ET Mi 109: Electrical Engineering for Autonomous Exploration Robots

EE3330TU Guiding & Radiating Structures

Entry Test: Solutions Complex Numbers

1. Given two complex numbers

$$V = 3 - j4$$
$$I = -(2 + j3)$$

(a) express V and I in polar form

$$|V| = |3 - j4| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5$$

V is in the second quadrant in the complex plane, thus

$$\angle V = \tan^{-1}(-4/3) = -0.93 \text{ rad (or } -53.1°)$$
$$V = 5e^{-j0.93}$$

$$|I| = |-2 - j3| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13} = 3.61$$

I is in the third quadrant in the complex plane, thus

$$\angle I = \tan^{-1}\left(\frac{3}{2}\right) - \pi = -2.16 \text{ rad (or } -123.7°)$$
$$I = 3.61e^{-j2.16}$$

(b) Calculate VI

$$VI = 5e^{-j0.93} \times 3.61e^{-j2.16} = 18.05e^{-j3.09}$$

(c) Calculate VI^*

$$VI^* = 5e^{-j0.93} \times 3.61e^{j2.16} = 18.05e^{j1.23}$$

(d) Calculate V/I

$$V/I = \frac{5e^{-j0.93}}{3.61e^{-j2.16}} = 1.39e^{j1.23}$$

(e) Calculate $\sqrt{I}$

$$\sqrt{I} = \sqrt{3.61e^{-j2.16}} = \pm\sqrt{3.61}e^{-j\frac{2.16}{2}} = \pm 1.9e^{-j1.08}$$

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2. Express the following complex functions in polar form:

$$z_1 = (4 - j3)^2 = \left(\sqrt{16 + 9}e^{j\tan^{-1}\left(-\frac{3}{4}\right)}\right)^2 = 25e^{-j1.29}$$

$$z_2 = (4 - j3)^{1/2} = (5e^{-j0.64})^{1/2} = \pm\sqrt{5}e^{-j0.32}$$

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3. Show that $\sqrt{2j} = \pm(1 + j)$

$$\sqrt{2j} = (2j)^{1/2} = \left(2e^{j\pi/2}\right)^{1/2} = \pm\sqrt{2}e^{j\pi/4}$$
$$= \pm\sqrt{2}\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)$$
$$= \pm\sqrt{2}\left(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right) = \pm(1 + j)$$

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4. Evaluate each of the following complex numbers and express the result in rectangular form (Re+jIm)

(a) $z = 4e^{j\pi/3}$

$$z = 4\left(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}\right) = 4\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = 2 + j2\sqrt{3}$$

(b) $z = \sqrt{3}e^{j3\pi/4}$

$$z = \sqrt{3}\left(\cos\frac{3\pi}{4} + j\sin\frac{3\pi}{4}\right) = \sqrt{3}\left(-\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right)$$
$$= -\frac{\sqrt{6}}{2}(1 - j)$$

(c) $z = 6e^{-j\pi/2}$

$$z = 6\left(\cos\frac{-\pi}{2} + j\sin\frac{-\pi}{2}\right) = 6(0 - j) = -6j$$

(d) $z = j^3$

$$z = j^2 \cdot j = -j$$

(e) $z = j^{-4}$

$$z = \frac{1}{j^4} = \frac{1}{j^2 \cdot j^2} = \frac{1}{(-1)\cdot(-1)} = 1$$

(f) $z = (1 - j)^3$

$$z = \left(\sqrt{2}e^{-j\frac{\pi}{4}}\right)^3 = 2\sqrt{2}\left(\cos\frac{-3\pi}{4} + j\sin\frac{-3\pi}{4}\right) =$$
$$= 2\sqrt{2}\left(-\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}\right) = -2 - 2j$$

(g) $z = (1 - j)^{1/2}$

$$z = \left(\sqrt{2}e^{-j\frac{\pi}{4}}\right)^{\frac{1}{2}} = \pm 1.19e^{-j\frac{\pi}{8}} = \pm 1.19\left(\cos\frac{\pi}{8} - j\sin\frac{\pi}{8}\right) = \pm 1.19(0.92 - j0.38) = \pm(1.09 - j0.45)$$

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5. If $z = -2 + j4$, determine the following quantities in polar form

(a) $1/z$

$$|z| = \sqrt{16 + 4} = 4.47 \qquad \angle z = \tan^{-1}(-2) + \pi = 2.03$$
$$\frac{1}{z} = \frac{1}{-2 + j4} = \frac{1}{4.47e^{j2.03}} = 0.22e^{-j2.03}$$

(b) z^3

$$z^3 = 4.47^3 e^{j2.03\cdot 3} = 89.31e^{j6.09} = 89.31e^{-j0.19}$$

(c) $|z|^2$

$$|z|^2 = 4.47^2 e^{j0} = 19.98e^{j0}$$

(d) $Im\{z\}$

$$Im\{z\} = 4e^{j0}$$

(e) $Im\{z^*\}$

$$Im\{z^*\} = -4 = 4e^{j\pi}$$

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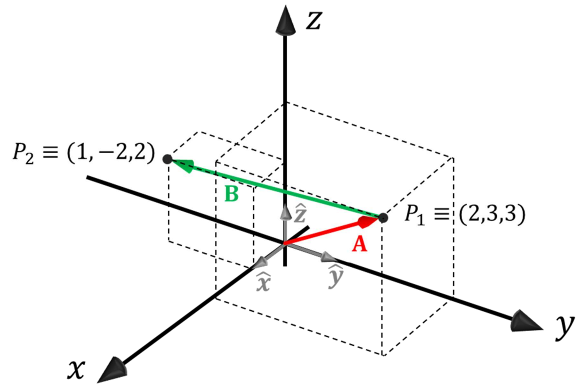
Entry Test: Solutions Vector Analysis



6. In Cartesian coordinates, vector $\mathbf{A}$ is directed from the origin to point $P_1(2,3,3)$, and vector $\mathbf{B}$ is directed from point P_1 to point $P_2(1,-2,2)$. Find

   (a) vector $\mathbf{A}$, its magnitude A, its unit vector $\hat{\boldsymbol{a}}$

$$\mathbf{A} = 2\hat{\boldsymbol{x}} + 3\hat{\boldsymbol{y}} + 3\hat{\boldsymbol{z}}$$

$$A = |\mathbf{A}| = \sqrt{2^2 + 3^2 + 3^2} = \sqrt{22}$$

$$\hat{\boldsymbol{a}} = \frac{\mathbf{A}}{A} = \frac{1}{\sqrt{22}}(2\hat{\boldsymbol{x}} + 3\hat{\boldsymbol{y}} + 3\hat{\boldsymbol{z}})$$

   (b) the angle that $\mathbf{A}$ makes with the y-axis

From the property of the scalar (inner) product

$$\mathbf{A} \cdot \hat{\boldsymbol{y}} = A\cos\theta$$

$$\mathbf{A} \cdot \hat{\boldsymbol{y}} = 3 \Rightarrow \theta = \cos^{-1}\frac{3}{\sqrt{22}} = 0.8768 \text{ rad} = 50.24°$$

   (c) vector $\mathbf{B}$

The vector directed from the origin to point P_2 is

$$\mathbf{C} = \hat{\boldsymbol{x}} - 2\hat{\boldsymbol{y}} + 2\hat{\boldsymbol{z}}$$

$$\mathbf{B} = \mathbf{C} - \mathbf{A} = (1-2)\hat{\boldsymbol{x}} + (-2-3)\hat{\boldsymbol{y}} + (2-3)\hat{\boldsymbol{z}}$$
$$= -\hat{\boldsymbol{x}} - 5\hat{\boldsymbol{y}} - \hat{\boldsymbol{z}}$$

   (d) the angle between $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{A} \cdot \mathbf{B} = A\,B\cos\alpha$$
$$B = |\boldsymbol{B}| = \sqrt{1^2 + 5^2 + 1^2} = \sqrt{27}$$
$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z = -2 - 15 - 3 = -20$$
$$\alpha = \cos^{-1}\frac{-20}{\sqrt{22}\sqrt{27}} = 2.5333 \text{ rad} = 145.15°$$

   (e) the vector product $\mathbf{A} \times \mathbf{B}$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ 2 & 3 & 3 \\ -1 & -5 & -1 \end{vmatrix} =$$
$$= \hat{\boldsymbol{x}}(-3+15) - \hat{\boldsymbol{y}}(-2+3) + \hat{\boldsymbol{z}}(-10+3) =$$
$$= 12\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}} - 7\hat{\boldsymbol{z}}$$

7. Given $\mathbf{A} = 2\hat{\boldsymbol{x}} - 3\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}$ and $\mathbf{B} = B_x\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} + B_z\hat{\boldsymbol{z}}$,

   (a) Find B_x and B_z if $\mathbf{A}$ is parallel to $\mathbf{B}$

Being parallel,

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ 2 & -3 & 1 \\ B_x & 2 & B_z \end{vmatrix} = 0$$
$$\hat{\boldsymbol{x}}(-3B_z - 2) - \hat{\boldsymbol{y}}(2B_z - B_x) + \hat{\boldsymbol{z}}(4 + 3B_x) = 0$$
$$B_z = -\frac{2}{3}, B_x = -\frac{4}{3}$$

   (b) Find a relation between B_x and B_z if $\mathbf{A}$ is perpendicular to $\mathbf{B}$

Being perpendicular,

$$\mathbf{A} \cdot \mathbf{B} = 2B_x - 6 + B_z = 0$$
$$B_z = 6 - B_x$$

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8. Given $\mathbf{A} = 3\hat{\boldsymbol{y}} + 4\hat{\boldsymbol{z}}$ and $\mathbf{B} = 4\hat{\boldsymbol{x}} - 10\hat{\boldsymbol{y}} + 5\hat{\boldsymbol{z}}$,

   (a) Find the vector component of $\mathbf{A}$ along $\mathbf{B}$

The component of $\mathbf{A}$ along $\mathbf{B}$ is $(\mathbf{A} \cdot \hat{\boldsymbol{b}})\hat{\boldsymbol{b}}$

$$\hat{\boldsymbol{b}} = \frac{\mathbf{B}}{|\mathbf{B}|} = \frac{4\hat{\boldsymbol{x}} - 10\hat{\boldsymbol{y}} + 5\hat{\boldsymbol{z}}}{\sqrt{4^2 + 10^2 + 5^2}} = \frac{4\hat{\boldsymbol{x}} - 10\hat{\boldsymbol{y}} + 5\hat{\boldsymbol{z}}}{11.87}$$
$$\mathbf{A} \cdot \hat{\boldsymbol{b}} = \frac{-30 + 20}{11.87} = -0.84$$
$$(\mathbf{A} \cdot \hat{\boldsymbol{b}})\hat{\boldsymbol{b}} = -0.84\frac{4\hat{\boldsymbol{x}} - 10\hat{\boldsymbol{y}} + 5\hat{\boldsymbol{z}}}{11.87}$$
$$= -0.28\hat{\boldsymbol{x}} + 0.71\hat{\boldsymbol{y}} - 0.35\hat{\boldsymbol{z}}$$

   (b) Determine a unit vector perpendicular to both $\mathbf{A}$ and $\mathbf{B}$

We know that $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is perpendicular to both vector, thus $\pm\hat{\boldsymbol{c}}$ is such a unit vector

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ 0 & 3 & 4 \\ 4 & -10 & 5 \end{vmatrix} =$$
$$= \hat{\boldsymbol{x}}(15 + 40) - \hat{\boldsymbol{y}}(0 - 16) + \hat{\boldsymbol{z}}(-12) =$$
$$= 55\hat{\boldsymbol{x}} + 16\hat{\boldsymbol{y}} - 12\hat{\boldsymbol{z}}$$

$$\pm\hat{\boldsymbol{c}} = \pm\frac{\mathbf{C}}{|\mathbf{C}|} = \pm\frac{55\hat{\boldsymbol{x}} + 16\hat{\boldsymbol{y}} - 12\hat{\boldsymbol{z}}}{\sqrt{55^2 + 16^2 + 12^2}} =$$
$$= \pm\frac{55\hat{\boldsymbol{x}} + 16\hat{\boldsymbol{y}} - 12\hat{\boldsymbol{z}}}{58.52} = \pm(0.94\hat{\boldsymbol{x}} + 0.27\hat{\boldsymbol{y}} - 0.21\hat{\boldsymbol{z}})$$