

The Effects of Dynamical Noises on the Identification of Chaotic Systems: with Application to Streamflow Processes

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Abstract

Three known chaotic system corrupted with different levels of dynamic noise were investigated for assessing the effects of dynamical noises on the identification of chaotic systems. It is found that when noise level is low, the chaotic attractor can still be well preserved and we can give basically correct estimate of correlation dimension, which indicates that even if we observed the existence of chaos in a time series, it does not necessarily mean determinism. On the other hand, in the presence of high-level dynamical noise, it is hard or even impossible to identify a chaotic system. Two streamflow processes are investigated as well for detecting the existence of chaotic behaviour. The existence of low-dimensional chaos in the streamflow series is not confirmed with the method of correlation dimension. Because the streamflow process usually suffers from strong natural and anthropogenic disturbances which are composed of both stochastic and deterministic components, consequently, it is not likely to identify the chaotic dynamics even if the streamflow process is indeed a low-dimension chaotic process under ideal circumstances (i.e., without any or only with small enough stochastic disturbances).

1. Introduction

Chaos is widely concerned in the last two decades, and chaotic mechanism of streamflows has gained quite a lot attention of the hydrology community. Most of the research in literature confirms the presence of chaos in the hydrologic time series (e.g., Rodriguez-Iturbe et al., 1989; Porporato and Ridolfi, 1997, 2003; Sivakumar, 2000, 2007; Elshorbagy et al., 2002). Consequently, some researchers (e.g., Sivakumar, 2000) believed that the dynamic structures of the seemingly complex hydrological processes, such as rainfall and runoff, might be better understood using nonlinear deterministic chaotic models than the stochastic ones. Meanwhile, some studies denied the existence of chaos in hydrologic processes (e.g., Wilcox et al., 1991; Koutsoyiannis and Pachakis, 1996; Pasternack, 1999), and there are many disputes

about the existence of low-dimensional chaos in hydrologic processes (e.g., Ghilardi and Rosso, 1990; Schertzer et al., 2002).

In this study, we would like to investigate the effects of dynamical noises on the identification of chaotic systems, and whether or not chaos exists in streamflow processes with correlation dimension method.

2. Correlation dimension method for detecting chaos

Correlation dimension method is most frequently employed to detect the existence of chaos. The basis of this method is multi-dimension state space reconstruction. The most commonly used method for reconstructing the state space is the time-delay coordinate method. In the time delay coordinate method, a scalar time series $\{x_1, x_2, \dots, x_N\}$ is converted to state vectors $X_t = (x_t, x_{t-\tau}, \dots, x_{t-(m-1)\tau})$ after determining two state space parameters: the embedding dimension m and delay time τ . To check whether chaos exists, the correlation exponent values are calculated against the corresponding embedding dimension values. If the correlation exponent leads to a finite value as embedding dimension increases, then the process under investigation is usually considered as being dominated by deterministic dynamics, and the saturated value is called correlation dimension.

The most commonly used algorithm for computing correlation dimension is Grassberger - Procaccia algorithm (Grassberger and Procaccia, 1983), modified by Theiler (1986). For a m -dimension phase-space, the modified correlation integral $C(r)$ is defined by (Theiler, 1986)

$$C(r) = \frac{2}{(M+1-w)(M-w)} \sum_{i=1}^M \sum_{j=i+w+1}^{M-i} H(r - \|X_i - X_j\|) \quad (1)$$

where $M = N - (m - 1)\tau$ is the number of embedded points in m -dimensional space; r the radius of a sphere centered on X_i ; $H(u) = 1$ for $u > 0$, and $H(u) = 0$ for $u \leq 0$; $\|\bullet\|$ denotes the sup-norm; $w (\geq 1)$ is the Theiler window to exclude those points which are temporally correlated.

For a finite dataset, there is a radius r below which there are no pairs of points, whereas at the other extreme, when the radius approaches the diameter of the cloud of points, the number of pairs will increase no further as the radius increases (saturation). The scaling region would be found somewhere between depopulation and saturation. When $\ln C(r)$ versus $\ln r$ is plotted for a given embedding dimension m , the range of $\ln r$ where the slope of the curve is approximately constant is the scaling region where fractal geometry is indicated. In this region $C(r)$ increase as a power of r , with the scaling exponent being the correlation dimension D . If the scaling region vanishes as m increases, then finite value of correlation dimension cannot be obtained, and the system under investigation is considered as non-chaotic. Local slopes of $\ln C(r)$ versus $\ln r$ plot can show scaling region clearly when it exists. Because the local slopes of $\ln C(r)$ versus $\ln r$ plot often fluctuate dramatically, to identify the scaling region more clearly, we can use Takens-Theiler estimator or smooth Gaussian kernel estimator to estimate correlation dimension (Hegger et al., 1999).

3. The effects of dynamical noises on the identification of chaotic systems

When analyzing the chaos properties in observational time series, we cannot avoid the problem of noise. There are two distinct types of noise: (1) measurement noise, which refers to the corruption of observations by errors that does not influence the evolution of the dynamical system; and (2) dynamical noise (including both input and output noises), which perturbs the system more or less at each time step. In the presence of dynamical noises, the time series is not a simple superposition of signal plus noise, but rather a signal modulated by the noise. With regard to an observed hydrologic series, its dynamics is inevitably contaminated not only by measurement noise, but also more significantly by dynamical noise, such as the disturbance of storm rain. What are the effects of dynamical noise on the estimation of characteristic quantities of chaotic systems? We will discuss this issue through experiments with three well-known chaotic systems: (1) Henon map (Henon, 1976), which has one attractor with an attraction basin nearly touching the attractor in several places; (2) Ikeda map (Ikeda, 1979), which has one chaotic attractor with a small attraction basin and a non-chaotic attractor with much larger attraction basin; (3) Mackey –Glass flow (Mackey and Glass, 1977), which has one attractor with unbounded attraction basin.

To analyze the effects of dynamical noises, we add different levels of dynamical noises to the known chaotic

systems by adding a noise item to the equations, and then check the phase state portraits and correlation dimensional estimates. The noise we add to the chaotic series are independently identically distributed Gaussian noise, with zero mean and variance of 2%, 5%, 10% and 100% of the original pure chaotic series. Besides the Gaussian noise, autoregressive (AR) dynamical noises of AR(1) structure have also been tested. Results show that, due to the impacts of the serial dependence of AR noises, the impacts of the AR noise on the dynamics of chaotic systems are slightly stronger than, but similar to, those of i.i.d. noises. To save space, the results are not displayed here.

3.1 Impacts of dynamical noises on Henon map

First, we consider the Henon map (Henon, 1976) of the form

$$\begin{cases} x_{n+1} = 1 - ax_n^2 + by_n \\ y_{n+1} = x_n \end{cases} \quad (2)$$

With $a = 1.4$ and $b = 0.3$, for initials such as $x = 0, y = 0.9$, Henon map has one strange chaotic attractor (Figure 1a). Experiments show that Henon map is sensitive to the disturbance of dynamical noises. Even 1% dynamical noise can lead the Henon map to infinity. Noises above 2% could easily push the orbit outside the basin of attraction, and the series goes to minus infinity exponentially. Figure 1b shows the portrait of a comparatively short Henon map series (1000 points) with 2% noise. It resembles the pure Henon map series in the appearance. However, with the evolution of the system, namely, the increase of the iterations of Henon map (e.g., >2000), it will surely go to infinity. The reason that such a low level as 2% of noise may lead the Henon map to infinity is that the boundary of the attraction basin of the Henon map nearly touches the attractor in several places, where very small disturbances will push the trajectory outside the basin. Therefore, in the presence of dynamical noise, for example, noises of 2% level, the Henon map rarely remains in the attractor after 1000 iterations. But as long as the noisy (but not too noisy) Henon map remains in the chaotic attractor, we can identify a clear scaling region on the Takens-Theiler estimate D_{TT} versus $\ln r$ plot (Figure 2), and give a finite correlation dimension estimate about 1.25 (for noise-free series, about 1.22). With 10% dynamical noise, the Henon map series usually start to grow exponentially to infinity within 20 iterations (as shown in Figure 1c). With 100% dynamical noise, the exponential growth starts within 10 steps. With such short series, low-dimension chaos cannot be identified.

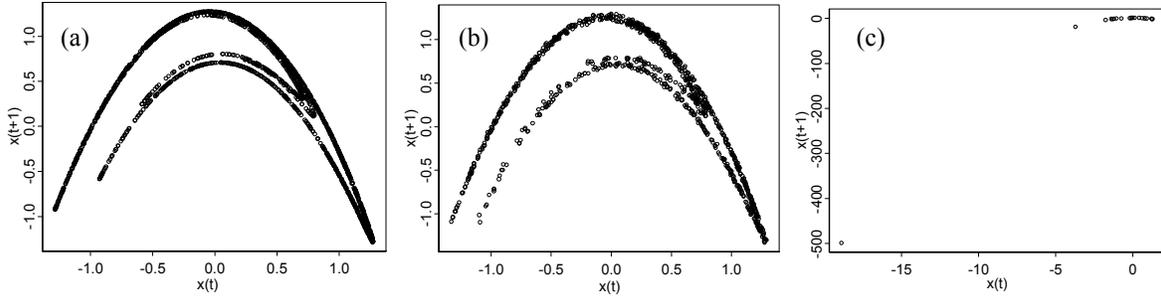


Figure 1 Henon map with (a) 0%; (b) 2% and (c) 10% Gaussian dynamical noise

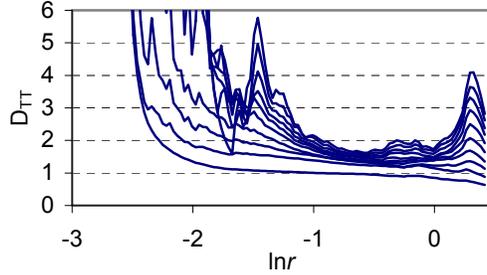


Figure 2 Correlation dimension estimate of Henon map series with 2% dynamical noise

3.2 Impacts of dynamical noises on Ikeda map

Next, we consider Ikeda map (Ikeda, 1979) of the form

$$z_{n+1} = \gamma + \mu \exp\left(i\beta - \frac{i\alpha}{1 + |z_n|^2}\right) z_n \quad (3)$$

where the z_n are complex variables. This map can be written as a two-dimensional system in the following form

$$\begin{cases} x_{n+1} = \gamma + \mu(x_n \cos \phi - y_n \sin \phi) \\ y_{n+1} = \mu(x_n \sin \phi + y_n \cos \phi) \end{cases} \quad (4)$$

where $\phi = \beta - \alpha / (1 + x_n^2 + y_n^2)$. With $\alpha = 6$, $\beta = 0.4$, $\gamma = 1$ and $\mu = 0.9$, for an initial such as $x = 0$ and $y = 0$, the map gives chaos (Figure 3a). It is known that Ikeda map has a chaotic attractor with a complex attraction basin and one stable point attractor centered at $(2.9721316, 4.145946)$. With 2% dynamical noise, the system will usually stay in the chaotic attractor first (Figure 3b), then move to point attractor as the system evolves. With the increase of noise level, the chance of staying in the chaotic attractors

decrease, meanwhile the chance of moving to the point attractor increase. When noise level reaches 10%, the system only stays in chaotic attractor for a short period of time (usually less than 1000 iterations), then move to the point attractor (Figure 3c). When the noise level is as high as 100%, the system usually escapes the chaotic attractor within 10 steps. Even for small noise level (e.g., 2% noise), with the evolution of the system (e.g., iterate the map more than 20000 times), the system will finally be trapped into the point attractor.

We also calculate the Takens-Theiler estimate D_{TT} of correlation dimension for Ikeda map series with 2% and 10% dynamical noise when the series stays in the chaotic attraction basin, and plot D_{TT} versus $\ln r$ in Figure 4. We can clearly identify the scaling region in Figure 4a, and estimate the correlation dimension about 1.85 (for pure series, about 1.69). But the scaling region in Figure 4b is not clearly discernable. From the vague region around $\ln r = -0.5$, the correlation dimension estimate for Ikeda series with 10% noise is about 2.12.

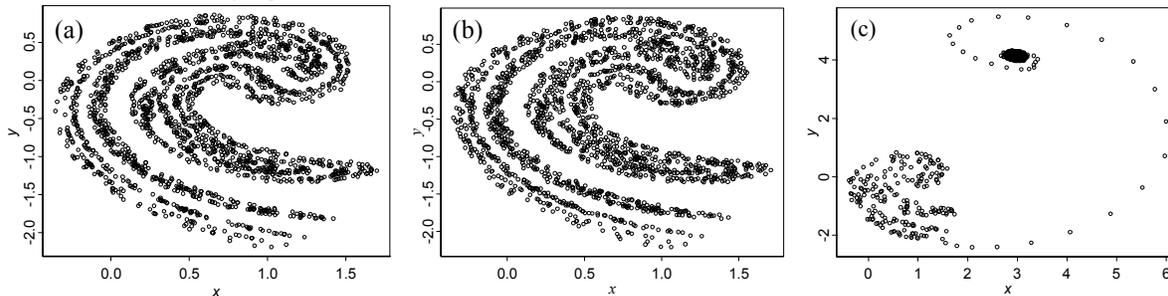


Figure 3 Ikeda map with (a) 0%; (b) 2% and (c) 10% Gaussian dynamical noise

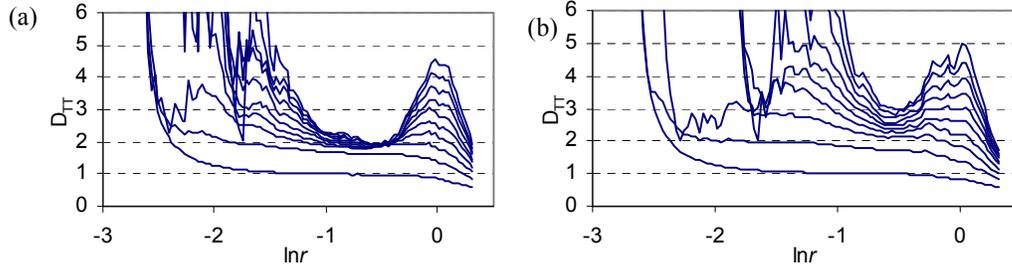


Figure 4 Correlation dimension estimate of Ikeda map with (a) 2% and (b) 10% dynamical noises

3.3 Impacts of dynamical noises on discretized Mackey-Glass flow

Finally, we consider Mackey-Glass delay differential equation (Mackey and Glass, 1977) of the form

$$\frac{dx}{dt} = \frac{ax_{t-\tau}}{1+x_{t-\tau}^c} - bx_t \quad (5)$$

where $x_{t-\tau}$ is the value of x at time $t-\tau$. It can be written as an approximate $m+1$ dimensional map in delay coordinates:

$$x_{n+1} = \frac{2m - b\tau}{2m + b\tau} x_n + \frac{a\tau}{2m + b\tau} \left(\frac{x_{n-m}}{1+x_{n-m}^c} + \frac{x_{n-m+1}}{1+x_{n-m+1}^c} \right) \quad (6)$$

With $a = 0.2$, $b = 0.1$ and $c = 10$, this map can generate time series with chaotic attractors of different dimension for $\tau > 16.8$. We choose $m = 30$, $\tau = 30$. The noise-free

series and the series with 2% and 10% dynamical noise are plotted in Figure 5a, 12b and 12c.

From Figure 5b, we see that the chaotic attractor is very clear with 2% Gaussian noise. Because the attraction basin of Mackey-Glass flow is unbounded, even for high level (e.g., 10%) of noises, the attractor still can be vaguely discerned (Figure 5c). But the attractor is not discernable anymore with the noise of 100% level (not shown in the figure).

We calculate the D_{TT} for discretized Mackey-Glass flow contaminated with 2% and 10% dynamical noise, plotted in Figure 6. With 2% noise, we can get an estimate of correlation dimension about 2.5 (correct correlation dimension is about 2.45). Although we can discern the attractor vaguely with noises as high as 10% as shown in Figure 5c, it is hard to define the scaling region in Figure 6b and hard to estimate the correlation dimension correctly. With the noise level as high as 100%, the scaling region is totally lost.

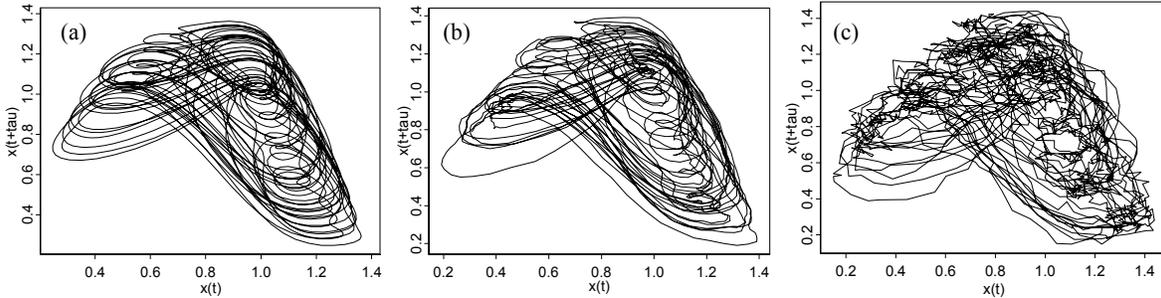


Figure 5 Mackey-Glass flow with (a) 0%; (b) 2% and (c) 10% Gaussian dynamical noise

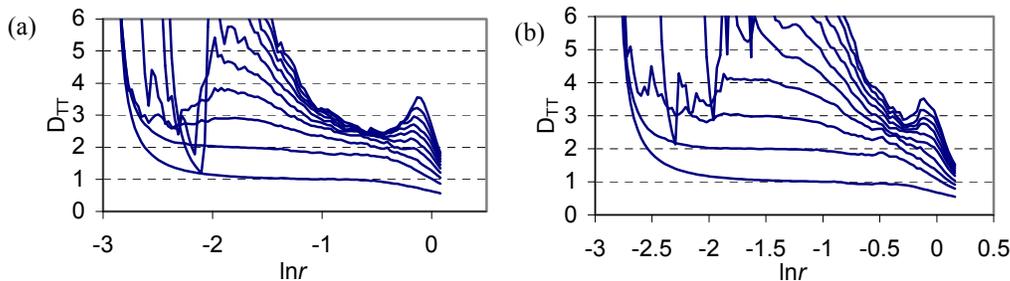


Figure 6 Estimate of correlation dimension for Mackey-Glass series with (a) 2% and (b) 10% dynamical noise

3.4 Analysis of the results for the cases of known chaotic processes

According to the above analyses, we have some remarks on the following two aspects:

(1) About the identification of chaotic system

Although the presence of noise limits the performance of many techniques of identification and prediction of chaotic systems, with low level (2%) Gaussian noise, the chaotic attractor can still be well preserved and we can give basically correct estimate of correlation dimension. However, in the presence dynamical noises, the estimate is biased to a higher value, and the higher the noise level, the larger the bias. When the level of dynamic noise is high (e.g., 10%), it is hard to identify the systems analyzed above, let alone in the presence of 100% level noise.

(2) About the property of chaotic system

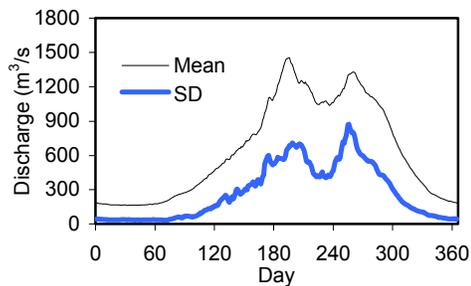
Although chaotic systems are widely considered as deterministic, in the presence of dynamical noise, the system may still possess chaotic behaviour. That means, chaos could be stochastic. In the presence of dynamical noise, whether or not the chaotic system remains in the chaotic attractor depends on the intensity of stochastic disturbances. If the disturbance is so strong as to push the orbit outside the chaotic attraction basin, then the system may go to infinity, or fall into neighboring non-chaotic attractors, or just lost the geometry of the chaotic attractor, and the system becomes non-chaotic.

Two factors affect whether the trajectories would escape the chaotic attractors for a chaotic system with stochastic noise. (1) The distance D_B between the boundary of attraction basin and that of the attractor. The larger the distance, the harder to push the trajectory away from the attractor. If the attraction basin is unbounded, then the system will never escape. (2) The distribution of the stochastic noise. For uniform distributed noise, if the upper limit is less than minimum D_B , then the stochastic chaotic system will never escape the chaotic attractor. For un-bounded (on whatever side) noise the dynamical systems would sooner or later escape the attractor basin and go to infinity or fall into other neighboring attractors as the evolution of the system goes to infinity.

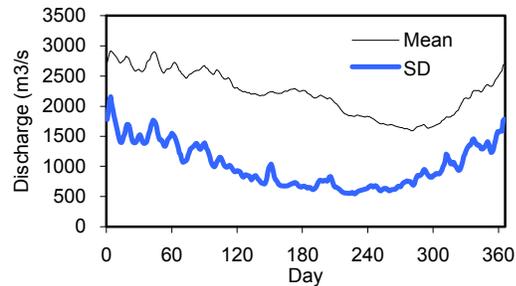
4. The case of streamflow processes

4.1 Data Used

The daily average discharge data of the upper Yellow River observed at Tangnaihai, which has a 133,650 km² drainage basin in the northeastern Tibet Plateau, and the daily discharge data of the Rhine River observed at Lobith, which has a drainage area is about 160,800 km², are used in the present study. The sizes of data are 45 years for the Yellow River and 96 years for the Rhine River. As shown in Figure 8, the streamflow of the Yellow River exhibits a much stronger seasonal variation than that of the Rhine River.



(a) Yellow River at Tangnaihai



(b) Rhine River at Lobith

Figure 8 Variation in daily mean and standard deviation of streamflow processes

4.2 Test for chaos in streamflow processes with correlation dimension method

4.2.1 Selection of delay time for streamflow processes

With the correlation dimension method for detecting chaos, the delay time τ should be determined first of all. The delay time τ is commonly selected by using the autocorrelation function (ACF) method where ACF first attains zeros or below a small value (e.g., 0.2 or 0.1), or the mutual information (MI) method (Fraser and Swinney, 1986) where the MI first attains a minimum.

Because of strong seasonality, for the Yellow River, ACF first attains zeros at the lag time of about 1/4 period,

namely, 91 days. The MI method gives similar estimates for τ to the ACF method, about approximately 1/4 annual period. We therefore select $\tau = 91$ for estimating correlation dimension for the streamflow series of the Yellow River. But for the Rhine River, the seasonality is not as obvious as that of the Yellow River. We cannot find the seasonal pattern in the ACF and MI of daily flow series of the Rhine River. If we determine the delay time according to the lags where ACF attains 0 or MI attains its minimum for the Rhine River, the lags would be about 200 days, which seems to be too large and would possibly make the successive elements of the state vectors in the embedded multi-dimensional state space almost

independent. Therefore we select the delay time equal to the lags before ACF attains 0.1, namely, $\tau = 92$.

4.2.2 Estimation of correlation dimension

In this study, the Theiler window w in Eq.(1) is set to be about half a year, namely, 182 days. The $\ln C(r)$ versus $\ln r$ plots of daily streamflow series of the two rivers are displayed in Figure 9, and the Takens-Theiler estimates (D_{TT}) of correlation dimension are displayed in Figure 10.

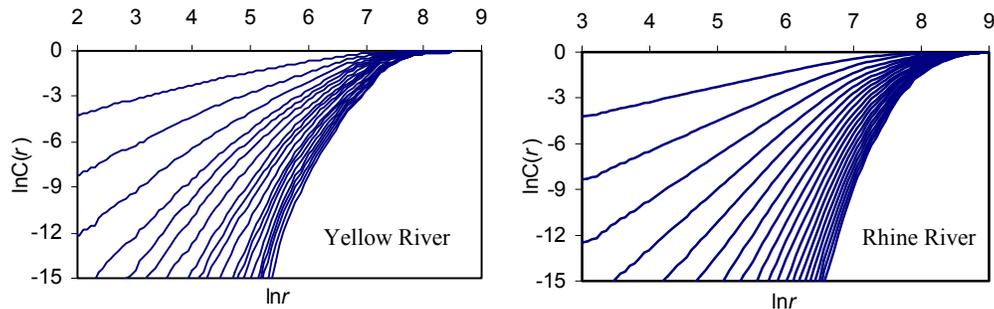


Figure 9 $\ln C(r)$ versus $\ln r$ plot for daily streamflow processes

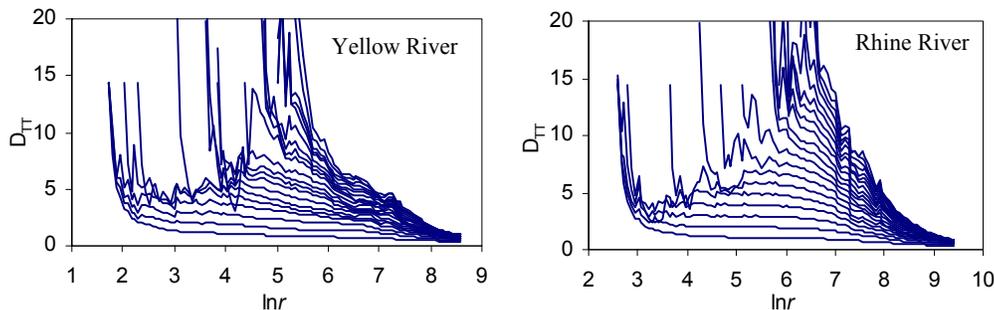


Figure 10 Takens-Theiler estimates of correlation dimension for daily streamflow processes

4.2 Discussion on the identifiability of chaotic behaviour in streamflow processes

With regard to an observed hydrologic series, its dynamics is inevitably contaminated by not only measurement noise, but also dynamical noise. For instance, a streamflow process, which is the major output of a watershed system, may be influenced by many factors, including external inputs (e.g., precipitation, temperature, solar radiation), other outputs (e.g., evaporation, transpiration), and various human interventions. These factors are generally composed of both deterministic components and stochastic components. Among all the factors, the precipitation is the dominant one which may disturb the streamflow process most significantly. In a flood event, it is very common that the flow generated by storm rainfall makes up over 50% of the total discharge. Assuming that base flow of a streamflow process is the noise-free time series, that 50% streamflow is generated by rainfall means 100% level of dynamical noise. Even if the streamflow process

We cannot find any obvious scaling region from Figure 10 for either Yellow River or Rhine River, whereas a clearly discernible scaling region is crucial to make a convincing and reliable estimate of correlation dimension (Kantz & Schreiber, 2003, pp 82-87). Because the presence of a finite correlation dimension from a time series is a must to infer the presence of chaos, therefore, we believe that the chaotic behaviour is not present or at least not identifiable here.

is chaotic, according to the experiments we made with the known chaotic systems, it is impossible to detect the chaotic characters with such intense disturbances.

One may argue that dynamical noise may be a higher dimensional part of the system dynamics. If viewing it in that way, then we may say, the streamflow process may be chaotic, but definitely not a low-dimensional chaotic process, because as a major output of watershed system, it is influenced by not only many dynamical inputs and other outputs of the watershed system, but also dynamical variations of many internal factors, such as the spatial-temporal variability of soil infiltration capacity. That means, if the streamflow process is chaotic, it would be a very high dimensional chaotic process, rather than a low-dimensional process as claimed by some researchers. Although it is possible that collective behavior of a huge number of external and internal degrees of freedom may lead to low-dimensional dynamics, it seems not the case for streamflow processes because we cannot observe finite correlation dimension in the streamflow processes we studied.

All in all, on one hand, due to high level of dynamic disturbances, it is not possible to identify the chaos in streamflow processes even if it exists; on the other hand, the existence of chaotic characteristics does not necessarily mean determinism, consequently, even if we found the presence of chaos in a streamflow process, we could not conclude that the streamflow process is deterministic. As pointed out by Schertzer et al. (2002), it is a questionable attempt to reduce complex systems to their low-dimensional caricatures, and there is no obvious reason that processes should be run by deterministic equations rather than by stochastic equations, since the former are merely particular cases of the latter.

5. Conclusions

Experiments with three well-known chaotic systems (i.e., Henon map, Ikeda map, discretized Mackey-Glass flow) show that, with low-level (e.g., 2% or less) Gaussian noise, especially low-level uniformly distributed noise, the chaotic attractor can still be well preserved and we can give basically correct estimate of correlation dimension. That indicates that even if we observed the existence of chaos in a time series, it does not necessarily mean determinism. A chaotic system with stochastic components behaves similar to a noise-free system when the stochastic disturbances are not strong. However, when the noise level is high (e.g., 10%), it is hard to correctly identify the chaotic system, let alone 100% level noise.

Correlation dimension method is applied to two daily streamflow series for identifying chaotic behaviour. It is found that there is no finite correlation dimension for both streamflow series of interest. We cannot conclude that there is no chaotic component exists in the streamflow processes, but at least the results of our study implies that the chaotic behaviour of streamflow processes cannot be identified clearly due to the existence of strong disturbance of various noises. On the other hand, because the existence of chaotic characteristics does not necessarily mean determinism, consequently, even if chaos exhibits in a streamflow process, we cannot conclude that the streamflow process is deterministic.

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