymbol{R}\boldsymbol{\xi}, \quad \textit{(The observations)} \tag{22}$$
$$\boldsymbol{R}^T \ddot{\boldsymbol{x}} = \boldsymbol{b}, \quad \textit{(The constraints)} \tag{23}$$
$$\boldsymbol{R}^T \boldsymbol{R}\boldsymbol{\xi} = \boldsymbol{b}, \quad \textit{(Determination of } \xi\textit{)} \tag{24}$$

The expression $\ddot{\boldsymbol{x}} = \boldsymbol{R}\boldsymbol{\xi}$ is trivial if $\boldsymbol{R}$ has rank three, since $\mathcal{R}(\boldsymbol{R}) = \mathbb{R}^3$ in that case. If rank$(\boldsymbol{R}) \leq 2$, the expression $\ddot{\boldsymbol{x}} = \boldsymbol{R}\boldsymbol{\xi}$ really means a special choice out of all possible vectors. Consider an alternative $\tilde{\ddot{\boldsymbol{x}}}$:

$$\tilde{\ddot{\boldsymbol{x}}} = \boldsymbol{R}\boldsymbol{\xi} + \boldsymbol{z}$$

then because of the constraints $\boldsymbol{R}^T \boldsymbol{z} = 0$. So $\boldsymbol{z} \perp \boldsymbol{R}\boldsymbol{\xi}$, and hence

$$\|\tilde{\ddot{\boldsymbol{x}}}\|^2 = \|\ddot{\boldsymbol{x}}\|^2 + \|\boldsymbol{z}\|^2$$

Therefore we can interprete the solution given by (24) and (22) as the minimum norm solution of (23).

So the observatons 1 and 2 lead to the following statement, introduced here as an axiom.

Axiom 3 (Law of least frustration) *If the motion of a mass-point has kinematic restrictions, the mass-point responds with minimal change of velocity.*

The frustration of the law of inertia can be quantified by any monotonic increasing function of the absolute acceleration. For practical reasons, we choose

Definition 11 (Frustration) *The frustration of a mass-point, that is prevented from moving according to the law of inertia is represented by*

$$F(\ddot{\boldsymbol{x}}) = \frac{1}{2}C\|\ddot{\boldsymbol{x}}\|^2 \tag{25}$$

in which C is an arbitrary positive constant. This function will be called the frustration function for the mass-point.

Axiom 3 can be reformulated as

> If a single mass-point m is subject to kinematical constraints, then $F(\ddot{\boldsymbol{x}})$ is minimal within these constraints.

We defined the 'state of rest' as a state in which material points have velocity zero, so their positions are fixed. A point that is in a state of rest in an inertial system $\mathcal{G}(\boldsymbol{u})$, is moving at uniform velocity $\boldsymbol{u}$ in $\mathcal{G}(\mathbf{0})$. Conversely: every mass-point that is moving uniformly with velocity $\boldsymbol{v}$, is in a state of rest in $\mathcal{G}(\boldsymbol{v})$. We can interprete the state of uniform motion as a *generalized state of rest*. If there are no frustrating events, the position $\boldsymbol{x}(t)$ and the velocity $\boldsymbol{v}(t) = \dot{\boldsymbol{x}}(t)$ at time t are completely determined by the initial position $\boldsymbol{x}(t_0)$ and the initial velocity $\boldsymbol{v}(t_0)$:

$$\boldsymbol{v}(t) = \boldsymbol{v}(t_0), \quad \boldsymbol{x}(t) = \boldsymbol{x}(t_0) + (t - t_0)\boldsymbol{v}(t_0) \tag{26}$$

For a system of material points, we call the position vectors and the velocities *state variables*.

If the motions of the points in a mechanical system are frustrated by external causes like dragging, or by internal causes, like mutual links, the state variables will not satisfy (26) anymore. In this section will be shown that, on the basis of the axioms, the accelerations $\ddot{\boldsymbol{x}}_k(t)$ are completely determined by the positions and velocities at time t. This results in a system of (coupled) differential equations, called the *equations of motion* for the system, from wich the time evolution of the state variables can be solved.

4.2 Mathematical prerequisite.

In building a mechanical theory based on axiom 3, we'll use an important tool from constrained optimization theory, known as *Lagrange's multiplier method*. It is probably well-known, but poorly explained in some elementary texts. Therefore a derivation of the method is given in the following lemma.
We restrict the analysis to the minimization of quadratic functions in $\mathbb{R}^N$ under k linear constraints.

Lemma 7 (Lagrange multiplier method) *Let $\boldsymbol{A}$ be an $N \times N$ positive definite matrix, $\boldsymbol{C}$ a real $N \times k$ matrix, and $\boldsymbol{p}$ and $\boldsymbol{b}$ vectors in $\boldsymbol{R}^N$ and $\boldsymbol{R}^k$ respectively, and let F be defined by $F(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} - \boldsymbol{p}^T\boldsymbol{x}$. Consider the problem:*

$$\text{Minimize} \quad F(\boldsymbol{x}) \tag{27}$$
$$\text{requiring} \quad \boldsymbol{C}^T\boldsymbol{x} = \boldsymbol{b} \tag{28}$$

(i) If this problem has a solution, this solution can be found in the following way:
1. Define the augmented *function $\widetilde{F}(\boldsymbol{x}) = F(\boldsymbol{x}) - \boldsymbol{\lambda}^T(\boldsymbol{C}^T\boldsymbol{x} - \boldsymbol{b})$, and minimize this function* formally,
2. Determine the multiplier vector $\boldsymbol{\lambda}$ from the constraint (28), using the formal solution $\boldsymbol{x}$.

(ii) The problem has a solution if and only if the constraining equations (28) are consistent.

(iii) If $\boldsymbol{x}$ is a solution, this solution is unique.

Proof: (i) Assume the problem has a solution $\boldsymbol{x}$. Then $F(\boldsymbol{x} + t\boldsymbol{u}) \geq F(\boldsymbol{x})$ for all real t, and all vectors $\boldsymbol{u}$ satisfying $\boldsymbol{C}^T\boldsymbol{u} = \boldsymbol{0}$. Working out $F(\boldsymbol{x} + t\boldsymbol{u})$, we get

$$F(\boldsymbol{x} + t\boldsymbol{u}) = F(\boldsymbol{x}) + t\boldsymbol{u}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{p}) + \frac{1}{2}t^2\boldsymbol{u}^T\boldsymbol{A}\boldsymbol{u} \geq F(\boldsymbol{x}),$$

for all t, and for all $\boldsymbol{u} \in \mathbb{R}^N$ satifying $\boldsymbol{C}^T\boldsymbol{u} = \boldsymbol{0}$. Since $\boldsymbol{u}^T\boldsymbol{A}\boldsymbol{u} \geq 0$, this implies

$$\boldsymbol{u}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{p}) = \boldsymbol{0}, \text{ for all vectors } \boldsymbol{u} \text{ in } \mathcal{N}(\boldsymbol{C}^T) \tag{29}$$

where $\mathcal{N}(\boldsymbol{C}^T)$ denotes the nullspace of $\boldsymbol{C}^T$, that is: all vectors satisfying $\boldsymbol{C}^T\boldsymbol{u} = \boldsymbol{0}$.
According to the projection theorem [5] each vector in $\mathbb{R}^N$ can be splitted into a component in $\mathcal{R}(\boldsymbol{C})$, the column space of $\boldsymbol{C}$, and a component perpendicular to $\mathcal{R}(\boldsymbol{C})$, i.e. a component in $\mathcal{N}(\boldsymbol{C}^T)$, the null space of $\boldsymbol{C}^T$. For the vector $\boldsymbol{A}\boldsymbol{x} - \boldsymbol{p}$, this splitting property then reads

$$\boldsymbol{A}\boldsymbol{x} - \boldsymbol{p} = \boldsymbol{C}\boldsymbol{\lambda} + \boldsymbol{z}, \quad \text{with} \quad \boldsymbol{C}^T\boldsymbol{z} = 0$$

Substitute this in (29)

$$0 = \boldsymbol{u}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{p}) = \boldsymbol{u}^T\boldsymbol{C}\boldsymbol{\lambda} + \boldsymbol{u}^T\boldsymbol{z} = \boldsymbol{u}^T\boldsymbol{z}, \text{ for all } \boldsymbol{u} \in \mathcal{N}(\boldsymbol{C}^T)$$

[5] Let $\boldsymbol{Q}$ be an $N \times k$ matrix with orthonormal columns, an let $\boldsymbol{x}$ be any vector in $\mathbb{R}^N$, then $\boldsymbol{v} = \boldsymbol{Q}\boldsymbol{Q}^T\boldsymbol{x}$ in the column space of $\boldsymbol{Q}$ and $\boldsymbol{w} = \boldsymbol{x} - \boldsymbol{v}$ performs such a splitting.

since $\boldsymbol{u}^T \boldsymbol{C}\boldsymbol{\lambda} = \boldsymbol{\lambda}^T \boldsymbol{C}^T \boldsymbol{u} = 0$ for all these $\boldsymbol{u}$. Choose $\boldsymbol{u} = \boldsymbol{z} \in \mathcal{N}(\boldsymbol{C}^T)$, we get $\boldsymbol{z}^T \boldsymbol{z} = 0$, implying $\boldsymbol{z} = \boldsymbol{0}$. It follows that if the constrained minimization problem (27) and (28) has a solution, this solution satisfies

$$\boldsymbol{A}\boldsymbol{x} - \boldsymbol{p} = \boldsymbol{C}\boldsymbol{\lambda} \tag{30}$$

Equation (30) can be obtained by setting to zero the partial derivatives of $\widetilde{F}$ with respect to the components of $\boldsymbol{x}$:

$$\frac{\partial \widetilde{F}}{\partial \boldsymbol{x}^T} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{p} - \boldsymbol{C}\boldsymbol{\lambda} = \boldsymbol{0}$$

For determining the vector $\boldsymbol{\lambda}$, we use the constraints.

$$\boldsymbol{C}^T \boldsymbol{x} = \boldsymbol{b} \Longrightarrow \boldsymbol{C}^T \boldsymbol{A}^{-1}(\boldsymbol{p} + \boldsymbol{C}\boldsymbol{\lambda}) = \boldsymbol{b}$$

and we may solve $\boldsymbol{\lambda}$ from

$$\boldsymbol{C}^T \boldsymbol{A}^{-1} \boldsymbol{C}\boldsymbol{\lambda} = \boldsymbol{b} - \boldsymbol{C}^T \boldsymbol{A}^{-1} \boldsymbol{p} \tag{31}$$

Practically we can solve the problem by solving $\boldsymbol{\lambda}$ from (31), followed by solving $\boldsymbol{x}$ from (30), which is in fact the block-Gaussian elimination procedure applied to the $(N+k) \times (N+k)$ linear system

$$\begin{bmatrix} \boldsymbol{A} & -\boldsymbol{C} \\ -\boldsymbol{C}^T & \boldsymbol{O} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{p} \\ -\boldsymbol{b} \end{bmatrix} \tag{32}$$

(ii) For consistency of this system is required that each $\boldsymbol{u}$ satisfying $\boldsymbol{u}^T \boldsymbol{C}^T \boldsymbol{A}^{-1} \boldsymbol{C} = \boldsymbol{0}^T$, also satisfies $\boldsymbol{u}^T(\boldsymbol{b} - \boldsymbol{C}^T \boldsymbol{A}^{-1} \boldsymbol{p}) = \boldsymbol{0}^T$. Now since $\boldsymbol{A}$ is positive definite, we have

$$\boldsymbol{u}^T \boldsymbol{C}^T \boldsymbol{A}^{-1} \boldsymbol{C} = \boldsymbol{0}^T \Longrightarrow \boldsymbol{u}^T \boldsymbol{C}^T \boldsymbol{A}^{-1} \boldsymbol{C}\boldsymbol{u} = 0 \Longrightarrow \boldsymbol{C}\boldsymbol{u} = \boldsymbol{0}$$

It follows that (31) is consistent if and only if $\boldsymbol{C}\boldsymbol{u} = \boldsymbol{0}$ implies $\boldsymbol{u}^T \boldsymbol{b} = \boldsymbol{0}$, meaning that $\boldsymbol{b}$ is in the column space of $\boldsymbol{C}^T$, or in other words if the constraining equations are consistent.

(iii) Finally let $\widetilde{\boldsymbol{x}}$ be another solution of the problem, then write $\delta\boldsymbol{x} = \widetilde{\boldsymbol{x}} - \boldsymbol{x}$, and we have

$$\boldsymbol{A}\delta\boldsymbol{x} = \boldsymbol{C}\delta\boldsymbol{\lambda}, \;\; \boldsymbol{C}^T \boldsymbol{A}^{-1} \boldsymbol{C}\delta\boldsymbol{\lambda} = \boldsymbol{0}$$

Then it follows $\boldsymbol{C}\delta\boldsymbol{\lambda} = \boldsymbol{0}$, and hence $\delta\boldsymbol{x} = \boldsymbol{0}$. So the solution for $\boldsymbol{x}$ is unique. □

Corollary 1 *The combined system (32) can also be obtained by puting to zero the partial derivatives of the augmented frustration function with respect to all variables $\ddot{\boldsymbol{x}}$ and $\boldsymbol{\lambda}$:*

$$\frac{\partial \widetilde{F}}{\partial \ddot{\boldsymbol{x}}} = \boldsymbol{0}, \;\; \frac{\partial \widetilde{F}}{\partial \boldsymbol{\lambda}} = \boldsymbol{0}$$

4.3 Constrained motion of one mass-point.

As a demonstration of the multiplier method, we again derive equations (24) and (22), for the problem of one mass m, dragged by N dragging points. Let the mass be located in $\boldsymbol{x}$, and the dragging points in $\boldsymbol{x}_k = \boldsymbol{x} + \boldsymbol{r}_k$, satisfying (20), (21). The problem can be described by

$$\begin{aligned} &\text{Minimize} && F(\boldsymbol{x}) && (33)\\ &\text{requiring} && \boldsymbol{R}^T\boldsymbol{x} = \boldsymbol{b} && (34) \end{aligned}$$

where the frustration function F, the constraint matrix $\boldsymbol{R}$ and the components of $\boldsymbol{b}$ are given by

$$F(\ddot{\boldsymbol{x}}) = \frac{1}{2}m\|\ddot{\boldsymbol{x}}\|^2 \quad (35)$$

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{r}_1 & \boldsymbol{r}_2 & \dots & \boldsymbol{r}_N \end{bmatrix} \quad (36)$$

$$b_k = \boldsymbol{r}_k^T\ddot{\boldsymbol{x}}_k + \|\dot{\boldsymbol{r}}_k\|^2, \; k = 1, 2, \dots, N \quad (37)$$

The augmented frustration function reads

$$\widetilde{F}(\ddot{\boldsymbol{x}}) = \frac{1}{2}m\|\ddot{\boldsymbol{x}}\|^2 - \left(\boldsymbol{R}^T\ddot{\boldsymbol{x}} - \boldsymbol{b}\right)^T\boldsymbol{\lambda}$$

Minimization results in the combined linear system

$$\begin{bmatrix} m\boldsymbol{I} & -\boldsymbol{R} \\ -\boldsymbol{R}^T & \boldsymbol{O} \end{bmatrix} \cdot \begin{bmatrix} \ddot{\boldsymbol{x}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ -\boldsymbol{b} \end{bmatrix} \quad (38)$$

Applying block-Gauss elimination on this system produces

$$\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{\lambda} = m\boldsymbol{b} \quad (39)$$

$$m\ddot{\boldsymbol{x}} = \boldsymbol{R}\boldsymbol{\lambda} = \sum_k \lambda_k\boldsymbol{r}_k \quad (40)$$

which is equivalent to the former result (24) and (22).
If the constraining equations are consistent, but overdetermined, then the solution for $\boldsymbol{\lambda}$ is not unique. The acceleration however is unique. Different combinations of multipliers lead to the same kinematic result.
The multipliers describe the *dynamical effect* of the forcing. In the case of uniquely determined λ's, that is if rank$(\boldsymbol{R}^T\boldsymbol{R}) = N$, each term $\lambda_k\boldsymbol{r}_k$ represents a momentum transfer from the constraining point $\boldsymbol{x}_k$ to the mass m. In all cases the vectors $\lambda_k\boldsymbol{r}_k$ *together* provide the change in momentum of the mass m. However, the dynamical meaning of the *individual* vectors $\lambda_k\boldsymbol{r}_k$ is not obvious in the overdetermined case. If we imagine a case that several people together try to move a heavy mass by dragging or pushing, then different solutions for the multipliers can be interpreted as different amounts of effort offerered by the individual draggers.
Weillustrate this by choosing a specified mechanical principle for the dragging devices. In a barbell, either mass is dragging the other simply by preventing the other mass to move accordingly the inertia law. We analyse a generalisation of the barbell: the spider.

Definition 12 (Spider) *A* spider *is an semi-rigid mass-point construction consisting of a central mass m_0, located in $\boldsymbol{x}_0$, which is connected to N masses $m_1, m_2, \ldots, m_N$ located in $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ respectively, such that $\|\boldsymbol{x}_k - \boldsymbol{x}_0\|$ is constant for all k.*

Lemma 8 (Spider motion) *Let the matrices $\boldsymbol{R}$ and $\boldsymbol{D}$, and the vector $\boldsymbol{b}$ be defined by*

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{r}_1 & \boldsymbol{r}_2 & \ldots & \boldsymbol{r}_N \end{bmatrix} \tag{41}$$

$$\boldsymbol{D} = \operatorname{diag}\left(\frac{m_0}{m_k}\|\boldsymbol{r}_k\|^2\right) \tag{42}$$

$$\boldsymbol{b} = \begin{bmatrix} \|\dot{\boldsymbol{r}}_1\|^2 & \|\dot{\boldsymbol{r}}_2\|^2 & \ldots & \|\dot{\boldsymbol{r}}_N\|^2 \end{bmatrix}^T \tag{43}$$

Then the equations of motion of a spider read

$$m_0\ddot{\boldsymbol{x}}_0 = \sum_{k=1}^{N}\mu_k\boldsymbol{r}_k = \boldsymbol{R}\boldsymbol{\mu} \tag{44}$$

$$m_k\ddot{\boldsymbol{x}}_k = -\mu_k\boldsymbol{r}_k \tag{45}$$

where $\boldsymbol{\mu}$ satisfies

$$(\boldsymbol{D} + \boldsymbol{R}^T\boldsymbol{R})\boldsymbol{\mu} = m_0\boldsymbol{b} \tag{46}$$

Proof: Each mass of a spider is a constrained mass, and therefore its response to the constraints is determined by minimization of its frustration. This leads to the following equations.

$$m_0\ddot{\boldsymbol{x}}_0 = \sum_{j=1}^{N}\lambda_j\boldsymbol{r}_j \tag{47}$$

$$m_k\ddot{\boldsymbol{x}}_k = -\mu_k\boldsymbol{r}_k,\ k = 1, 2, \ldots, N \tag{48}$$

with

$$\boldsymbol{r}_k^T(\ddot{\boldsymbol{x}}_k - \ddot{\boldsymbol{x}}_0) + \|\dot{\boldsymbol{r}}_k\|^2 = 0,\ k = 1, 2, \ldots, N \tag{49}$$

The conservation of momentum requires

$$\sum_{j=1}^{N}\mu_j\boldsymbol{r}_j = \sum_{j=1}^{N}\lambda_j\boldsymbol{r}_j \tag{50}$$

by wich relation equations (47) and (48) reduce to (44) and (45), which is a part of the lemma.

To verify (46), we substitute (44) and (45) into (49), which leads to the following equations for the multipliers μ_k:

$$\|\dot{\boldsymbol{r}}_k\|^2 = \boldsymbol{r}_k^T \left[\frac{\boldsymbol{r}_k}{m_k}\mu_k + \sum_{j=1}^{N} \frac{\boldsymbol{r}_j}{m_0}\mu_j \right], \ k = 1, 2, \ldots, N \tag{51}$$

Using the definitions (41), (42) and (43), this can be written as

$$(\boldsymbol{D} + \boldsymbol{R}^T\boldsymbol{R})\boldsymbol{\mu} = m_0\boldsymbol{b} \tag{52}$$

In this system are $\boldsymbol{D}$ and $\boldsymbol{R}^T\boldsymbol{R}$ positive definite and positive semi-definite respectively, hence the system is uniquely solvable. So the point-wise minimal frustration argument, together with the conservation of momentum produce the equations of motion for the spider □

In the spider, the constraints are practically *dynamic* rather then kinematic. This is because the kinematical restrictions for m_0 are related to those of the other masses, and vice versa. In contrast with the rather abstract multiple dragging experiment, the individual vectors $\mu_k\boldsymbol{r}_k$ exactly mean the 'amount' of momentum that is received by m_0 from m_k.
We next show that also in the original multiple dragging experiment, the vectors $\lambda_k\boldsymbol{r}_k$ perform momentum input from $\boldsymbol{x}_k$ to the central point $\boldsymbol{x}$.

Lemma 9 (Substitution lemma) *Let the system (38) be consistent, and let $\boldsymbol{\lambda}$ and $\ddot{\boldsymbol{x}}$ be a solution, then for each k, the constraining action by $\boldsymbol{x}_k$ can be performed by any positive mass $\widetilde{m}_k$, rigidly located in a point $\widetilde{\boldsymbol{x}}_k$ somewhere on the line through $\boldsymbol{x}$ and $\boldsymbol{x}_k$, with a suitable velocity such that $\widetilde{m}_k\ddot{\widetilde{\boldsymbol{x}}}_k = -\lambda_k\boldsymbol{r}_k$.*

Proof: Any point $\widetilde{\boldsymbol{x}}_k$ on the line through $\boldsymbol{x}$ and $\boldsymbol{x}_k$, is described by

$$\widetilde{\boldsymbol{x}}_k = \boldsymbol{x} + \tau(\boldsymbol{x}_k - \boldsymbol{x}) = \boldsymbol{x} + \tau\boldsymbol{r}_k$$

Let τ be constant in time, then

$$\dot{\widetilde{\boldsymbol{x}}}_k = \dot{\boldsymbol{x}} + \tau\dot{\boldsymbol{r}}_k, \ \ddot{\widetilde{\boldsymbol{x}}}_k = \ddot{\boldsymbol{x}} + \tau\ddot{\boldsymbol{r}}_k \tag{53}$$

As long as (53) is satisfied, the mass in $\boldsymbol{x}$ is 'not aware' of which point $\widetilde{\boldsymbol{x}}$ is actually responsible for the forcing, so the solutions $\boldsymbol{\lambda}$ and $\ddot{\boldsymbol{x}}$ are not affected.
The acceleration of $\widetilde{\boldsymbol{x}}$ satisfies

$$(\ddot{\widetilde{\boldsymbol{x}}}_k - \ddot{\boldsymbol{x}}){\scriptstyle\bullet}\tau\boldsymbol{r}_k + \tau^2\|\dot{\boldsymbol{r}}\|^2 \tag{54}$$

Assume a mass $\widetilde{m}_k$ is located in $\widetilde{\boldsymbol{x}}_k$, and performs a constraint on the motion of m, then similarly the motion of $\widetilde{m}_k$ is constrained by $\boldsymbol{x}$. According to the law of minimal frustration we have

$$\widetilde{m}_k\ddot{\widetilde{\boldsymbol{x}}}_k = -\mu_k\widetilde{\boldsymbol{r}}_k = -\tau\mu_k\boldsymbol{r}_k \Longrightarrow \ddot{\widetilde{\boldsymbol{x}}}_k = \frac{\tau\mu_k\boldsymbol{r}_k}{\widetilde{m}_k} \tag{55}$$

We now find a value for τ such that $\tau\mu_k = \lambda_k$. Substituting the expression for $\ddot{\widetilde{\boldsymbol{x}}}_k$ in (54), with $\tau\mu_k = \lambda_k$ we get

$$\left(\frac{-\lambda_k \boldsymbol{r}_k}{\widetilde{m}_k} - \ddot{\boldsymbol{x}}\right) \bullet \tau \boldsymbol{r}_k + \tau^2 \|\dot{\boldsymbol{r}}\|^2 = 0$$

from which follows

$$\tau = \frac{(\lambda_k \boldsymbol{r}_k + \widetilde{m}_k \ddot{\boldsymbol{x}}) \bullet \boldsymbol{r}_k}{\widetilde{m}_k \|\dot{\boldsymbol{r}}_k\|^2} \tag{56}$$

The acceleration $\ddot{\boldsymbol{x}}$ satisfies $m\ddot{\boldsymbol{x}} = \boldsymbol{R\lambda}$, and this can now be written as

$$m\ddot{\boldsymbol{x}} = \sum_{j \neq k} \lambda_j \boldsymbol{r}_j - m_k \ddot{\widetilde{\boldsymbol{x}}}_k$$

So the constraining action of $\boldsymbol{x}_k$ can be performed by the mass $\widetilde{m}_k$ located in $\boldsymbol{x} + \tau \boldsymbol{r}_k$, with τ satisfying (56). □

4.4 General principle of least frustration.

Each point in a semi-rigid mass-point construction is constrained by the motions of the masses to which it is 'connected'. Let $\mathcal{A}_k$ denote the 'adjancy set' for $\boldsymbol{x}_k$, that is the set of indices l for which $\boldsymbol{x}_l$ is connected to $\boldsymbol{x}_k$. Then obviously $l \in \mathcal{A}_k \iff k \in \mathcal{A}_l$. By $k \sim l$ we mean $\boldsymbol{x}_k$ and $\boldsymbol{x}_l$ are connected.
Let $\boldsymbol{r}_{k,l} = \boldsymbol{x}_l - \boldsymbol{x}_k$, then $\boldsymbol{r}_{k,l}$ is constant in time whenever $l \in \mathcal{A}_k$. Then the acceleration of the mass m_k in $\boldsymbol{x}_k$ satisfies

$$m_k \ddot{\boldsymbol{x}}_k = \sum_{l \in \mathcal{A}_k} \mu_{k,l} (\boldsymbol{x}_l - \boldsymbol{x}_k) \tag{57}$$

where the multipliers $\mu_{k,l}$ follow from the requirements $\ddot{\boldsymbol{r}}_{k,l} \bullet \boldsymbol{r}_{k,l} + \|\dot{\boldsymbol{r}}_{k,l}\|^2 = 0$. If the semi-rigid mass-point construction is a barbell consisting of points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, then the conservation of momentum requires $\mu_{1,2} = \mu_{2,1}$. We prove that this also holds for every couple of connected points in an arbitrary semi-rigid mass-point construction .

Lemma 10 (Local conservation of momentum) *If in a free moving semi-rigid mass-point construction , with masses m_k in points $\boldsymbol{x}_k$, the equations (57) hold, then $\mu_{k,l} = \mu_{l,k}$ for all $k \sim l$.*

Proof: In a free moving semi-rigid mass-point construction , the (total) momentum is constant, so $\sum m_k \ddot{\boldsymbol{x}}_k = \boldsymbol{0}$. Consider two connected points $\boldsymbol{x}_k$ and $\boldsymbol{x}_l$. Apply the substitution lemma on $\boldsymbol{x}_k$, *and* on $\boldsymbol{x}_l$. Then the constraining action by $\boldsymbol{x}_l$ on the mass in $\boldsymbol{x}_k$ is now performed by a moving mass $\widetilde{m}_l$ in $\widetilde{\boldsymbol{x}}_l = \boldsymbol{x}_k + \tau_l \boldsymbol{r}_{k,l}$, with momentum change $\widetilde{m}_l \ddot{\widetilde{\boldsymbol{x}}}_l = -\mu_{k,l} \boldsymbol{r}_{k,l}$. Similarly, a mass $\widetilde{m}_k$ in $\widetilde{\boldsymbol{x}}_k = \boldsymbol{x}_l + \tau_k \boldsymbol{r}_{l,k}$, with a momentum change $\widetilde{m}_k \ddot{\widetilde{\boldsymbol{x}}}_k = -\mu_{l,k} \boldsymbol{r}_{l,k}$ performs the constraint on $\boldsymbol{x}_l$.

Since nothing else changes, still $\dot{\boldsymbol{p}} = \sum_i m_i \ddot{\boldsymbol{x}}_i = \mathbf{0}$. The modified system with the replacing masses $\widetilde{m}_k$ and $\widetilde{m}_l$ *is a free moving semi-rigid mass-point construction as well*, so $\dot{\boldsymbol{p}} + \widetilde{m}_k \ddot{\widetilde{\boldsymbol{x}}}_k + \widetilde{m}_l \ddot{\widetilde{\boldsymbol{x}}}_l = \mathbf{0}$. Hence

$$\widetilde{m}_k \ddot{\widetilde{\boldsymbol{x}}}_k + \widetilde{m}_l \ddot{\widetilde{\boldsymbol{x}}}_l = \mathbf{0} \Longrightarrow -\mu_{l,k}\boldsymbol{r}_{l,k} - \mu_{k,l}\boldsymbol{r}_{k,l} = \mathbf{0}$$

Since $\boldsymbol{r}_{k,l} = -\boldsymbol{r}_{l,k}$ this implies

$$\mu_{l,k} = \mu_{k,l} \tag{58}$$

□

We now can prove that the accelerations of the masses in any semi-rigid mass-point construction can be obtained by constrained minimization of a frustration function $F(\ddot{\boldsymbol{x}}_1, \ddot{\boldsymbol{x}}_2, \ldots, \ddot{\boldsymbol{x}}_N)$.

Theorem 4 (General principle of minimal frustration.) *For all $N \geq 1$, the equations of motion of a semi-rigid mass-point construction of size N can be obtained by constrained minimization of its frustration function*

$$F(\ddot{\boldsymbol{x}}_1, \ddot{\boldsymbol{x}}_2, \ldots, \ddot{\boldsymbol{x}}_N) = \frac{1}{2}\sum_{k=1}^{N} m_k \|\ddot{\boldsymbol{x}}_k\|^2$$

taking account of all kinematic constraints.

Proof: We analyse the equations (57) in more detail. Each point in the construction is constrained by all points in its adjacency set, and the constraints can be described by:

$$G_{k,l}(\ddot{\boldsymbol{x}}_k, \ddot{\boldsymbol{x}}_l) = (\ddot{\boldsymbol{x}}_k - \ddot{\boldsymbol{x}}_l)\boldsymbol{\cdot}\boldsymbol{r}_{k,l} - \|\dot{\boldsymbol{r}}_{k,l}\|^2 = 0, \text{ for all } l \text{ in } \mathcal{A}_k \tag{59}$$

where $\boldsymbol{r}_{k,l} = \boldsymbol{x}_l - \boldsymbol{x}_k$. For each point $\boldsymbol{x}_k$, we can use the minimization of its own augmented frustration function:

$$\widetilde{F}_k(\ddot{\boldsymbol{x}}_k) = \frac{1}{2} m_k \|\ddot{\boldsymbol{x}}_k\|^2 - \sum_{l \in \mathcal{A}_k} \mu_{k,l} G_{k,l}(\ddot{\boldsymbol{x}}_k, \ddot{\boldsymbol{x}}_l) \tag{60}$$

and minimization of this function produces (57):

$$m_k \ddot{\boldsymbol{x}}_k = \sum_{l \in \mathcal{A}_k} \mu_{k,l} \frac{\partial G_{k,l}(\ddot{\boldsymbol{x}}_k, \ddot{\boldsymbol{x}}_l)}{\partial \ddot{\boldsymbol{x}}_k} = \sum_{l \in \mathcal{A}_k} \mu_{k,l} \boldsymbol{r}_{k,l} \tag{61}$$

where the multipliers $\mu_{k,l}$ must be determined by the constraints.
If we count all multipliers, then we get twice the total number of constraints, but according to lemma 10 the multipliers are symmetric, so the effective number of multipliers equals the number of constraints, and we have exactly enough equations to determine all $\mu_{k,l}$.
Now consider the function

$$\widetilde{F}(\ddot{\boldsymbol{x}}_1, \ddot{\boldsymbol{x}}_2, \ldots, \ddot{\boldsymbol{x}}_N) = \sum_k \frac{1}{2} m_k \|\ddot{\boldsymbol{x}}_k\|^2 - \sum_{l \sim k} \mu_{k,l} G_{k,l}(\ddot{\boldsymbol{x}}_k, \ddot{\boldsymbol{x}}_l) \tag{62}$$

where the summation is taken over all *linked pairs*, i.e. every connection occurs once. We'll show that $\widetilde{F}$ is the augmented frustration function for the semi-rigid mass-point construction .
$\widetilde{F}$ depends on $\ddot{\boldsymbol{x}}_k$ via $\frac{1}{2}m_k\|\ddot{\boldsymbol{x}}_k\|^2$, and all links containing $\boldsymbol{x}_k$. These links are precisely the pairs $[\boldsymbol{x}_k, \boldsymbol{x}_l]$ with $l \in \mathcal{A}_k$. Therefore

$$\frac{\partial \widetilde{F}}{\partial \ddot{\boldsymbol{x}}_k} = m_k\ddot{\boldsymbol{x}}_k - \sum_{l\in\mathcal{A}_k} \mu_{k,l}\frac{\partial G_{k,l}(\ddot{\boldsymbol{x}}_k, \ddot{\boldsymbol{x}}_l)}{\partial \ddot{\boldsymbol{x}}_k} = m_k\ddot{\boldsymbol{x}}_k - \sum_{l\in\mathcal{A}_k} \mu_{k,l}\boldsymbol{r}_{k,l}$$

and this is zero because of (61).
So the partial derivatives of $\widetilde{F}(\ddot{\boldsymbol{x}}_1, \ddot{\boldsymbol{x}}_2, \ldots, \ddot{\boldsymbol{x}}_N)$ are zero, and hence $\widetilde{F}$ is minimized, in other words: F is minimized under the constraints represented by $G_{k,l}(\ddot{\boldsymbol{x}}_k, \ddot{\boldsymbol{x}}_l) = 0$ for all linked points.
Now finally assume that also external constraints in directions $\boldsymbol{n}_1, \boldsymbol{n}_2, \ldots, \boldsymbol{n}_N$ are present. According to lemma 9, we can replace these by moving mass-points. The resulting system is a free semi-rigid mass-point construction , so the solution is obtainable by constrained minimization of the frustration function. Later, replace the extra masses by the original forcings. This leads to the final augmented frustration function:

$$\widetilde{F}(\ddot{\boldsymbol{x}}_1, \ddot{\boldsymbol{x}}_2, \ldots, \ddot{\boldsymbol{x}}_N) = \sum_{k=1}^{N}\frac{1}{2}m_k\|\ddot{\boldsymbol{x}}_k\|^2 - \sum_{k\sim l}\mu_{k,l}G_{k,l}(\ddot{\boldsymbol{x}}_k, \ddot{\boldsymbol{x}}_l) - \sum_{k=1}^{N}\lambda_k\ddot{\boldsymbol{x}}_k\boldsymbol{\cdot}\boldsymbol{n}_k \tag{63}$$

Hence $F(\ddot{\boldsymbol{x}}_1, \ddot{\boldsymbol{x}}_2, \ldots)$ is a frustration function for the system. □

The generalized principle of minimal frustration is similar to a principle formulated by C.F. Gauss in 1829 [3]:

Principle of least constraint. *Let a mechanical system consist of N masses m_k, located in the points $\boldsymbol{x}_k$, and let $\boldsymbol{F}_k$ denote the force working on m_k, then for the true motion of this system the expression*

$$Z = \sum_{k=1}^{N}\frac{1}{2}m_k\|\ddot{\boldsymbol{x}}_k - \frac{\boldsymbol{F}_k}{m_k}\|^2 \tag{64}$$

has a minimum value.

Working out the squared norm in (64), we get

$$Z = \sum_{k=1}^{N}\left(\frac{1}{2}m_k\|\ddot{\boldsymbol{x}}_k\|^2 - \boldsymbol{F}_k\boldsymbol{\cdot}\ddot{\boldsymbol{x}}_k + \frac{1}{2}\frac{\|\boldsymbol{F}_k\|^2}{m_k}\right)$$

and apart from the constant last term, this expresssion is equivalent to the augmented frustration function, with $\lambda_k\boldsymbol{n}_k$ replaced by the forces $\boldsymbol{F}_k$.
If the masses are free, we obviously get $m_k\ddot{\boldsymbol{x}}_k = \boldsymbol{F}_k$ for all points, which is Newtons second law. If however the points have restricted freedom, then minimizing (64) can be regarded as a least squares approximation for Newtons second law.
Gauss derived his principle on the basis of d'Alemberts principle, which in turn is based on the priciple of virtual work in statics.

4.5 Forces, Newtons laws.

In practice, only simple constraining actions on the motion of a material point can be described as a proper *dragging* in which one component of the acceleration is prescribed:

$$\ddot{\boldsymbol{x}} \cdot \boldsymbol{r}_k = \ddot{\boldsymbol{x}}_k \cdot \boldsymbol{r}_k - \|\dot{\boldsymbol{r}}_k\|^2$$

where $\ddot{\boldsymbol{x}}_k$ is prescribed, and $\boldsymbol{r}_k$ and $\dot{\boldsymbol{r}}_k$ are state variables and therefore known. The links between 'connected' mass-points in a semi-rigid mass-point construction show mutual dragging, but in these cases only the *direction vectors* of the constraints are prescribed, whereas the magnitudes follow from the conservation of momentum. A similar thing happens in the case of a forcing of a mass-point's motion by a lot (say more then three) draggings. We already saw that such situations may be inconsistent, but in practice it is not unusual that a lot of persons move a heavy object by pushing and pulling in various directions. This can be considered as *attempts* of dragging actions, in which the *directions* of the dragging are prescribed.
This results in vectors $\lambda_j \boldsymbol{n}_j$, representing the input of momentum in the construction, where $\boldsymbol{n}_j$ are the prescribed directions. We call them forces.

Definition 13 (Force) *A force $\boldsymbol{F}$ acting in a point $\boldsymbol{x}$ of a mechanical construction is the input of momentum into the system via the point $\boldsymbol{x}$.*

Equation (40) can be interpreted as *'the momentum change of a mass-point equals the sum of forces acting on this mass-point'*.
Since all semi-rigid mass-point constructions satisfy theorem 4.4, the accelerations of the masses follow from minimization of the augmented frustration function, leading to the equations of motion

$$m_k \ddot{\boldsymbol{x}}_k = \sum_j \lambda_{k,j} \boldsymbol{n}_{k,j} + \sum_1^N \mu_{k,l} (\boldsymbol{x}_l - \boldsymbol{x}_k) \tag{65}$$

in which the multipliers $\lambda_{k,j}$ and $\mu_{k,l}$ can be determined from the external and internal constraints respectively. Using the terminology with forces, we call $\lambda_{k,j} \boldsymbol{n}_{k,j} = \boldsymbol{F}_{k,j}$ external forces, and $\mu_{k,l}(\boldsymbol{x}_l - \boldsymbol{x}_k) = \boldsymbol{F}_{k,l} = -\boldsymbol{F}_{l,k}$ internal forces

Resultant of forces. In equation (65), obviously the forces $\boldsymbol{F}_{k,j} = \lambda_{k,j} \boldsymbol{n}_{k,j}$ could be replaced by one single force $\boldsymbol{F}_k = \sum_j \boldsymbol{F}_{k,j}$, the *resultant force*. So a combination of forces, acting in one point, is equivalent with the resultant force, acting in this point.

Total momentum transfer, Newton's second and third laws. Since the internal multipliers $\mu_{k,l}$ in (65) are symmetric, the sum of the corresponding terms is zero:

$$\dot{\boldsymbol{p}} = \sum_k m_k \ddot{\boldsymbol{x}}_k = \sum_k (\sum_j \lambda_{k,j} \boldsymbol{n}_{k,j}) + \sum_{k \sim l} \mu_{k,l}(\boldsymbol{x}_l - \boldsymbol{x}_k) = \sum_k \boldsymbol{F}_k$$

which can be interpreted as *Newton's second law*. According to definition 13, the vector $\mu_{k,l}\boldsymbol{r}_{k,l}$ can be considered as the force that is applied to $\boldsymbol{x}_k$ by $\boldsymbol{x}_l$. The symmetry $\mu_{k,l} = \mu_{l,k}$ can be interpreted as *Newton's third law*, of action and reaction.

Energy and work. The change of energy of the mass m_k satisfies:

$$\frac{d}{dt}\left(\frac{1}{2} m_k \|\dot{\boldsymbol{x}}_k\|^2\right) = \dot{\boldsymbol{x}}_k \boldsymbol{\cdot} \left(\boldsymbol{F}_k + \sum_{l=1}^{N} \boldsymbol{F}_{k,l}\right)$$

In which $\boldsymbol{F}_{k,l} = \mu_{k,l}(\boldsymbol{x}_l - \boldsymbol{x}_k)$. Hence, after summation over k, the internal terms vanish, and we have the transfer of total energy:

$$\frac{dW}{dt} = \sum_k \dot{\boldsymbol{x}}_k \boldsymbol{\cdot} \boldsymbol{F}_k$$

The righthand side is the *work per time unit*, done by the external forces.

Moment of momentum, force moment. There is a third global quantity which can be calculated without knowledge of the internal forces. In this relation, vector products play a role. Consider for any time-dependent vector $\boldsymbol{x}$ the product $\boldsymbol{x} \times \ddot{\boldsymbol{x}}$

$$\boldsymbol{x} \times \ddot{\boldsymbol{x}} = \frac{d}{dt}(\boldsymbol{x} \times \dot{\boldsymbol{x}}) - (\dot{\boldsymbol{x}} \times \dot{\boldsymbol{x}}) = \frac{d}{dt}(\boldsymbol{x} \times \dot{\boldsymbol{x}})$$

Apply this to the equations (65):

$$\frac{d}{dt}(m_k \boldsymbol{x}_k \times \dot{\boldsymbol{x}}_k) = m_k \boldsymbol{x}_k \times \ddot{\boldsymbol{x}}_k = \boldsymbol{x}_k \times \boldsymbol{F}_k + \boldsymbol{x}_k \times \sum_l \mu_{k,l}(\boldsymbol{x}_l - \boldsymbol{x}_k)$$

Summation over all k yields

$$\frac{d}{dt}\left[\sum_k m_k \boldsymbol{x}_k \times \dot{\boldsymbol{x}}_k\right] = \sum_k \boldsymbol{x}_k \times \boldsymbol{F}_k + \sum_k \sum_l \mu_{k,l} \boldsymbol{x}_k \times (\boldsymbol{x}_l - \boldsymbol{x}_k)$$

The last term can be written as

$$\sum_{k,l} \mu_{k,l} \boldsymbol{x}_k \times (\boldsymbol{x}_l - \boldsymbol{x}_k) = \frac{1}{2}\left[\sum_{k,l} \mu_{k,l}(\boldsymbol{x}_k - \boldsymbol{x}_l) \times (\boldsymbol{x}_l - \boldsymbol{x}_k)\right] = \boldsymbol{0}$$

and we get

$$\sum_{k=1}^{N} m_k \boldsymbol{x}_k \times \ddot{\boldsymbol{x}}_k = \frac{d}{dt}\left(\sum_{k=1}^{N} m_k \boldsymbol{x}_k \times \dot{\boldsymbol{x}}_k\right) = \sum_{k=1}^{N} \boldsymbol{x}_k \times \boldsymbol{F}_k \tag{66}$$

Define the force moments $\boldsymbol{L}_k$, and the system's moment of momentum $\boldsymbol{b}$ by

$$\boldsymbol{L}_k = \boldsymbol{x}_k \times \boldsymbol{F}_k, \;\; \boldsymbol{b} = \sum_{k=1}^{N} m_k \boldsymbol{x}_k \times \dot{\boldsymbol{x}}_k$$

then we can interprete (66) as *'the change of the moment of momentum of a system equals the total force moment acting upon the system'*:

$$\dot{\boldsymbol{b}} = \sum_{k=1}^{N} \boldsymbol{x}_k \times \boldsymbol{F}_k = \sum_{k=1}^{N} \boldsymbol{L}_k = \boldsymbol{L} \tag{67}$$

Moving a force to another point of action. Suppose we change the point of action $\boldsymbol{x}$ of a force $\boldsymbol{F}$ to a point $\boldsymbol{x}'$. This has no influence on the momentum transfer, but in the balance of moment of momentum with the external force moments, there may be changes. In order to avoid these changes, we must require

$$\boldsymbol{L}' = \boldsymbol{L} \Longrightarrow \boldsymbol{x}' \times \boldsymbol{F} = \boldsymbol{x} \times \boldsymbol{F}$$

Hence $(\boldsymbol{x}' - \boldsymbol{x}) \times \boldsymbol{F} = \boldsymbol{0}$, or $\boldsymbol{x}' = \boldsymbol{x} + \tau\boldsymbol{F}$ for some τ. This implies an important property of forces:

A force may, and may only be transported along its line of action.

5 Rigid bodies, Euler equations, and the remains of statics.

5.1 Description of rigidity.

A finite rigid body is a semi-rigid mass-point construction in which all masses have fixed mutual distances. For N mass-points, this implies the occurrence of $\frac{1}{2}N(N-1)$ constraints (of which a lot may be removable). Therefore the internal forces are not at all defined uniquely. However, a body is an example of the possibility to resolve the constraints implicitly, such that the use of an *augmented* frustration function is no longer required.
Suppose $\boldsymbol{x}_c$ is some fixed 'central point' of the body, then every point $\boldsymbol{x}_k$ of the body can be written as

$$\boldsymbol{x}_k = \boldsymbol{x}_c + \boldsymbol{r}_k$$

where $\|\boldsymbol{r}_k\|$ is constant in time for all k. The position and orientation of the body is determined by the momentary central vector $\boldsymbol{x}_c(t)$, and by an orthogonal matrix $\boldsymbol{Q}(t)$:

$$\boldsymbol{r}_k(t) = \boldsymbol{Q}(t)\boldsymbol{r}_k(0)$$

Now

$$\boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I} \Longrightarrow \frac{d}{dt}(\boldsymbol{Q}\boldsymbol{Q}^T) = \dot{\boldsymbol{Q}}\boldsymbol{Q}^T + \boldsymbol{Q}\dot{\boldsymbol{Q}}^T = \dot{\boldsymbol{Q}}\boldsymbol{Q}^T + (\dot{\boldsymbol{Q}}\boldsymbol{Q}^T)^T = \boldsymbol{O}$$

Therefore $\dot{\boldsymbol{Q}}\boldsymbol{Q}^T = \boldsymbol{\Omega}$ is a skew symmetric matrix:

$$\boldsymbol{\Omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

and by right multiplication with $\boldsymbol{Q}$, we get:

$$\dot{\boldsymbol{Q}} = \boldsymbol{\Omega}\boldsymbol{Q}, \;\; \dot{\boldsymbol{r}}_k = \dot{\boldsymbol{Q}}\boldsymbol{r}_k(0) = \boldsymbol{\Omega}\boldsymbol{Q}\boldsymbol{r}_k(0) = \boldsymbol{\Omega}\boldsymbol{r}_k = \boldsymbol{\omega} \times \boldsymbol{r}_k \;\; \text{(See (2))}$$

The velocity of a particular point $\boldsymbol{x}_k$ of the body is completely determined by the velocity $\boldsymbol{v}_c$ and the angular velocity $\boldsymbol{\omega}$:

$$\dot{\boldsymbol{x}}_k = \dot{\boldsymbol{x}}_c + \boldsymbol{\omega} \times \boldsymbol{r}_k \tag{68}$$

The acceleration is similarly expressed by

$$\ddot{\boldsymbol{x}}_k = \ddot{\boldsymbol{x}}_c + \ddot{\boldsymbol{r}}_k = \ddot{\boldsymbol{x}}_c + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_k + \boldsymbol{\omega} \times \dot{\boldsymbol{r}}_k \tag{69}$$

5.2 Derivation of Euler's equations.

From now on, we choose the mass centre $\boldsymbol{x}_c$ as central point:

$$\boldsymbol{x}_c = \frac{\sum_k m_k \boldsymbol{x}_k}{\sum_k m_k} = \frac{\sum_k m_k \boldsymbol{x}_k}{M}$$

in which M is the total mass of the body. This implies $\sum_k m_k \boldsymbol{r}_k = \boldsymbol{0}$, which is useful for working out the formulas.

The frustration of the body is a function of $\ddot{\boldsymbol{x}}_c$, $\boldsymbol{\omega}$, and $\dot{\boldsymbol{\omega}}$. Since $\boldsymbol{\omega}$ is a state variable, it won't vary in the minimization of the frustration. So effectively

$$F(\ddot{\boldsymbol{x}}_1, \ddot{\boldsymbol{x}}_2, \ldots, \ddot{\boldsymbol{x}}_N) = \frac{1}{2}\sum_{k=0}^{N} m_k \|\ddot{\boldsymbol{x}}_k\|^2 = \Phi(\ddot{\boldsymbol{x}}_c, \dot{\boldsymbol{\omega}})$$

Without calculating the explicit formula for Φ, we can calculate its variation:

$$\delta\Phi = \sum_{k=1}^{N} m_k \ddot{\boldsymbol{x}}_k \boldsymbol{\cdot} \delta\ddot{\boldsymbol{x}}_k = \sum_{k=1}^{N} m_k \ddot{\boldsymbol{x}}_k \boldsymbol{\cdot} (\delta\ddot{\boldsymbol{x}}_c + \delta\dot{\boldsymbol{\omega}} \times \boldsymbol{r}_k)$$

Using the scalar triple product identity $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a})$, we can rearrange the second term in the righthandside,

$$\begin{aligned}\delta\Phi &= \left(\sum_{k=1}^{N} m_k \ddot{\boldsymbol{x}}_k\right)\cdot\delta\ddot{\boldsymbol{x}}_c + \left(\sum_{k=1}^{N} m_k \boldsymbol{r}_k \times \ddot{\boldsymbol{x}}_k\right)\cdot\delta\dot{\boldsymbol{\omega}} \\ &= \dot{\boldsymbol{p}}\cdot\delta\ddot{\boldsymbol{x}}_c + \dot{\boldsymbol{b}}\cdot\delta\dot{\boldsymbol{\omega}}\end{aligned}$$

Minimization of Φ then yields

$$\frac{\partial\Phi}{\partial\ddot{\boldsymbol{x}}_c} = \dot{\boldsymbol{p}} = \boldsymbol{0} \qquad (70)$$

$$\frac{\partial\Phi}{\partial\dot{\boldsymbol{\omega}}} = \dot{\boldsymbol{b}} = \boldsymbol{0} \qquad (71)$$

For rigid bodies, the moment of momentum $\boldsymbol{b}$ is also called *angular momentum*, and denoted by $\boldsymbol{H}$.

$$\boldsymbol{H} = \sum \boldsymbol{r}_k \times m_k \dot{\boldsymbol{x}}_k \qquad (72)$$

This can be expressed in terms of $\boldsymbol{x}_c$ and $\boldsymbol{\omega}$:

$$\boldsymbol{H} = \sum \boldsymbol{r}_k \times m_k \dot{\boldsymbol{x}}_k = \sum m_k \boldsymbol{r}_k \times (\dot{\boldsymbol{x}}_c + \boldsymbol{\omega}\times\boldsymbol{r}_k) = \sum m_k \boldsymbol{r}_k \times (\boldsymbol{\omega}\times\boldsymbol{r}_k)$$

since $\sum m_k \boldsymbol{r}_k = \boldsymbol{0}$. With help of the matrix iterpretation of the vector triple product this can be written as $\boldsymbol{H} = \boldsymbol{J}\boldsymbol{\omega}$ where

$$\boldsymbol{J} = \sum m_k \left(\|\boldsymbol{r}_k\|^2 \boldsymbol{I} - \boldsymbol{r}_k \boldsymbol{r}_k^T\right)$$

is the *tensor of inertia*
Next assume constraints are working in points $\boldsymbol{x}_j$ of the body

$$\boldsymbol{n}_j\cdot\ddot{\boldsymbol{x}}_j = \alpha_j,\;\; j = 1, 2, \ldots, m$$

We translate these constraints to $\ddot{\boldsymbol{x}}_c$ and $\dot{\boldsymbol{\omega}}$:

$$\begin{aligned}\boldsymbol{n}_j\cdot\ddot{\boldsymbol{x}}_j &= \boldsymbol{n}_j\cdot(\ddot{\boldsymbol{x}}_c + \dot{\boldsymbol{\omega}}\times\boldsymbol{r}_j + \boldsymbol{\omega}\times\dot{\boldsymbol{r}}_j) = \alpha_j \\ &= \boldsymbol{n}_j\cdot\ddot{\boldsymbol{x}}_c + (\boldsymbol{r}_j\times\boldsymbol{n}_j)\cdot\dot{\boldsymbol{\omega}} + \boldsymbol{n}_j\cdot(\boldsymbol{\omega}\times\dot{\boldsymbol{r}}_j)\end{aligned}$$

The variation with respect to $\ddot{\boldsymbol{x}}_c$ and $\dot{\boldsymbol{\omega}}$ reads

$$\delta(\boldsymbol{n}_j\cdot\ddot{\boldsymbol{x}}_j) = \boldsymbol{n}_j\cdot\delta\ddot{\boldsymbol{x}}_c + (\boldsymbol{r}_j\times\boldsymbol{n}_j)\cdot\delta\dot{\boldsymbol{\omega}}$$

Now consider the augmented frustration function for the constrained rigid body:

$$\widetilde{\Phi}(\ddot{\boldsymbol{x}}, \dot{\boldsymbol{\omega}}) = \Phi(\ddot{\boldsymbol{x}}_c, \dot{\boldsymbol{\omega}}) - \sum_{j=1}^{m} \lambda_j \left(\boldsymbol{n}^T \ddot{\boldsymbol{x}}_c + (\boldsymbol{r}_j\times\boldsymbol{n}_j)^T \dot{\boldsymbol{\omega}}\right)$$

Putting to zero the partial derivatives with respect to $\ddot{\boldsymbol{x}}_c$ and $\dot{\boldsymbol{\omega}}$ we get

$$\begin{aligned}\frac{\partial\widetilde{\Phi}}{\partial\ddot{\boldsymbol{x}}_c} &= \dot{\boldsymbol{p}} - \sum_{j=1}^{N} \lambda_j \boldsymbol{n}_j = \boldsymbol{0} \\ \frac{\partial\widetilde{\Phi}}{\partial\dot{\boldsymbol{\omega}}} &= \dot{\boldsymbol{H}} - \sum_{j=1}^{N} \boldsymbol{r}_j \times (\lambda_j \boldsymbol{n}_j) = \boldsymbol{0}\end{aligned}$$

Putting $\lambda_j \boldsymbol{n}_j = \boldsymbol{F}_j$, the force acting in $\boldsymbol{x}_j$, these equations become Euler's equations for the motion of a rigid body:

$$\dot{\boldsymbol{p}} = \sum_{j=1}^{N} \boldsymbol{F}_j = \boldsymbol{F} \tag{73}$$

$$\dot{\boldsymbol{H}} = \sum_{j=1}^{N} \boldsymbol{r}_j \times \boldsymbol{F}_j = \boldsymbol{L} \tag{74}$$

So the change of linear momentum and angular momentum equal the total applied force $\boldsymbol{F}$ and the total applied force moment $\boldsymbol{L}$ respectively.

6 Discussion and acknowledgements.

Classical point mechanics can be founded on the principle of relativity, together with two rather obvious kinematic principles, based on everyday experience. A striking side-effect of this set-up is the way in which mass and momentum enter the theory.
The present analysis follows initially more ore less *method A* in E.A. Desloge's paper [2]. In fact, we do one step before the start of *method A*, namely proving the existence of mass and momentum, with their properties.
C. Truesdell [7] criticises the fact that according to many physicists the law of moment of momentum *follows from Newton's laws*. In derivations of this law, one assumes that internal forces in a rigid body *are directed centrally from point to point*. Whether or not this is true, there is not much knowledge on these internal forces in general.
Truesdell's remedy: Choose the conservation of linear momentum *and* of moment of momentum as basic axioms. The disadvantage is that these axioms are not really 'Euclid-like' from the self-evidenceness point of view.
In the present analysis, the internal forces $\mu_{k,l}\boldsymbol{r}_{k,l}$ occur as multipliers in a mathematical problem; the direction of these vectors is a consequence of the constraining direction, their (skew) symmetry is required by the conservation of linear momentum. So apparently, the mutual 'forces' are 'central', but may have nothing to do with the way how points act upon other points physically. So in the present analysis, this property of the mutual forces is not an assumption about *physical* forces.

One can imagine that the true equations of motion of a semi-rigid mass-point construction depend on the actual multipliers (say the actual internal forces). In lemma 7 however is proved that the solution of a consistently linearly constrained minimization problem is unique, even if the multipliers are not. In the analysis of the rigid body, the multipliers are completely absent: the internal forces ore not relevant.
So our approach doesn't suffer from the shortcomings of the 'central force' assumption, and is based on a really simple axiom, the law of minimal frustration.

A remark can be made on the derivation of the momentum and energy properties of semi-rigid mass-point constructions . Probably many mechanists would accept actual-formal equality for momentum and energy without lemma 3. The author has been a bit scrupulous because of the pretention of trying to be 'Euclidean'.

Finally, the present axioms may certainly be considered as self-evident, because they reflect common experience of many people not trained in mechanics. They also are attractive from philosophic point of view. The law of decrease reflects the common feeling that nothing can be obtained from nothing. The law of minimal frustration reflects another common principle, not only in physics, but also in dayly life: *'Do not more than strictly necessary'*.

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