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Point Forces in Elasticity Equation and Their Alternatives in Multi Dimensions

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Abstract

We consider several mathematical issues regarding models that simulate forces exerted by cells. Since the size of cells is much smaller than the size of the domain of computation, one often considers point forces, modelled by Dirac Delta distributions on boundary segments of cells. In the current paper, we treat forces that are directed normal to the cell boundary and that are directed toward the cell centre. Since it can be shown that there exists no smooth solution, at least not in H^1 for solutions to the governing momentum balance equation, we analyse the convergence and quality of approximation. Furthermore, the expected finite element problems that we get necessitate to scrutinize alternative model formulations, such as the use of smoothed Dirac distributions, or the so-called smoothed particle approach as well as the so-called hole approach where cellular forces are modelled through the use of (natural) boundary conditions. In this paper, we investigate and attempt to quantify the conditions for consistency between the various approaches. This has resulted into error analyses in the H^1-norm of the numerical solution based on Galerkin principles that entail Lagrangian basis functions. The paper also addresses well-posedness in terms of existence and uniqueness. The current analysis has been performed for the linear steady-state (hence neglecting inertia and damping) momentum equations under the assumption of Hooke's law.

1 Introduction

Wound healing is a complicated process of a bunch of cellular events contributing to resurfacing, reconstitution and restoration of the tensile strength of injured skin. Significant damage of dermal tissue often leads to skin contraction. If the contraction of the skin near a joint is large then it may result into a decrease of functionality. If the patient's daily life is impacted as result of the contraction, then one speaks of a contracture.

In order to improve the patient's quality of life, one aims at reducing the contractile behavior of the skin. In order to reduce the severity of the contraction, one needs to know the physiological dynamics and time evolution of the underlying biological mechanisms. According to [5, 7, 10], the contraction starts developing during the proliferative phase of wound healing. This proliferative phase sets in after the inflammatory phase, in which the immune system is clearing up the debris that resulted from the damage. The proliferative phase usually sets in during the second day post-wounding, and commonly lasts during two to four weeks. Besides the closure of the epidermis (that is the top layer of skin), the proliferative phase is characterized by ingress of fibroblasts from the surrounding undamaged tissue and differentiation to myofibroblasts, and by the regeneration of collagen by the (myo)fibroblasts. Despite the relatively quick closure of the epidermis, often the restoration of the underlying dermis is still in progress. After closure of the epidermis, the damaged region in the dermis is referred to

as a scar instead of a wound. Next to the regeneration of collagen, the (myo)fibroblasts exert contractile forces on their direct surroundings, which will result into contraction of the scar tissue. In human skin, wound volume reductions of 5 - 10% are commonly observed [6].

The current manuscript contains an extension of the work in [8], which treats a model for the contractile forces that are exerted by the (myo)fibroblasts. The forces are distinguished into two categories: (1) temporary forces that are exerted as long as the (myo)fibroblasts are actively pulling; and (2) permanent or plastic forces, which are imaginary forces that are introduced to describe the localized plastic deformations of the tissue. This formalism was firstly developed by [14], and later extended by [3]. The formalism is based on the point forces, which are mathematically incorporated by means of linear combinations of Dirac Delta distributions. The irregular nature of Dirac Delta distributions make the solution to the elliptic boundary value problem from the balance of momentum have a singular solution in the sense that for dimensionality higher than one, no formal solutions in the finite-element space H^1 exist. Although in classical finite-element strategies, one uses for instance piecewise linear Lagrangian elements, of which the basis functions are in H^1, and therewith one attempts to approximate the solution, which is not in H^1, as well as possibly by a function in H^1. In [2], the convergence of finite-element solutions by means of piecewise linear Lagrangian elements in multiple dimensions has been demonstrated. In our earlier studies [11, 12], we proved the convergence of solutions obtained by regularization of Dirac Delta distributions, the so-called smoothed particle approach and the so-called 'hole' approach to the solution obtained by Dirac Delta distributions in the one- and two dimensional cases. In the one-dimensional case, for the sake of completeness, we start with the presentation of force equilibrium with point forces in one dimension, the equations are given by

$$-\frac{d\sigma}{dx} = f, \qquad \text{Equation of Equilibrium,} \tag{1.1}$$

$$\epsilon = \frac{du}{dx}, \qquad \text{Strain-Displacement Relation,} \tag{1.2}$$

$$\sigma = E\epsilon, \qquad \text{Constitutive Equation.} \tag{1.3}$$

To simplify the equation, we use $E = 1$ here, the equations above can be combined to the one-dimensional Laplace equation:

$$-\frac{d^2u}{dx^2} = f. \tag{1.4}$$

We assume that there is a biological cell with size h and centre position c in the computational domain such that $0 < c - h/2 < c < c + h/2 < L$. Then the force is given by $f = \delta(x - (c - h/2)) - \delta(x - (x + h/2))$. Combined with homogeneous Dirichlet boundary conditions:

$$u(0) = 0, \ u(L) = 0,$$

the Galerkin form is given by

$$\begin{cases} \text{Find } u_h \in H_0^1((0, L)), \text{ such that} \\ \int_\Omega \nabla u_h \nabla \phi_h d\Omega = \phi_h(c - \frac{h}{2}) - \phi_h(c + \frac{h}{2}), \\ \text{for all } \phi_h \in H_0^1((0, L)). \end{cases}$$

The exact solution is

$$u(x) = \frac{hx}{L} + (x - (c + \frac{h}{2}))_+ - (x - (c - \frac{h}{2}))_+,$$

where $(x)_+ = \max\{0, x\}$. Note that in one dimension, the solution is piecewise linear and hence in $H^1(\Omega)$, however not in $H^2(\Omega)$. Since most conventional errors are expressed in the

L^2–norm of the second derivative of the solution, one may not apriorily expect very accurate finite element solutions.

In the current manuscript we extend the results to general dimensionality. The boundary value problem is stated in Section 2. The 'hole' approach and the smoothed particle approach are developed in Section 3. Furthermore, we prove consistency between the alternatives and the immersed boundary approach in multi dimensions. Section 4 displays some conclusions and discussions.

2 Elasticity Equation with Point Sources in Multi Dimensions

Let Ω be a bounded domain in $\mathbb{R}^n$, then we consider the following balance of momentum where inertial effects have been neglected:

$$- \nabla \cdot \boldsymbol{\sigma} = \boldsymbol{f}. \tag{2.1}$$

Here $\boldsymbol{\sigma}$ denotes the stress tensor and $\boldsymbol{f}$ represents a body force that is exerted within Ω. We consider a linear, homogeneous, isotropic and continuous material; hence, Hooke's Law is used here for the relation between the stress and strain tensors:

$$\boldsymbol{\sigma} = \frac{E}{1+\nu} \left\{ \boldsymbol{\epsilon} + \text{tr}(\boldsymbol{\epsilon}) \left[\frac{\nu}{1-2\nu} \right] \boldsymbol{I} \right\}, \tag{2.2}$$

where E is the stiffness of the computational domain, ν is Poisson's ratio and $\boldsymbol{\epsilon}$ is the infinitesimal Eulerian strain tensor:

$$\boldsymbol{\epsilon} = \frac{1}{2} \left[\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T \right]. \tag{2.3}$$

Within the domain of computation, Ω, we consider the presence of a biological cell, which occupies the portion Ω_C that is completely embedded within Ω (hence Ω_C is a strict subset of Ω). The boundary of the cell is divided into surface elements. On the centre of each surface element, a point force by means of Dirac Delta distribution, is exerted in the direction of the normal vector that is directed inward into the cell. This results into (see [14]):

$$\boldsymbol{f}_t = \sum_{j=1}^{N_S} P(\boldsymbol{x}, t) \boldsymbol{n}(\boldsymbol{x}) \delta(\boldsymbol{x} - \boldsymbol{x}_j(t)) \Delta S(\boldsymbol{x}_j(t)), \tag{2.4}$$

where N_S is the number of surface elements of the cell, $P(\boldsymbol{x}, t)$ is the magnitude of the pulling force exerted at point $\boldsymbol{x}$ and time t per unit of measure (being area in $\mathbb{R}^3$ or length in $\mathbb{R}^2$), $\boldsymbol{n}(\boldsymbol{x})$ is the unit inward pointing normal vector (towards the cell centre) at position $\boldsymbol{x}$, $\boldsymbol{x}_j(t)$ is the midpoint on surface element j of the cell at time t and $\Delta S(\boldsymbol{x}_j)$ is the measure of the surface element j. In the general model where we use this principle, we consider transient effect due to migration and possible deformation of the cells. However, since we predominantly focus on the mathematical issues in the current manuscript, we will not consider any time-dependencies and hence t will be removed from the expressions in the remainder of the paper.

In the n-dimensional case, we are solving the boundary value problems described in Eq (2.1), (2.2) and (2.3). The body-force is given in Eq (2.4). Therefore, the boundary value problem that we are going to consider is given by

$$(BVP) \begin{cases} - \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{x}) = \sum_{j=1}^{N_S} P(\boldsymbol{x}_j) \boldsymbol{n}(\boldsymbol{x}_j) \delta(\boldsymbol{x} - \boldsymbol{x}_j) \Delta S(\boldsymbol{x}_j), & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0}, & \text{on } \partial\Omega. \end{cases}$$

Next to this boundary value problem, we consider the continuous immersed boundary counterpart, given by

$$(BVP_\infty) \begin{cases} -\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{x}) = \int_{S_c} P(\boldsymbol{x}')\boldsymbol{n}(\boldsymbol{x}')\delta(\boldsymbol{x} - \boldsymbol{x}')dS(\boldsymbol{x}'), & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0}, \text{ on } \partial\Omega. \end{cases}$$

Due to the irregular nature of the Dirac Delta distributions, the solutions do not exist in H^1. We attempt to approximate the solution by the functions in H^1 via the Galerkin form of (BVP). In this manuscript, piecewise linear Lagrangian basis functions are selected. Further, the convergence of the finite-element solutions using linear Lagrangian elements in general dimensionality has been proved in [2].

To construct the Galerkin form, we introduce the bilinear form $a(.,.)$:

$$a(\boldsymbol{u}_h, \boldsymbol{\phi}_h) = \int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}_h) : \nabla\boldsymbol{\phi}_h d\Omega = \int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}_h) : \boldsymbol{\epsilon}(\boldsymbol{\phi}_h)d\Omega, \tag{2.5}$$

The last step is motivated by symmetry of the stress tensor $\boldsymbol{\sigma}$. Since the solution $\boldsymbol{u}$ is not in $\boldsymbol{H}^1(\Omega)$, we consider a subspace of $\boldsymbol{H}^1(\Omega)$, which is defined as $\boldsymbol{V_h}(\Omega) = \text{Span}\{\boldsymbol{\phi^1}, \boldsymbol{\phi^2}, \dots, \boldsymbol{\phi^N}\}$ [13]. Here, $\boldsymbol{\phi^i}$ for $i = \{1, 2, \dots, N\}$ is the linear Lagrangian basis function, which is piecewise smooth and continuous over Ω. Hence, these basis functions are in $\boldsymbol{H}^1$. Subsequently, the Galerkin form is

$$(GF) \begin{cases} \text{Find } \boldsymbol{u}_h \in \boldsymbol{V_h}(\Omega), \text{ such that} \\ a(\boldsymbol{u}_h, \boldsymbol{\phi}_h) = (\boldsymbol{\phi}_h, \boldsymbol{f}_t) = \sum_{j=1}^{N_S} P(\boldsymbol{x}_j)\boldsymbol{n}(\boldsymbol{x}_j)\boldsymbol{\phi}_h(\boldsymbol{x}_j)\Delta S(\boldsymbol{x}_j), \\ \text{for all } \boldsymbol{\phi}_h \in \{\boldsymbol{\phi^1}, \boldsymbol{\phi^2}, \dots, \boldsymbol{\phi^N}\} \subset \boldsymbol{V_h}(\Omega). \end{cases}$$

We further consider the solution to the continuous immerse boundary problem, with the following Galerkin form:

$$(GF_\infty) \begin{cases} \text{Find } \boldsymbol{u}_h \in \boldsymbol{V_h}(\Omega), \text{ such that} \\ a(\boldsymbol{u}_h, \boldsymbol{\phi}_h) = (\boldsymbol{\phi}_h, \boldsymbol{f}_t) = \int_{S_c} P(\boldsymbol{x}')\boldsymbol{n}(\boldsymbol{x}')\boldsymbol{\phi}_h(\boldsymbol{x}')dS(\boldsymbol{x}'), \\ \text{for all } \boldsymbol{\phi}_h \in \{\boldsymbol{\phi^1}, \boldsymbol{\phi^2}, \dots, \boldsymbol{\phi^N}\} \subset \boldsymbol{V_h}(\Omega). \end{cases}$$

Before we proceed to claim the existence and the uniqueness of the Galerkin solution in (GF), we need to state Korn's Inequality in multiple dimensions:

Lemma 2.1. (Korn's Second Inequality[4]) *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, connected domain. Then there exists a positive constant K, such that for any vector-valued function $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$,*

$$\int_\Omega ||\boldsymbol{\epsilon}(\boldsymbol{u})||^2 d\Omega \geqslant K\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}^2.$$

We note that Korn's Second Inequality can be generalised to cases in which the boundary condition $\boldsymbol{u} = \boldsymbol{0}$ is imposed only on a non-zero measure part of the boundary. Using Korn's Second inequality gives the following lemma:

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, connected domain. Then there exists a positive constant K, such that for any vector-valued function $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$,*

$$a(\boldsymbol{u}, \boldsymbol{u}) = \int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{u})d\Omega \geqslant K\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}^2.$$

4

Proof. The lemma directly follows from the definition of the stress tensor, let $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$:

$$
\begin{aligned}
a(\boldsymbol{u}, \boldsymbol{u}) &= \int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{u}) d\Omega \\
&= \int_\Omega \frac{E}{1+\nu} \left\{ \boldsymbol{\epsilon}(\boldsymbol{u}) + \text{tr}(\boldsymbol{\epsilon}(\boldsymbol{u})) \frac{\nu}{1-2\nu} \boldsymbol{I} \right\} : \boldsymbol{\epsilon}(\boldsymbol{u}) d\Omega \\
&= \int_\Omega \frac{E}{1+\nu} ||\boldsymbol{\epsilon}(\boldsymbol{u})||^2 + \frac{E\nu}{(1+\nu)(1-2\nu)} (\text{tr}(\boldsymbol{\varepsilon}(\boldsymbol{u})))^2 d\Omega \\
&\geqslant \frac{E}{1+\nu} K ||\boldsymbol{u}||^2_{\boldsymbol{H}^1(\Omega)}.
\end{aligned}
$$

The last step follows from Lemma 2.1. Hence redefining $K := \frac{E}{1+\nu} K$ concludes the proof the lemma. $\square$

Herewith coerciveness of the linear form $a(.,.)$ has been demonstrated, which is needed for the proof of existence and uniqueness of the Galerkin finite-element solution.

Theorem 2.1. *Let $\{\boldsymbol{\phi^i}\}$ be piecewise Lagrangian basis field functions and let $\boldsymbol{F}$ be a vector in $\mathbb{R}^n$ with unit length, further let $P \in C(\overline{\Omega})$, and let $|P| \leqslant M_2$ for some $M_2 > 0$. We define $\boldsymbol{V}_h(\Omega) = \text{Span}\{\boldsymbol{\phi^1}, \boldsymbol{\phi^2}, \dots, \boldsymbol{\phi^N}\} \subset \boldsymbol{H}_0^1(\Omega)$, then*

- $\exists ! \, \boldsymbol{u}_h^G(\boldsymbol{x}; \boldsymbol{x}'; \boldsymbol{F}) \in \boldsymbol{V}_h(\Omega)$ *such that* $a(\boldsymbol{u}_h, \boldsymbol{\phi}_h) = \boldsymbol{F}(\boldsymbol{x}') \cdot \boldsymbol{\phi}_h(\boldsymbol{x}')$ *for all* $\boldsymbol{\phi}_h \in \boldsymbol{V}_h$;

- $\exists ! \, \boldsymbol{u}_h \in \boldsymbol{V}_h(\Omega)$ *such that* $a(\boldsymbol{u}_h, \boldsymbol{\phi}_h) = \sum_{j=1}^{N_S} P(\boldsymbol{x}_j) \boldsymbol{n}(\boldsymbol{x}_j) \boldsymbol{\phi}_h(\boldsymbol{x}_j) \Delta S(\boldsymbol{x}_j)$ *for all* $\boldsymbol{\phi}_h \in \boldsymbol{V}_h$, *and* $\boldsymbol{u}_h = \sum_{j=1}^{N_S} P(\mathbf{x}_j) \mathbf{u}_h^G(\mathbf{x}; \mathbf{x}_j; \mathbf{n}(\mathbf{x}_j)) \Delta S(\mathbf{x}_j)$;

- $\exists ! \, \boldsymbol{u}_h \in \boldsymbol{V}_h(\Omega)$ *such that* $a(\boldsymbol{u}_h, \boldsymbol{\phi}_h) = \int_{S_c} P(\boldsymbol{x}') \boldsymbol{n}(\boldsymbol{x}') \boldsymbol{\phi}_h(\boldsymbol{x}') dS(\boldsymbol{x}')$ *for all* $\boldsymbol{\phi}_h \in \boldsymbol{V}_h$, *and* $\boldsymbol{u}_h = \int_{S_c} P(\mathbf{x}') \mathbf{u}_h^G(\mathbf{x}; \mathbf{x}'; \mathbf{n}(\mathbf{x}')) dS(\boldsymbol{x}')$;

Proof. • It is immediately clear that $a(.,.)$ is a bilinear form. We have $\boldsymbol{V}_h \subset \boldsymbol{H}_0^1(\Omega)$, and $a(.,.)$ is bounded in $\boldsymbol{H}_0^1(\Omega)$ (see for instance [1]). Furthermore, Lemma 2.2 says that $a(.,.)$ is coercive in $\boldsymbol{H}_0^1(\Omega)$. Regarding the right-hand side, we have $|\boldsymbol{\phi}_h| \leq M_1$ for some $M_1 > 0$ since $\boldsymbol{\phi}_h$ is a Lagrangian function, and hence the magnitude of the right-hand side can be bounded from above by

$$
|\boldsymbol{F} \cdot \boldsymbol{\phi}_h(\boldsymbol{x}')| \leqslant M_1.
$$

Note that $||\boldsymbol{F}|| = 1$. Hence the right-hand side is bounded, and herewith from Lax-Milgram's Lemma, the existence and uniqueness of $\boldsymbol{u}_h^G(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{n}(\boldsymbol{x}'))$ follows.

• Existence and uniqueness follow analogously, only boundedness of the right-hand side, which is a linear functional in $\boldsymbol{\phi}_h \in \boldsymbol{V}_h(\Omega)$ has to be checked:

$$
\begin{aligned}
|\sum_{j=1}^{N_S} P(\boldsymbol{x}_j) \boldsymbol{n}(\boldsymbol{x}_j) \cdot \boldsymbol{\phi}_h(\boldsymbol{x}_j) \Delta S(\boldsymbol{x}_j)| &\leqslant \sum_{j=1}^{N_S} |P(\boldsymbol{x}_j)| \cdot ||\boldsymbol{n}(\boldsymbol{x}_j)|| \cdot ||\boldsymbol{\phi}_h(\boldsymbol{x}_j)|| \Delta S(\boldsymbol{x}_j) \\
&= \sum_{j=1}^{N_S} |P(\boldsymbol{x}_j)| \cdot ||\boldsymbol{\phi}_h(\boldsymbol{x}_j)|| \Delta S(\boldsymbol{x}_j) \\
&\leqslant M_1 M_2 \sum_{j=1}^{N_S} \Delta S(\boldsymbol{x}_j).
\end{aligned}
$$

Note that $\boldsymbol{n}$ has unit length. The summation gives the polygonal length or polyhedral area of the cell boundary. Hence the right-hand side is bounded, then by Lax-Milgram's Lemma, existence and uniqueness follow. Further by substitution, it follows that that

$$a(\boldsymbol{u}_h, \boldsymbol{\phi}_h) = a(\sum_{j=1}^{N_S} P(\boldsymbol{x}_j)\boldsymbol{u}_h^G(\boldsymbol{x}, \boldsymbol{x}_j, \boldsymbol{n}(\boldsymbol{x}_j))\Delta S(\boldsymbol{x}_j), \boldsymbol{\phi}_h)$$

$$= \sum_{j=1}^{N_S} P(\boldsymbol{x}_j)a(\boldsymbol{u}_h^G(\boldsymbol{x}, \boldsymbol{x}_j, \boldsymbol{n}(\boldsymbol{x}_j)), \boldsymbol{\phi}_h)\Delta S(\boldsymbol{x}_j)$$

$$= \sum_{j=1}^{N_S} P(\boldsymbol{x}_j)\boldsymbol{n}(\boldsymbol{x}_j) \cdot \boldsymbol{\phi}_h(\boldsymbol{x}_j)\Delta S(\boldsymbol{x}_j).$$

The last step uses the first part of the theorem, and finally proves the assertion.

- We proceed similarly, by boundedness of the right-hand side:

$$|\int_{S_c} P(\boldsymbol{x}')\boldsymbol{n}(\boldsymbol{x}') \cdot \boldsymbol{\phi}_h(\boldsymbol{x}')dS(\boldsymbol{x}')| \leqslant M_1 M_2 |S_c|,$$

which again shows that the right-hand side is a bounded linear functional in $\boldsymbol{V}_h(\Omega)$. We proceed by substitution:

$$a(\boldsymbol{u}_h, \boldsymbol{\phi}_h) = a(\int_{S_c} P(\boldsymbol{x}')\boldsymbol{u}_h^G(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{n}(\boldsymbol{x}'))dS(\boldsymbol{x}'), \boldsymbol{\phi}_h)$$

$$= \int_{S_c} P(\boldsymbol{x}')a(\boldsymbol{u}_h^G(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{n}(\boldsymbol{x}')), \boldsymbol{\phi}_h)dS(\boldsymbol{x}')$$

$$= \int_{S_c} P(\boldsymbol{x}')\boldsymbol{n}(\boldsymbol{x}') \cdot \boldsymbol{\phi}_h(\boldsymbol{x}')dS(\boldsymbol{x}').$$

Note that, formally, it was not necessary to prove boundedness, because coerciveness implies uniqueness and the existence was proved by construction and by combining the result for the existence of $\boldsymbol{u}_h^G$. $\qquad\square$

Further in two and three dimensional case, the convergence between the solution to (GF) and (GF_∞) can be proved. Similar work has been done in [9] regarding Stokes problem with the Delta distribution term.

Theorem 2.2. *Let S_c be a polygon or polyhedron embedded in $\Omega \subset \mathbb{R}^n$ and let $P(\boldsymbol{x})$ be sufficiently smooth. Further, let $\boldsymbol{x}_j$ be the midpoint of surface element $\Delta S(\boldsymbol{x}_j)$. Denote $\boldsymbol{u}_h^{\Delta S}$ as the Galerkin solution to (GF) and the $\boldsymbol{u}_h^\infty$ as the Galerkin solution to (GF_∞), respectively. In two dimensions, for any $\boldsymbol{x} \notin S_c$, there exists a positive constant C, such that for each component of $\boldsymbol{u}_h^\infty$ we have*

$$|\boldsymbol{u}_h^{\Delta S} - \boldsymbol{u}_h^\infty| \leqslant C\Delta S_{max}^2,$$

where $\Delta S_{max} = \max\{\Delta S(\boldsymbol{x}_j)\}$ for any $j = \{1, 2, \cdots, N_S\}$. In three dimensions, for any $\boldsymbol{x} \notin S_c$, there exists a positive constant C, such that for each component of $\boldsymbol{u}_h^\infty$ we have

$$|\boldsymbol{u}_h^{\Delta S} - \boldsymbol{u}_h^\infty| \leqslant Ch_{max}^2,$$

where h_{max} is the maximal diameter among all the triangular elements over S_c.

Proof. Away from S_c, the function u_h^G is smooth, and since $P(\boldsymbol{x})$ is smooth as well, the integrand, given by $P(\boldsymbol{x})u_h^G(\boldsymbol{x};\boldsymbol{x}';\boldsymbol{n})$ is smooth as well. For ease of notation, we set $f(\boldsymbol{x}) = P(\boldsymbol{x})u_h^G(\boldsymbol{x};\boldsymbol{x}';\boldsymbol{n})$. We start with the 2D-case. Given the $i-th$ boundary element $\Delta\Gamma_i$ on S_c with the endpoints $\boldsymbol{x}_i$ and $\boldsymbol{x}_{i+1}$ and we denote its midpoint by $\boldsymbol{x}_{i+1/2}$, where $i \in \{1, 2, \cdots, N_S\}$. We consider

$$\boldsymbol{x}(s) = \boldsymbol{x}_{i+1/2} + s\frac{\boldsymbol{x}_{i+1} - \boldsymbol{x}_i}{2}, \quad -1 \leqslant s \leqslant 1,$$

Hence, $\boldsymbol{x}(0) = \boldsymbol{x}_{i+1/2}$ and $\boldsymbol{x}'(s) = \frac{1}{2}(\boldsymbol{x}_{i+1} - \boldsymbol{x}_i)$, and subsequently

$$\|\boldsymbol{x}'(s)\| = \frac{1}{2}\|\boldsymbol{x}_{i+1} - \boldsymbol{x}_i\|.$$

We calculate the contribution over ΔS_i to the integral, where Taylor's Theorem and the Mean Value Theorem for integration are used to warrant the existence of a $\hat{s} \in (-1, 1)$, such that

$$\int_{\Delta S_i} f(\boldsymbol{x})dS = \int_{-1}^{1} f(\boldsymbol{x}(s))\|\boldsymbol{x}'(s)\|ds$$

$$= \frac{1}{2}\|\boldsymbol{x}_{i+1} - \boldsymbol{x}_i\| \int_{-1}^{1} f(\boldsymbol{x}(s))ds$$

$$\text{(Taylor Expansion)} = \frac{1}{2}\|\boldsymbol{x}_{i+1} - \boldsymbol{x}_i\| \int_{-1}^{1} f(\boldsymbol{x}(0)) + s\frac{\boldsymbol{x}_{i+1} - \boldsymbol{x}_i}{2}\nabla f(\boldsymbol{x}(s))|_{s=0}$$

$$+ \frac{1}{2}s^2(\frac{\boldsymbol{x}_{i+1} - \boldsymbol{x}_i}{2})^T\boldsymbol{H}(\boldsymbol{x}(\hat{s}))|(\frac{\boldsymbol{x}_{i+1} - \boldsymbol{x}_i}{2})$$

$$= \frac{1}{2}\|\boldsymbol{x}_{i+1} - \boldsymbol{x}_i\|[2f(\boldsymbol{x}_{i+1/2}) + 0 + \frac{1}{12}(\boldsymbol{x}_{i+1} - \boldsymbol{x}_i)^T\boldsymbol{H}(\boldsymbol{x}(\hat{s}))|(\boldsymbol{x}_{i+1} - \boldsymbol{x}_i)]$$

$$= \|\boldsymbol{x}_{i+1} - \boldsymbol{x}_i\|f(\boldsymbol{x}_{i+1/2}) + \frac{1}{24}\|\boldsymbol{x}_{i+1} - \boldsymbol{x}_i\|(\boldsymbol{x}_{i+1} - \boldsymbol{x}_i)^T\boldsymbol{H}(\boldsymbol{x}(\hat{s}))|(\boldsymbol{x}_{i+1} - \boldsymbol{x}_i),$$

where $\boldsymbol{H}(\boldsymbol{x}(s))$ is the Hessian matrix of $f(\boldsymbol{x}(s))$. Therefore, we obtain that

$$\left|\int_{\Delta S_i} f(\boldsymbol{x})dS - \|\boldsymbol{x}_{i+1} - \boldsymbol{x}_i\|f(\boldsymbol{x}_{i+1/2})\right| = \frac{1}{24}\|\boldsymbol{x}_{i+1} - \boldsymbol{x}_i\| \cdot |(\boldsymbol{x}_{i+1} - \boldsymbol{x}_i)^T\boldsymbol{H}(\boldsymbol{x}(\hat{s}))|(\boldsymbol{x}_{i+1} - \boldsymbol{x}_i)|$$

$$\leqslant \frac{1}{24}\|\boldsymbol{x}_{i+1} - \boldsymbol{x}_i\|\tilde{K}\|\boldsymbol{x}_{i+1} - \boldsymbol{x}_i\|^2.$$

Since $f(\boldsymbol{x}) \in C^2(\Omega)$, it follows that there exists a $\tilde{K} > 0$, such that

$$|(\boldsymbol{x}, \boldsymbol{H}(\boldsymbol{x}))| \leqslant \tilde{K}\|\boldsymbol{x}\|^2.$$

Therefore, considering the summation of the boundary elements over S_C,

$$\left|\int_{\Delta S_i} f(\boldsymbol{x})dS - \sum_{i=1}^{N_S} \|\boldsymbol{x}_{i+1} - \boldsymbol{x}_i\|f(\boldsymbol{x}_{i+1/2})\right| \leqslant \sum_{i=1}^{N_S} \frac{1}{24}\|\boldsymbol{x}_{i+1} - \boldsymbol{x}_i\|\tilde{K}\|\boldsymbol{x}_{i+1} - \boldsymbol{x}_i\|^2$$

$$\leqslant \frac{1}{24}\tilde{K}\Delta S_{max}^2 \sum_{i=1}^{N_S} \|\boldsymbol{x}_{i+1} - \boldsymbol{x}_i\|$$

$$\leqslant \frac{1}{24}\tilde{K}\Delta S_{max}^2|S_c|,$$

where $\Delta S_{max} = \max_{i\in\{1,...,N_S\}} \|\boldsymbol{x}_{i+1} - \boldsymbol{x}_i\|$ is the maximal length of the line segment over S_C, and $|S_c|$ is the perimeter of the polygon S_c. It can be concluded that there exists a positive constant K, such that

$$|\boldsymbol{u}_h^\infty - \boldsymbol{u}_h^{\Delta S}| \leqslant K\Delta S_{max}^2.$$

In three dimensions, the surface element is a triangle. We map the triangle in (x, y, z)-space to the reference triangle in (s, t)-space with points $(0, 0), (0, 1)$ and $(1, 0)$. Suppose there is a surface element e_j with nodal points $\boldsymbol{x_1}, \boldsymbol{x_2}$ and $\boldsymbol{x_3}$, then the centre point of e_j is $\boldsymbol{x}_c = (\boldsymbol{x_1} + \boldsymbol{x_2} + \boldsymbol{x_3})/3$. The map from the reference triangle e_0 to the physical triangle e_j is given by

$$\boldsymbol{x}(s, t) = \boldsymbol{x_1}(1 - s - t) + s\boldsymbol{x_2} + t\boldsymbol{x_3}, \quad 0 \leqslant s, t \leqslant 1.$$

For any function $f(\boldsymbol{x}) \in C^2(\Omega)$, the integral over the original triangle is given by

$$\int_{e_j} f(\boldsymbol{x}) d\boldsymbol{x} = \int_{e_0} f(\boldsymbol{x}(s, t)) |\sqrt{\det(\boldsymbol{J}^T \boldsymbol{J})}| d(s, t),$$

where $\boldsymbol{J}$ is the Jacobian matrix, given by

$$\boldsymbol{J} = \frac{\partial(x, y, z)}{\partial(s, t)} = \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \\ z_2 - z_1 & z_3 - z_1 \end{pmatrix},$$

and $\sqrt{|\det(\boldsymbol{J}^T \boldsymbol{J})|}$ is twice the area of the original triangle e_j, i.e.

$$|\Delta_j| := \sqrt{|\det(\boldsymbol{J}^T \boldsymbol{J})|} = ||(\boldsymbol{x_2} - \boldsymbol{x_1}) \times (\boldsymbol{x_3} - \boldsymbol{x_1})||.$$

We conduct the same process as for the two dimensional case, we obtain, where $\boldsymbol{x}(\frac{1}{3}, \frac{1}{3}) = \boldsymbol{x}_c$ coincides with the midpoint of element e_j, and where Taylor's Theorem for multi-variate functions is used:

$$\int_{e_j} f(\boldsymbol{x}) d\boldsymbol{x} = \int_{e_0} f(\boldsymbol{x}(s, t)) |\Delta_j| d(s, t)$$

$$= |\Delta_j| \int_{e_0} f(\boldsymbol{x}(s, t)) d(s, t)$$

$$= |\Delta_j| \int_{e_0} f(\boldsymbol{x}_c) + (\boldsymbol{x}(s, t) - \boldsymbol{x}_c) \cdot \nabla f(\boldsymbol{x}_c) + \frac{1}{2}(\boldsymbol{x}(s, t) - \boldsymbol{x}_c)^T \boldsymbol{H}(\boldsymbol{x}(\hat{s}, \hat{t}))(\boldsymbol{x}(s, t) - \boldsymbol{x}_c) d(s, t)$$

$$= |\Delta_j| [\frac{1}{2} f(\boldsymbol{x}_c) + 0 + \frac{1}{2} \int_{e_0} (\boldsymbol{x}(s, t) - \boldsymbol{x}_c)^T \boldsymbol{H}(\boldsymbol{x}(\hat{s}, \hat{t}))(\boldsymbol{x}(s, t) - \boldsymbol{x}_c) d(s, t)].$$

Due to $f(\boldsymbol{x}) \in C^2(\Omega)$, then for the Hessian matrix of $f(\boldsymbol{x})$, there exists $\tilde{K} > 0$, such that

$$|(\boldsymbol{x}, \boldsymbol{H}(\boldsymbol{x}))| \leqslant \tilde{K} \|\boldsymbol{x}\|^2.$$

It yields

$$\left| \int_{e_j} f(\boldsymbol{x}) d\boldsymbol{x} - \frac{|\Delta_j|}{2} f(\boldsymbol{x}_c) \right| \leqslant \left| \frac{|\Delta_j|}{2} \int_{e_0} (\boldsymbol{x}(s, t) - \boldsymbol{x}_c)^T \boldsymbol{H}(\boldsymbol{x}(\hat{s}, \hat{t}))(\boldsymbol{x}(s, t) - \boldsymbol{x}_c) d(s, t) \right|$$

$$\leqslant \frac{|\Delta_j|}{4} \tilde{K} h_{max}^2,$$

where h_{max}^2 is the largest diameter in the original triangle e_j. Considering all the surface elements over S_C, we compute

$$\left| \int_{S_c} f(\boldsymbol{x}) d\boldsymbol{x} - \sum_{j=1}^{N_S} \frac{|\Delta_j|}{2} f(\boldsymbol{x}_j) \right| \leqslant \frac{\tilde{K}}{4} h_{max}^2 \sum_{j=1}^{N_S} \frac{|\Delta_j|}{2}$$

$$\leqslant \frac{\tilde{K}}{4} h_{max}^2 |S_c|,$$

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where h_{max}^2 is the maximal diameter among all the surface element (i.e. triangle) and $|S_c|$ is the sum of the measure (area in $\mathbb{R}^3$) of all the surface elements over S_c. Therefore, in three dimensions, we can conclude that there exists a positive constant K, such that for the unique Galerkin solution to both (GF) and (GF_∞),

$$|\boldsymbol{u}_h^\infty - \boldsymbol{u}_h^{\Delta S}| \leqslant Kh_{max}^2.$$

$\square$

The above proof and theorem can easily be extended to higher dimensionalities.

3 Alternative Approaches for Elasticity Equation with Point Sources in Multi Dimensions

3.1 The 'Hole' Approach

We compare the immersed boundary method with an alternative description, where we use a force boundary condition over the cell boundary. The area within the cell is not taken into account, and only stresses, strains and displacements over the region outside the cell are computed. We consider the balance of momentum over $\Omega \setminus \overline{\Omega}_C$. This gives the following boundary value problem:

$$(BVP_H) \begin{cases} -\nabla \cdot \boldsymbol{\sigma} = 0, & \text{in } \Omega \backslash \Omega_C, \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = P(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x}), & \text{on } S_C, \\ \boldsymbol{u} = \boldsymbol{0}, & \text{on } \partial\Omega, \end{cases}$$

Taking $\beta \geqslant 0$, we adjust the immersed boundary method such that

$$E(\boldsymbol{x}) = \begin{cases} \beta E, & \text{in } \Omega_C, \\ E, & \text{in } \Omega \setminus \Omega_C. \end{cases}$$

Let $D \subset \Omega$, then we introduce the following notation:

$$a_{D,E}(\boldsymbol{u}, \boldsymbol{v}) := \int_D \boldsymbol{\sigma}_E(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v})d\Omega,$$

where

$$\boldsymbol{\sigma}_E := \frac{E(\boldsymbol{x})}{1+\nu} \left\{ \boldsymbol{\epsilon} + \text{tr}(\boldsymbol{\epsilon}) \left[\frac{\nu}{1-2\nu} \right] \boldsymbol{I} \right\}$$

Regarding the adjusted immersed boundary approach, we have the following Galerkin form

$$(GF_\beta) \begin{cases} \text{Find } \boldsymbol{u}_h^\beta \in V_h(\Omega) \text{ such that for all } \boldsymbol{\phi}_h \in V_h(\Omega), \text{ we have} \\ \beta a_{\Omega_C,E}(\boldsymbol{u}_h^\beta, \boldsymbol{\phi}_h) + a_{\Omega \setminus \Omega_C,E}(\boldsymbol{u}_h^\beta, \boldsymbol{\phi}_h) = \int_{S_c} P(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{\phi}_h dS(\boldsymbol{x}). \end{cases}$$

For the hole approach, we have the following Galerkin form

$$(GF_H) \begin{cases} \text{Find } \boldsymbol{u}_h^H \in V_h(\Omega \setminus \Omega_C) \text{ such that, for all } \boldsymbol{\phi}_h \in V_h(\Omega \setminus \Omega_C), \text{ we have} \\ a_{\Omega \setminus \Omega_C,E}(\boldsymbol{u}_h^H, \boldsymbol{\phi}_h) = \int_{S_c} P(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{\phi}_h dS(\boldsymbol{x}). \end{cases}$$

We will prove that the adjusted immersed boundary method is a perturbation of the hole approach:

Proposition 3.1. *Let $\boldsymbol{u}_h^H$ and $\boldsymbol{u}_h^{\beta}$, respectively, satisfy Galerkin forms (GF_H) and (GF_{β}), then there is a $C > 0$ such that $||\boldsymbol{u}_h^H - \boldsymbol{u}_h^{\beta}||_{H^1(\Omega)} \leqslant C\sqrt{\beta}$.*

Proof. Let $\boldsymbol{v} = \boldsymbol{u}_h^{\beta} - \boldsymbol{u}_h^H$, then subtraction of problems (GF_H) and (GF_{β}) gives

$$a_{\Omega\backslash\Omega_C,E}(\boldsymbol{v}, \boldsymbol{\phi}_h) = -\beta a_{\Omega_C,E}(\boldsymbol{u}_h^{\beta}.\boldsymbol{\phi}_h).$$

The left-hand side is a bounded and coercive form on which we can apply Korn's Inequality. Furthermore, boundedness of the right-hand side in $V_h(\Omega \backslash \Omega_C)$ follows by application of the Cauchy-Schwartz Inequality, hence there is an $L > 0$ such that $a_{\Omega_C,E}(\boldsymbol{u}_h^{\beta}.\boldsymbol{\phi}_h) \leqslant L$. Herewith, we arrive at

$$-\beta L \leqslant a_{\Omega\backslash\Omega_C,E}(\boldsymbol{v}, \boldsymbol{\phi}_h) \leqslant \beta L, \text{ for all } \boldsymbol{\phi}_h \in V_h(\Omega \backslash \Omega_C).$$

Using Korn's Inequality, and upon setting $\boldsymbol{\phi}_h = v$ in $\Omega \backslash \Omega_C$, we arrive at

$$K||\boldsymbol{v}||_{H^1\Omega\backslash\Omega_C}^2 \leqslant a_{\Omega\backslash\Omega_C,E}(\boldsymbol{v}, \boldsymbol{v}) \leqslant \beta L$$

$$\Rightarrow ||\boldsymbol{v}||_{H^1\Omega\backslash\Omega_C} \leqslant C\sqrt{\beta}, \text{ where } C = \sqrt{\frac{L}{K}}.$$

$\square$

For the case of a spring-force boundary condition on $\partial\Omega$ one can derive a compatibility condition. To this extent, we consider the following boundary value problems, for the hole problem:

$$(BVP_H') \begin{cases} -\nabla \cdot \boldsymbol{\sigma} = 0, & \text{in } \Omega\backslash\Omega_C, \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = P(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x}), & \text{on } S_C, \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} + \kappa\boldsymbol{u} = \boldsymbol{0}, & \text{on } \partial\Omega, \end{cases}$$

and for the immersed boundary problem:

$$(BVP_I') \begin{cases} -\nabla \cdot \boldsymbol{\sigma} = \int_{S_c} P(\boldsymbol{x}')\boldsymbol{n}(\boldsymbol{x}')\delta(\boldsymbol{x} - \boldsymbol{x}')dS(\boldsymbol{x}'), & \text{in } \Omega\backslash\Omega_C, \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} + \kappa\boldsymbol{u} = \boldsymbol{0}, & \text{on } \partial\Omega, \end{cases}$$

Next we give a proposition regarding compatibility for the hole approach and the immersed boundary method for the case of a spring boundary condition:

Proposition 3.2. *Let $\boldsymbol{u}_H$ and $\boldsymbol{u}_I$, respectively, be solutions to the 'hole-approach', see (BVP_H') and to the immersed boundary approach, see (BVP_I'). Let S_C denote the boundary of the biological cell, over which internal forces are exerted, and let $\partial\Omega$ be the outer boundary of Ω. Then*

$$\int_{\partial\Omega} \kappa\boldsymbol{u}_H dS = \int_{\partial\Omega} \kappa\boldsymbol{u}_I dS = \int_{S_C} P(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x})dS$$

.

Proof. To prove that the above equation holds true, we integrate the PDE of both approaches over the computational domain Ω.

For the immersed boundary approach, we get

$$-\int_{\Omega} \nabla \cdot \boldsymbol{\sigma} d\Omega = \int_{\Omega} \sum_{j=1}^{N_S} P(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x})\delta(\boldsymbol{x} - \boldsymbol{x}_j)\Delta S(\boldsymbol{x}_j)d\Omega,$$

then after applying Gauss Theorem in the left-hand side and simplifying the right-hand side, we obtain

$$-\int_{\partial\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{n}(\boldsymbol{x})dS = \sum_{j=1}^{N_S} P(\boldsymbol{x}_j)\boldsymbol{n}(\boldsymbol{x}_j)\Delta S(\boldsymbol{x}_j).$$

By substituting the Robin's boundary condition and letting $N_S \to \infty$, i.e. $\Delta S(\boldsymbol{x}_j) \to 0$, the equation becomes

$$\int_{\partial\Omega} \kappa\boldsymbol{u_I}dS = \int_{S_C} P(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x})dS. \tag{3.1}$$

Subsequently, we do the same thing for the hole approach. Then, we get

$$-\int_{\Omega} \nabla \cdot \boldsymbol{\sigma} d\Omega = 0,$$

and we apply Gauss Theorem:

$$-\int_{\partial\Omega \cup S_C} \boldsymbol{\sigma} \cdot \boldsymbol{n}(\boldsymbol{x})dS = 0,$$

which implies

$$-\int_{\partial\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{n}(\boldsymbol{x})dS - \int_{S_C} \boldsymbol{\sigma} \cdot \boldsymbol{n}(\boldsymbol{x})dS = 0.$$

Using the boundary conditions, we get

$$\int_{\partial\Omega} \kappa\boldsymbol{u_H}dS = \int_{S_C} P(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x})dS,$$

which is exactly the same as Eq (3.1). Hence we proved that

$$\int_{\partial\Omega} \kappa\boldsymbol{u_H}dS = \int_{\partial\Omega} \kappa\boldsymbol{u_I}dS = \int_{S_C} P(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x})dS.$$

$\square$

Hence, the two different approaches are consistent in the sense of global conservation of momentum and therefore the results from both approaches should be comparable.

3.2   The Smoothed Particle Approach

The Gaussian distribution is used here as an approximation for the Dirac Delta distribution. Hereby, we show that in the n-dimensional case, the Gaussian distribution is a proper approximation for the Dirac Delta distribution.

Lemma 3.1. *For an open domain* $\Omega = (x_{1,1}, x_{1,2}) \times (x_{2,1}, x_{2,2}) \times \cdots \times (x_{n,1}, x_{n,2}) \subset \mathbb{R}^n, n \geqslant 2,$
let
$$\delta_\varepsilon(\boldsymbol{x} - \boldsymbol{x}') = \frac{1}{(2\pi\varepsilon^2)^{n/2}} \exp\{-\frac{\|\boldsymbol{x} - \boldsymbol{x}'\|^2}{2\varepsilon^2}\},$$

where $\boldsymbol{x}' = (x'_1, \ldots, x'_n) \in \Omega,$ *then*
(i) $\lim_{\varepsilon\to 0^+} \delta_\varepsilon(\boldsymbol{x} - \boldsymbol{x}') \to 0,$ *for all* $\boldsymbol{x} \neq \boldsymbol{x}'$;

(ii) Let $f(\boldsymbol{x}) \in C^2(\mathbb{R}^d)$ *and* $\|f(\boldsymbol{x})\| \leqslant M < +\infty,$ *then there is a* $C > 0$ *such that*

$$|\int_\Omega \delta_\varepsilon(\boldsymbol{x} - \boldsymbol{x}')f(\boldsymbol{x})d\Omega - f(\boldsymbol{x}')| \leqslant C\varepsilon^2 \ as \ \varepsilon \to 0^+.$$

Proof. (i) Since $\boldsymbol{x} \neq \boldsymbol{x}'$, $\lim_{\varepsilon \to 0^+} \exp\{-\frac{\|\boldsymbol{x} - \boldsymbol{x}'\|^2}{2\varepsilon^2}\} \to 0$. Thus,

$$\lim_{\varepsilon \to 0^+} \delta_\varepsilon(\boldsymbol{x} - \boldsymbol{x}') \to 0, \text{ for all } \boldsymbol{x} \neq \boldsymbol{x}'.$$

(ii) Now we consider

$$\int_\Omega \delta_\varepsilon(\boldsymbol{x} - \boldsymbol{x}') f(\boldsymbol{x}) d\Omega = \int_\Omega \frac{1}{(2\pi\varepsilon^2)^{n/2}} \exp\{-\frac{\|\boldsymbol{x} - \boldsymbol{x}'\|^2}{2\varepsilon^2}\} f(\boldsymbol{x}) d\Omega.$$

Firstly, we integrate over the infinite domain:

$$\int_{\mathbb{R}^n} \delta_\varepsilon(\boldsymbol{x} - \boldsymbol{x}') f(\boldsymbol{x}) d\Omega = \frac{1}{(2\pi\varepsilon^2)^{n/2}} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\{-\frac{\|\boldsymbol{x} - \boldsymbol{x}'\|^2}{2\varepsilon^2}\} f(\boldsymbol{x}) dx_n \cdots dx_1$$

$$= \frac{1}{(2\pi\varepsilon^2)^{n/2}} \int_{-\infty}^{+\infty} \exp\{-\frac{(x_1 - x_1')^2}{2\varepsilon^2}\} \cdots \int_{-\infty}^{+\infty} \exp\{-\frac{(x_n - x_n')^2}{2\varepsilon^2}\} f(\boldsymbol{x}) dx_n \cdots dx_1.$$

Again let $s_i = \frac{(x_i - x_i') - \frac{x_{i,1} + x_{i,2}}{2}}{\sqrt{2}\varepsilon}$, and furthermore $\xi_i = s_i + \frac{x_{i,1} + x_{i,2}}{2}, i = \{1, 2, \ldots, n\}$. We denote $\boldsymbol{x_1} = (x_{1,1}, x_{2,1}, \ldots, x_{n,1})$, $\boldsymbol{x_2} = (x_{1,2}, x_{2,2}, \ldots, x_{n,2})$ and $\boldsymbol{x}' = (x_1', x_2' \ldots, x_n')$. By Taylor Expansion, $f(\boldsymbol{x})$ can be rewritten as

$$f(\boldsymbol{x}) = f(\sqrt{2}\varepsilon\boldsymbol{s} + \frac{\boldsymbol{x_1} + \boldsymbol{x_2}}{2} + \boldsymbol{x}')$$

$$= f(\boldsymbol{x}') + \nabla f(\boldsymbol{x}')(\sqrt{2}\varepsilon\boldsymbol{s} + \frac{\boldsymbol{x_1} + \boldsymbol{x_2}}{2}) + \frac{1}{2!}(\sqrt{2}\varepsilon\boldsymbol{s} + \frac{\boldsymbol{x_1} + \boldsymbol{x_2}}{2})^T \boldsymbol{H}(\boldsymbol{x}')(\sqrt{2}\varepsilon\boldsymbol{s} + \frac{\boldsymbol{x_1} + \boldsymbol{x_2}}{2}) + \mathcal{O}(\varepsilon^3)$$

$$= f(\boldsymbol{x}') + \nabla f(\boldsymbol{x}')\sqrt{2}\varepsilon(\boldsymbol{s} + \frac{\boldsymbol{x_1} + \boldsymbol{x_2}}{2\sqrt{2}\varepsilon}) + \varepsilon^2(\boldsymbol{s} + \frac{\boldsymbol{x_1} + \boldsymbol{x_2}}{2\sqrt{2}\varepsilon})^T \boldsymbol{H}(\boldsymbol{x}')(\sqrt{2}\varepsilon\boldsymbol{s} + \frac{\boldsymbol{x_1} + \boldsymbol{x_2}}{2\sqrt{2}\varepsilon}) + \mathcal{O}(\varepsilon^3)$$

$$= f(\boldsymbol{x}') + \nabla f(\boldsymbol{x}')\sqrt{2}\varepsilon\boldsymbol{\xi} + \varepsilon^2\boldsymbol{\xi}^T \boldsymbol{H}(\boldsymbol{x}')\boldsymbol{\xi} + \mathcal{O}(\varepsilon^3)$$

where $\boldsymbol{H}(\boldsymbol{x}')$ is Hessian matrix of $f(\boldsymbol{x})$. For any non-negative integer d,

$$\int_{-\infty}^{+\infty} z^d e^{-z^2} dz = \begin{cases} 0, & \text{if } d \text{ is odd}, \\ \Gamma(\frac{d+1}{2}), & \text{if } d \text{ is even}. \end{cases}$$

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First we calculate

$$\int_{\mathbb{R}^n} \delta_\varepsilon(\boldsymbol{x} - \boldsymbol{x}')f(\boldsymbol{x})d\Omega = \frac{1}{(2\pi\varepsilon^2)^{n/2}} \int_{-\infty}^{+\infty} \exp\{-\frac{(x_1-x_1')^2}{2\varepsilon^2}\} \cdots \int_{-\infty}^{+\infty} \exp\{-\frac{(x_n-x_n')^2}{2\varepsilon^2}\}f(\boldsymbol{x})dx_n \cdots dx_1$$

$$= \frac{1}{\pi^{n/2}} \int_{-\infty}^{+\infty} \exp\{(-s_1 + \frac{x_{1,1}+x_{1,2}}{2})^2\} \cdots \int_{-\infty}^{+\infty} \exp\{(-s_n + \frac{x_{n,1}+x_{n,2}}{2})^2\}$$

$$f(\sqrt{2}\varepsilon\boldsymbol{s} + \frac{\boldsymbol{x_1} + \boldsymbol{x_2}}{2} + \boldsymbol{x}')ds_n \cdots ds_1$$

$$= \frac{1}{\pi^{n/2}} \int_{-\infty}^{+\infty} e^{-\xi_1^2} \cdots \int_{-\infty}^{+\infty} e^{-\xi_n^2} f(\sqrt{2}\varepsilon\boldsymbol{\xi} + \boldsymbol{x}')d\xi_n \cdots d\xi_1$$

$$= \frac{1}{\pi^{n/2}} \int_{-\infty}^{+\infty} e^{-\xi_1^2} \cdots \int_{-\infty}^{+\infty} e^{-\xi_n^2} [f(\boldsymbol{x}') + \nabla f(\boldsymbol{x}')\sqrt{2}\varepsilon\boldsymbol{\xi} + \varepsilon^2\boldsymbol{\xi}^T\boldsymbol{H}(\boldsymbol{x}')\boldsymbol{\xi} + \mathcal{O}(\varepsilon^3)]d\xi_n \cdots d\xi_1$$

$$= \frac{f(\boldsymbol{x}')}{\pi^{n/2}} \int_{-\infty}^{+\infty} e^{-\xi_1^2} \cdots \int_{-\infty}^{+\infty} e^{-\xi_n^2} d\xi_n \cdots d\xi_1 + \frac{\sqrt{2}\varepsilon}{\pi^{n/2}} \int_{-\infty}^{+\infty} e^{-\xi_1^2}\xi_1 f'_{x_1}(\boldsymbol{x}') \cdots \int_{-\infty}^{+\infty} e^{-\xi_n^2}\xi_n f'_{x_n}(\boldsymbol{x}')d\xi_n \cdots d\xi_1$$

$$+ \frac{\varepsilon^2}{\pi^{n/2}} \int_{-\infty}^{+\infty} e^{-\xi_1^2}(\sqrt{2}\xi_1 + f''_{x_1,x_1}(\boldsymbol{x}')\xi_1^2 + \sum_{i=1,i\neq 1}^{n} f''_{x_1,x_i}(\boldsymbol{x}')\xi_1\xi_i) \cdots \int_{-\infty}^{+\infty} e^{-\xi_n^2}(\sqrt{2}\xi_1$$

$$+ (f''_{x_n,x_n})(\boldsymbol{x}')\xi_n^2 + \sum_{i=1,i\neq n}^{n} f''_{x_n,x_i}(\boldsymbol{x}')\xi_n\xi_i)d\xi_n \cdots d\xi_1 + \mathcal{O}(\varepsilon^3)$$

$$= f(\boldsymbol{x}') + \frac{\varepsilon^2}{\sqrt{\pi}}\Gamma(\frac{3}{2})\sum_{i=1}^{d} f''_{x_i,x_i}(\boldsymbol{x}') + \mathcal{O}(\varepsilon^3)$$

$$\to f(\boldsymbol{x}'), \text{ as } \varepsilon \to 0^+.$$

For the integral over the given domain $\Omega = (x_{1,1}, x_{1,2}) \times \cdots \times (x_{n,1}, x_{n,2})$, it can be written as

$$\int_{x_{1,1}}^{x_{1,2}} \cdots \int_{x_{n,1}}^{x_{n,2}} dx_n \cdots dx_1 = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} dx_n \cdots dx_1 - \sum_{i=1}^{n} \int_{x_{1,1}}^{x_{1,2}} \cdots \int_{-\infty}^{x_{i,1}} \cdots \int_{x_{n,1}}^{x_{n,2}} dx_n \cdots dx_1$$

$$- \sum_{i=1}^{n} \int_{x_{1,1}}^{x_{1,2}} \cdots \int_{x_{i,2}}^{+\infty} \cdots \int_{x_{n,1}}^{x_{n,2}} dx_n \cdots dx_1$$

$$= (\sqrt{2}\varepsilon)^n \left[\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} ds_n \cdots ds_1 - \sum_{i=1}^{n} \int_{s_{1,1}}^{s_{1,2}} \cdots \int_{-\infty}^{s_{i,1}} \cdots \int_{s_{n,1}}^{s_{n,2}} ds_n \cdots ds_1 \right.$$

$$\left. - \sum_{i=1}^{n} \int_{\xi_{1,1}}^{\xi_{1,2}} \cdots \int_{\xi_{i,2}}^{+\infty} \cdots \int_{\xi_{n,1}}^{\xi_{n,2}} d\xi_n \cdots d\xi_1 \right]$$

$$= (\sqrt{2}\varepsilon)^n \left[\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} d\xi_n \cdots d\xi_1 - \sum_{i=1}^{n} \int_{\xi_{1,1}}^{\xi_{1,2}} \cdots \int_{-\infty}^{\xi_{i,1}} \cdots \int_{\xi_{n,1}}^{\xi_{n,2}} d\xi_n \cdots d\xi_1 \right.$$

$$\left. - \sum_{i=1}^{n} \int_{\xi_{1,1}}^{\xi_{1,2}} \cdots \int_{\xi_{i,2}}^{+\infty} \cdots \int_{\xi_{n,1}}^{\xi_{n,2}} d\xi_n \cdots d\xi_1 \right],$$

where $\xi_{i,1} = \frac{x_{i,1}-x_i'}{\sqrt{2}\varepsilon}$ and $\xi_{i,2} = \frac{x_{i,2}-x_i'}{\sqrt{2}\varepsilon}$. Therefore,

$$\left|\int_\Omega \delta_\varepsilon(\boldsymbol{x}-\boldsymbol{x}')f(\boldsymbol{x})d\Omega - f(\boldsymbol{x}')\right| = \left|f(\boldsymbol{x}') + \frac{\varepsilon^2}{\sqrt{\pi}}\Gamma(\frac{3}{2})\sum_{i=1}^d f''_{x_i,x_i}(\boldsymbol{x}') + \mathcal{O}(\varepsilon^3)\right.$$

$$-\frac{1}{\pi^{n/2}}\left[\sum_{i=1}^n \int_{\xi_{1,1}}^{\xi_{1,2}} e^{-\xi_1}\cdots\int_{-\infty}^{\xi_{i,1}} e^{-\xi_i}\cdots\int_{\xi_{n,1}}^{\xi_{n,2}} e^{-\xi_n} f(\sqrt{2}\varepsilon\boldsymbol{\xi}+\boldsymbol{x}')d\xi_n\cdots d\xi_1\right.$$

$$\left.+\sum_{i=1}^n \int_{\xi_{1,1}}^{\xi_{1,2}} e^{-\xi_1}\cdots\int_{\xi_{i,2}}^{+\infty} e^{-\xi_i}\cdots\int_{\xi_{n,1}}^{\xi_{n,2}} e^{-\xi_n} f(\sqrt{2}\varepsilon\boldsymbol{\xi}+\boldsymbol{x}')d\xi_n\cdots d\xi_1\right] - f(\boldsymbol{x}')\Bigg|$$

$$\leqslant \left|\frac{\varepsilon^2}{\sqrt{\pi}}\Gamma(\frac{3}{2})\sum_{i=1}^d f''_{x_i,x_i}(\boldsymbol{x}') + \mathcal{O}(\varepsilon^3)\right| + \frac{M}{2^{n-1}}\sum_{j=1}^n \prod_{i=1,i\neq j}^n [\mathrm{erf}(\xi_{j,2})-\mathrm{erf}(\xi_{j,1})+2]\times$$

$$[\mathrm{erf}(\xi_{i,2})-\mathrm{erf}(\xi_{i,1})]$$

$$\leqslant \left|\frac{\varepsilon^2}{\sqrt{\pi}}\Gamma(\frac{3}{2})\sum_{i=1}^d f''_{x_i,x_i}(\boldsymbol{x}') + \mathcal{O}(\varepsilon^3)\right| + \frac{M}{2}\sum_{j=1}^n [\mathrm{erf}(\xi_{j,1})-\mathrm{erf}(\xi_{j,2})+2] \to 0, \text{ as } \varepsilon\to 0^+,$$

since $\|f(\boldsymbol{x})\| < M < +\infty$, $\xi_{i,1} \to -\infty$ and $\xi_{i,2} \to \infty$ respectively. Using $1 - \mathrm{erf}(y) < \frac{2}{\sqrt{\pi}}\exp(-y)$ for $y > 0$ and the fact that $\exp(y) < \frac{1}{y^\alpha}$ as $y \to \infty$, we see that the second term approximates zero faster than the first term. Hence, we conclude that there is a $C > 0$ such that

$$|\int_\Omega \delta_\varepsilon(\boldsymbol{x}-\boldsymbol{x}')f(\boldsymbol{x})d\Omega - f(\boldsymbol{x}')| \leqslant C\varepsilon^2 \text{ as } \varepsilon \to 0^+.$$

$\square$

As a remark we add that setting $f(\boldsymbol{x}) = 1$, immediately shows that there is a $C > 0$ such that

$$|\int_\Omega \delta_\varepsilon(\boldsymbol{x}-\boldsymbol{x}')d\Omega - 1| \leqslant C\varepsilon^2 \text{ as } \varepsilon \longrightarrow 0^+.$$

Using the result above, we start with analysing different approaches with only one relatively big cell in the computational domain. According to the model described in Eq (2.4), the forces released on the boundary of the cell are the superposition of point forces on the midpoint of each surface element. For example, if we use a square shape to approximate the biological cell, then the forces are depicted in Figure 3.1. Therefore, in n dimensional case ($n > 1$), if the biological cell is a n-dimensional hypercube, then the forces can be rewritten as

$$\boldsymbol{f}_t = \sum_{j=1}^{N_S} P(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{x}_j)\Delta S(\boldsymbol{x}_j)$$

$$= P\sum_{i=1}^n \{\boldsymbol{e_i}(\Delta x)^{n-1}[\delta(x_1 - x_1', \ldots, x_i - (x_i' + \frac{\Delta x}{2}), \ldots, x_n - x_n') \tag{3.2}$$

$$- \delta(x_1 - x_1', \ldots, x_i - (x_i' - \frac{\Delta x}{2}), \ldots, x_n - x_n')]\},$$

where $\boldsymbol{e_i}$ is the standard basis vector with 1 in the i-th coordinate and $0's$ elsewhere, and Δx is the length of cell boundary in each coordinate. For the smoothed force approach, we set $\delta(\boldsymbol{x}) \approx \delta_\varepsilon(\boldsymbol{x})$. The force is given by

$$\boldsymbol{f}_\varepsilon = P\sum_{i=1}^n \{\boldsymbol{e_i}(\Delta x)^{n-1}[\delta_\varepsilon(x_1 - x_1', \ldots, x_i - (x_i' + \frac{\Delta x}{2}), \ldots, x_n - x_n')$$

$$- \delta_\varepsilon(x_1 - x_1', \ldots, x_i - (x_i' - \frac{\Delta x}{2}), \ldots, x_n - x_n')]\}. \tag{3.3}$$

Following the same process in two dimensions [12] and thanks to the continuity of Gaussian distribution, as $\Delta x \to 0$, the force converges to

$$\boldsymbol{f}_{SP} = P(\Delta x)^n \nabla \delta_\varepsilon (\boldsymbol{x} - \boldsymbol{x}'). \tag{3.4}$$
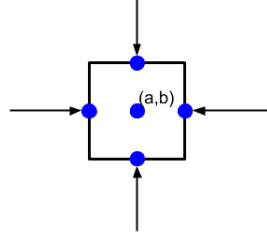
Figure 3.1: We consider a rectangular shape cell in two dimensions, with the centre position at (a, b). The forces exerted on the boundary are indicated by arrows

Theorem 3.1. *Let $\boldsymbol{u}_h \subset \boldsymbol{V}_h(\Omega)$ be the Galerkin solution to the problem*

$$(BVP) \begin{cases} \text{Find } \boldsymbol{u}_h \in V_h(\Omega) \text{ such that} \\ a(\boldsymbol{u}_h, \boldsymbol{\phi}_h) = \displaystyle\int_\Omega \boldsymbol{f}_t \boldsymbol{\phi}_h d\Omega, \text{ for all } \boldsymbol{\phi}_h \in V_h(\Omega), \end{cases}$$

and $\boldsymbol{u}_h^\varepsilon$ be the Galerkin solution to

$$(BVP_\varepsilon) \begin{cases} \text{Find } \boldsymbol{u}_h^\varepsilon \in V_h(\Omega) \text{ such that} \\ a(\boldsymbol{u}_h^\varepsilon, \boldsymbol{\phi}_h) = \displaystyle\int_\Omega \boldsymbol{f}_\varepsilon \boldsymbol{\phi}_h d\Omega, \text{ for all } \boldsymbol{\phi}_h \in V_h(\Omega). \end{cases}$$

Then there is an $L_1 > 0$ such that $||\boldsymbol{u}_h^\varepsilon - \boldsymbol{u}_h||_{H^1(\Omega)} \leqslant L_1 (\Delta x)^{(n-1)/2} \varepsilon$.

Proof. Using bilinearity of $a(.,.)$ gives upon setting $\boldsymbol{w} = \boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon$ the following equation:

$$a(\boldsymbol{w}, \boldsymbol{\phi}_h) = \int_\Omega (\boldsymbol{f}_t - \boldsymbol{f}_\epsilon) \cdot \boldsymbol{\phi}_h d\Omega.$$

Using the result from Lemma 3.1 and the Triangle Inequality, bearing in mind that $||\boldsymbol{e_i}|| = 1$ and that the basis field functions $\boldsymbol{\phi}_h$ are bounded, and after some algebraic manipulations, we can write the right-hand side as

$$| \int_\Omega (\boldsymbol{f}_t - \boldsymbol{f}_\epsilon) \cdot \boldsymbol{\phi}_h d\Omega| \leqslant C(\Delta x)^{n-1} \varepsilon^2. \tag{3.5}$$

Coerciveness, see Lemma 2.2, and using $\boldsymbol{\phi}_h = \boldsymbol{w}$, gives

$$K||\boldsymbol{w}||_{H_1(\Omega)}^2 \leqslant a(\boldsymbol{w}, \boldsymbol{w}) \leqslant C(\Delta x)^{n-1} \varepsilon^2,$$

hence there is an $L_1 > 0$ such that $||\boldsymbol{w}||_{H_1(\Omega)} \leqslant L (\Delta x)^{(n-1)/2} \varepsilon$, which immediately implies that

$$||\boldsymbol{u}_h - \boldsymbol{u}_h^\varepsilon||_{H_1(\Omega)} \leqslant L_1 (\Delta x)^{(n-1)/2} \varepsilon$$

$\square$

Theorem 3.2. *Let $\boldsymbol{u}_h^\varepsilon$ be the solution to the boundary value problems in Eq (3.1), and $\boldsymbol{u}_h^{SP}$ the solution to*

$$(BVP_{SP}) \begin{cases} \text{Find } \boldsymbol{u}_h^{SP} \in V_h(\Omega) \text{ such that} \\ a(\boldsymbol{u}_h^{SP}, \boldsymbol{\phi}_h) = \int_\Omega \boldsymbol{f}_{SP}\boldsymbol{\phi}_h d\Omega, \\ \text{for all } \boldsymbol{\phi}_h \in V_h(\Omega). \end{cases} \tag{3.6}$$

Then there is an $L_2 > 0$ such that

$$\frac{1}{(\Delta x)^n}||\boldsymbol{u}_h^{SP} - \boldsymbol{u}_h^\epsilon||_{H^1(\Omega)} \leqslant L_2 \frac{(\Delta x)^2}{\varepsilon^3}.$$

Proof. Using bilinearity of $a(.,.)$ gives upon setting $\boldsymbol{w} = \boldsymbol{u}_h^\varepsilon - \boldsymbol{u}_h^{SP}$ the following equation:

$$a(\boldsymbol{w}, \boldsymbol{\phi}_h) = \int_\Omega (\boldsymbol{f}_\varepsilon - \boldsymbol{f}_{SP}) \cdot \boldsymbol{\phi}_h d\Omega.$$

Using Taylor's Theorem for multivariate functions on smoothed delta distributions, we get the following result for the right-hand side:

$$\int_\Omega (\boldsymbol{f}_\epsilon - \boldsymbol{f}_{SP}) \cdot \boldsymbol{\phi}_h d\Omega = \int_\Omega \frac{P(\boldsymbol{x}')}{48}(\Delta x)^{n+2} \sum_{i=1}^n \boldsymbol{e}_i \frac{\partial^3 \delta_\epsilon(\hat{\boldsymbol{x}} - \boldsymbol{x}')}{\partial x_i^3} \cdot \boldsymbol{\phi}_h d\Omega, \tag{3.7}$$

for $\hat{\boldsymbol{x}}$ between $\boldsymbol{x}$ and $\boldsymbol{x}'$. The magnitude of the above expression can be estimated from above by

$$|\int_\Omega (\boldsymbol{f}_\epsilon - \boldsymbol{f}_{SP}) \cdot \boldsymbol{\phi}_h d\Omega| \leqslant \frac{P(\boldsymbol{x}')}{48}(\Delta x)^{n+2}||\sum_{i=1}^n \boldsymbol{e}_i \frac{\partial^3 \delta_\epsilon}{\partial x_i^3}||_{L^\infty(\Omega)}||\boldsymbol{\phi}_h||_{H^1(\Omega)}. \tag{3.8}$$

Using Lemma 2.2, this gives

$$K||\boldsymbol{w}||_{H^1(\Omega)}^2 \leqslant a(\boldsymbol{w}, \boldsymbol{w}) \leqslant \frac{P(\boldsymbol{x}')}{48}(\Delta x)^{n+2}||\sum_{i=1}^n \boldsymbol{e}_i \frac{\partial^3 \delta_\epsilon}{\partial x_i^3}||_{L^\infty(\Omega)}||\boldsymbol{\phi}_h||_{H^1(\Omega)}.$$

Division by $K||\boldsymbol{w}||_{H^1(\Omega)}$ gives

$$||\boldsymbol{w}||_{H^1(\Omega)} \leqslant \frac{P(\boldsymbol{x}')}{48}(\Delta x)^{n+2}||\sum_{i=1}^n \boldsymbol{e}_i \frac{\partial^3 \delta_\epsilon}{\partial x_i^3}||_{L^\infty(\Omega)}.$$

We bear in mind that $\frac{\partial^3 \delta_\epsilon}{\partial x_i^3} = \mathcal{O}(\varepsilon^{-3})$, this implies that there is an $L_2 > 0$ such that

$$\frac{1}{(\Delta x)^n}||\boldsymbol{u}_h^{SP} - \boldsymbol{u}_h^\epsilon||_{H^1(\Omega)} \leqslant L_2 \frac{(\Delta x)^2}{\varepsilon^3}.$$

$\square$

With the two theorems above, we have proved that the solution to (BVP_ε) converges to the solution to (BVP), and the solution to (BVP_{SP}) converges to the solution to (BVP_ε). Hence, we can derive the following theorem:

Theorem 3.3. *Let $\boldsymbol{u}_h$ be the Galerkin solution to (BVP) and $\boldsymbol{u}_h^{SP}$ be the solution to (BVP_{SP}), let $\varepsilon = \mathcal{O}(\Delta x)^p$ and $\Delta x \to 0$. If $0 < p < (2+n)/3$ then $\boldsymbol{u}_h^{SP}$ converges to $\boldsymbol{u}_h$ in the H^1–norm, and $\boldsymbol{u}_h^{SP}$ converges to $\boldsymbol{u}$ in the H^1–norm.*

Proof. Denote $\boldsymbol{u}_h$ and $\boldsymbol{u}_h^{SP}$ to be the Galerkin solution to (BVP_ε) and (BVP_{SP}). Firstly, we consider

$$
\begin{aligned}
\|\boldsymbol{u}_h - \boldsymbol{u}_h^{SP}\| &= \|\boldsymbol{u}_h - \boldsymbol{u}_h^{\boldsymbol{\varepsilon}} + \boldsymbol{u}_h^{\boldsymbol{\varepsilon}} - \boldsymbol{u}_h^{SP}\| \\
&\leqslant \|\boldsymbol{u}_h - \boldsymbol{u}_h^{\boldsymbol{\varepsilon}}\| + \|\boldsymbol{u}_h^{\boldsymbol{\varepsilon}} - \boldsymbol{v}_h^{\boldsymbol{\varepsilon}}\| \\
&= L_1(\Delta x)^{(n-1)/2}\varepsilon + L_2\frac{(\Delta x)^{2+n}}{\varepsilon^3} \\
&= L_1(\Delta x)^{(n-1)/2+p} + L_2(\Delta x)^{2+n-3p} \longrightarrow 0, \\
&\quad \text{as } \Delta x \longrightarrow 0, \text{ if } 0 < p < (2+n)/3.
\end{aligned}
$$

From this inequality, we conclude that the finite element solution of the smooth particle method converges to the solution of the immersed boundary method upon letting $\Delta x \to 0$ and choosing $\varepsilon = \mathcal{O}((\Delta x)^p)$ for $0 < p < (2+p)/3$. [2] demonstrated that the Galerkin solution of the immersed boundary method converges in H^1 norm to the exact solution of the immersed boundary method. Using

$$
\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{u}_h^{SP}\|_{H^1(\Omega)} &\leqslant \|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1(\Omega)} + \|\boldsymbol{u}_h - \boldsymbol{u}_h^{SP}\|_{H^1(\Omega)} \\
&\to 0, \text{ as } \Delta x \longrightarrow 0,
\end{aligned}
$$

upon letting $\Delta x \to 0$ and choosing $\varepsilon = \mathcal{O}((\Delta x)^p)$ for $0 < p < (2+p)/3$. $\qquad\square$

4 Conclusion

For the dimensionality exceeding one, the existence of Dirac Delta distributions in the elasticity equation results into a singular solution. We analyse the solutions based on Galerkin approximations with Lagrangian basis functions for different approaches that are consistent if biological cell sizes and smoothness parameters tend to zero. We have shown that the smoothed particle approach is consistent with the immersed boundary approach. The current paper has investigated and extended earlier findings to multi-dimensionality.

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