

Improved Dynamic Bounds In Monte Carlo Simulations

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In this paper, the concept of Dynamic Bounds (DB) is reviewed, the formula for prediction of its efficiency is described, and the method is improved. The improved dynamic bounds (IDB) are clarified. In this case, a polynomial function is assumed for the response of the system in the indistinct area according to the imposed constraints under the maximum uncertain conditions. Besides, by additional simulations, prior boundary uncertainties are recovered. Therefore, according to the required accuracy of the model and acceptable tolerance, number of simulations is determined. As a result, this method can be more efficiently applied in a multidimensional Monte Carlo simulations. This conclusion is demonstrated by comparing of results with the Classical Monte Carlo (CMC), dynamic bounds (DB), and improved dynamic bounds (IDB) for a model of impact water waves as an example.¹

Nomenclature

x_i	A random variable
\bar{x}	A vector of random variables
$G(\bar{x})$	Limit state function
x_{ij}	A realization from x_i
LSF	Limit State Function
s_{ut}	Set of upper thresholds
s_{lt}	Set of lower thresholds
ξ	Horizontal local axes
η	Vertical local axes

I. Introduction

Probabilistic approach provides a better understanding of failure mechanisms, occurrence probabilities, as well as consequences of failure in engineering works; however, to achieve these advantages a well defined model of the structure together with a robust reliability technique are needed. Nevertheless, the complex engineering problems with a complicated boundary conditions usually are modeled with the finite elements method providing an accurate but implicit limit state function (LSF) of behavior.² Therefore, taking the advantages of probabilistic techniques needs a smart way of coupling of advanced engineering models with reliability methods. This coupling has been the topic of many researches in the structural reliability field to

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get the benefits of not only the probabilistic approach but also of accurate models. In this paper, we are not going to discuss different approaches which are suggested for this application; nevertheless, the results of the method suggested in this paper is compared with the results of Classical Monte Carlo (CMC), dynamic bounds (DB), and improved dynamic bounds (IDB).

In fact, the purpose of this research is to provide a cheap and accurate way of this coupling by taking the advantages of every realization of preference function for the next step. A key assumption is that the model has monotonicity or anti-monotonicity and then the concept of dynamic thresholds is discussed. The Dynamic Bounds (DB) method is explained in Section II, and it is tried explain it in an easy and practical way. Finally, a comparison between the results of DB and other reliability techniques are presented and discussed. The results show that the DB is a robust technique and provides a high order of accuracy without doing an enormous number of calculations efforts for the problems with the monotonicity or anti-monotonicity regarding its variables.

II. Dynamic Bounds (DB)

A. Monotonicity

A function is called monotone with respect to a variable when increasing or decreasing of that variable causes increasing or decreasing of the outputs. Monotonicity is a property of a function or a system that implicitly presents additional assumptions. It is known for any true logical monotone system that it will continue to be true by increasing its variable, x_i , as far as Equation 3 or 4 holds. Therefore, it can respectively be called a monotonic increasing or monotonic decreasing function with respect to its variable or variables.

$$G(\bar{x}) = G(x_1, \dots, x_n). \quad (1)$$

$$h_i(x) = G(x_1, x_2, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n). \quad (2)$$

$$x_{m+1} \leq x_m \Rightarrow h_i(x_{m+1}) \leq h_i(x_m). \quad (3)$$

$$x_{m+1} \leq x_m \Rightarrow h_i(x_{m+1}) \geq h_i(x_m). \quad (4)$$

If a function like $G(\bar{x})$ is not monotonic with respect to all of its variables, it is possible to transform it to a completely monotone function. If $G(\bar{x})$ is a multidimensional function, its variables can be considered in three sets of variables which are the sets of monotonically increasing, decreasing and none monotonic one. Removing the last set, a monotonic function can extracted. This transformation is useful when the the variables of the third are not influential variables considering final results.

Monotonicity is a property that can normally be found in most of the engineering problems. The fact that a structure fails by increasing loads, or a structure gets more stability by increasing its strength presents a simple demonstration of the monotonic behavior. Therefore, assuming of a monotonic function doesn't usually limit the domain of application; and its advantages can be taken into account in different fields of engineering.

During the rest of this paper, we assume a monotonic behavior for the LSE, and the monotonically increasing or decreasing behavior can be logically interpreted by the reader. However, the presented equations are based on a monotonically increasing function.

B. Thresholds

The threshold concept is widely used in engineering language and usually determines the difference between levels. This concept divides responses of a model into several subsets with the common desired properties and provides a logical judgment possible. The factor of safety, F_s , is a famous representation of this concept. It defines a ratio of resistance parameters over driving forces, $F_s = Resistance/Force$. Generally one (or zero) is a threshold for the safety factor. In this case, it can determine whether the structure is stable or unstable by getting a value of F_s above or below one (or zero), respectively. The concept of threshold is interesting from the point of view of stability; if a model is above its thresholds and its strength parameters are being increased then the model would remain stable because of its monotonic behavior. A model above the threshold also remains stable by decreasing of the driving forces. In other words, the model remain stable by decreasing of the variable which the model has a monotonically decreasing behavior with respect to them.

However, the calculation of a threshold with an implicit limit state equation needs extra efforts. The thresholds are usually determined by trial and error methods, and the related algorithms for both explicit and implicit functions are trivial. This process is usually done by using a ratio for reduction of all variables. Therefore, we can approach to a threshold that determines the stability or failure of a model. Because we approach to that threshold we get two points which are called upper and lower thresholds: $x_{ut}^{(0)}$ and $x_{lt}^{(0)}$. In this notation ut stands for upper threshold, lt stands for the lower threshold, and (0 stands for the reduction of all variables with a same ratio. Transforming the axis to the level of $x_{ut}^{(0)}$, then we explore the effects of each variable. As a result of $G(\bar{x}) = G(x_1, \dots, x_n)$, there are n upper thresholds produced and called $x_{ut}^{(1)}, \dots, x_{ut}^{(n)}$. By considering the cross terms, a set of $x_{ut}^{(pq)}$ can be defined given the conditions that $p \neq q$ and $p \leq q$ especially when a high level of accuracy is required. To overcome this problem, we define an upper threshold and a lower threshold for a monotonically increasing function of $G(x)$ according to equations 5 and 6.

$$X_{ut} = \{x \in \mathfrak{R}^n : x \geq x_{ut} \longrightarrow G(\bar{x}) \geq 0\}. \quad (5)$$

$$X_{lt} = \{x \in \mathfrak{R}^n : x \geq x_{lt} \longrightarrow G(\bar{x}) \leq 0\}. \quad (6)$$

C. Dynamic bounds

We define a stable set which stores every set of variables mapped into positive values of the limit state equation, $G(\bar{x})$ as presented in Equation 7. With the same concept, a set of variables in \mathfrak{R}^n is defined which accumulates every set of variables mapped to the negative values of limit state equation as presented in Equation 8.

$$S = \{x \in \mathfrak{R}^n : G(\bar{x}) \geq 0\}. \quad (7)$$

$$U = \{x \in \mathfrak{R}^n : G(\bar{x}) < 0\}. \quad (8)$$

Having monotonicity in the limit state function can help us to define two bounds which are called as upper and lower bounds as a set of upper and lower thresholds. As a result of these two bounds, the whole rang of the LSF ($G(\bar{x})$), is divided into three parts which are the safe part where $G(\bar{x}) \geq 0$, the failure part where $G(\bar{x}) < 0$, and the part in between which is defined as an unqualified part. It is called the unqualified part because it is a region of combination of safe and failure. In other words, it means that in order to get the value of the LSF in the this part, unqualified part, we need to evaluate the LSF over this range. This concept is in detail in the following reference.¹

As a matter of fact, any combination of the variables which are bigger than the upper threshold, x_{ut} , for a monotone relation and lower than x_{lt} will mapped to the safe area by the preference function as presented in Equation 9.

$$\forall_{i=1}^n \begin{cases} \bar{x}_i \geq x_{iut}^-, & \text{Monotonically increasing LSF} \\ \bar{x}_i \leq x_{iut}^-, & \text{A.Monotonically decreasing LSF} \end{cases} \Rightarrow G(\bar{x}) \geq 0. \quad (9)$$

In the other, hand any combination of the variables which are smaller than the lower threshold, x_{lt} , for a monotonically increasing LSF and bigger than , x_{lt} , for the a monotonically decreasing LSF will be mapped to the failure area by the LSF as presented in Equation 10.

$$\forall_{i=1}^n \begin{cases} \bar{x}_i \leq x_{ilt}^-, & \text{Monotonically increasing LSF} \\ \bar{x}_i \geq x_{ilt}^-, & \text{Monotonically decreasing LSF} \end{cases} \Rightarrow G(\bar{x}) < 0 \quad (10)$$

There were two limit bounds presented by Equations 9 and 10 in the domain of limit state function, $G(\bar{x})$, which can be probabilistically related to the range of LSF. Then, it can be concluded that those limit bounds can be dynamic; in fact, those limit bounds not only are not limited to two bounds but also they can update each other by every realization.

III. The most uncertain responses

The range of a limit state function which is divided into the stable, unstable, and unqualified region encourages us implementing the dynamic bounds technique. The stability of the model can be predicted in the stable or unstable region. But, in the unqualified region, there is no explicit data. Nevertheless, there is some implicit information in this region which can help us to define the upper and lower bounds for the response of the model. This extra information can be the monotonic behavior of the model, the order of the response, and the first, second, or higher order derivatives.

A. One dimensional response

1. first, second, ... and n^{th} order response

We assume that points A and B in Figure 1 are two points from the lower and upper bounds of dynamic limit bounds of the LSF. Therefore, we have explicit information about the area that its coordinates are smaller than point A and higher than point B, but there is no information for the area in between. In other words, $G(\bar{x} \leq [A]) < 0$ and $G(\bar{x} \geq [B]) \geq 0$. Having this knowledge, if we assume that the response of our limit state function is linear, we can directly draw a line from A to B and predict the behavior of our model in the unqualified region. As a result, any randomly generated number can be judged whether it is in stable or unstable area. However, we do not have this information and even we do not know the behavior of the model whether it is second order, third order, or higher order polynomial; nevertheless, we assume that its response can be modeled by a polynomial.

Let's first assign a new orthogonal coordinates on the point A, located on the lower response boundary; then, in a two dimensional space we will have, $A : [\xi = 0, \eta = 0]$ and $B : [\xi = m, \eta = n]$. Also, there is another source of information which can help us for prediction of the behavior of the model in this area which is the monotonicity (increasing or decreasing) of the limit state function regarding the input variables which implies that $\frac{d(G(\bar{x}))}{dx_1} \geq 0$ and $\frac{d(G(\bar{x}))}{dx_2} \geq 0$. Now, assuming the response of the model is second order continuous and smooth, we are looking for the most uncertain response of the LSF. Equation 11 presents a second order polynomial considered as a possible response of the LSF. This function should fit in to the points A and B. Meanwhile, its integral should be minimized in order to get maximum uncertainty. Equation 14 presents the most uncertain situation (or a lower bound of response surface). This equation is a result of a minimization process to get the response of the LSF in the most uncertain conditions.

$$\eta = a\xi^2 + b\xi + c \quad (11)$$

$$A : [\xi = 0, \eta = 0] \implies c = 0$$

$$B : [\xi = m, \eta = n] \implies b = \frac{n - am^2}{m}$$

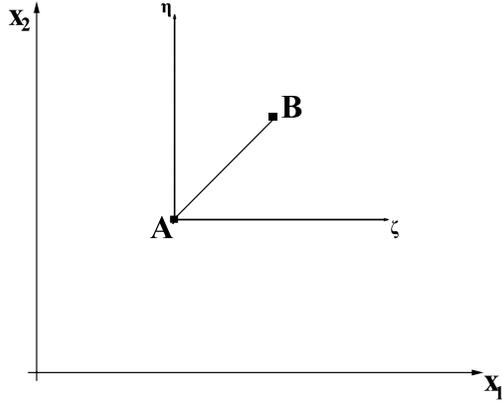
$$I = \int_0^m \eta d\xi = \frac{1}{3}am^3 + \frac{1}{2}\left(\frac{n}{m} - am\right)m^2 \quad (12)$$

$$\frac{\partial I}{\partial m} = 0 \implies a = \frac{n}{m^2}, b = 0, c = 0 \quad (13)$$

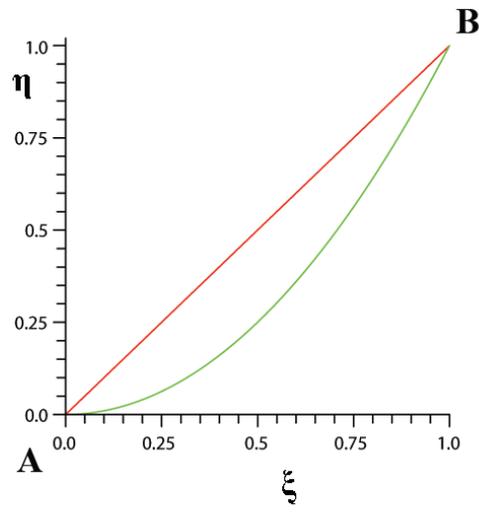
$$\implies \eta = \frac{n}{m^2}\xi^2 \quad (14)$$

Figure 1(b) presents the conditions in which the deviation of the response is maximized assuming a second order polynomial function as a response of the limit state function. Besides, Figure 2(a) presents its upper and lower bound. As a result of these assumptions the lower and upper bounds can be extended to the indistinct area; this concept reduces the Monte Carlo simulations efforts and saves a lot of costs for the probabilistic finite elements analysis with a limited number of variables.

Further investigations about the higher order response of polynomials can be obtained. For illustration, Equation 15 presents the maximum uncertainty bound for the third order response. Figure 2(b) presents the most uncertain response of the LSF, assuming a second, third, or higher order response.



(a) Linear response



(b) Lower bound of the second order response

Figure 1. (a) A and B are two points located on the lower and upper limit bounds, respectively in a two dimensional space of $\bar{x} = \{x_1, x_2\}$. (b) The second order response of the limit state function in the most uncertain condition.

$$\begin{aligned}\eta &= a\xi^3 + b\xi^2 + c\xi + d \\ I &= \int_0^m \eta d\xi = \frac{1}{4}am^4 + \frac{1}{3}bm^3 + \frac{1}{2}cm^2 \\ \frac{\partial I}{\partial m} &= 0 \implies a = \frac{n}{m^3}, b = 0, c = 0, d = 0\end{aligned}$$

$$\implies \eta = \frac{n}{m^3}\xi^3 \quad (15)$$

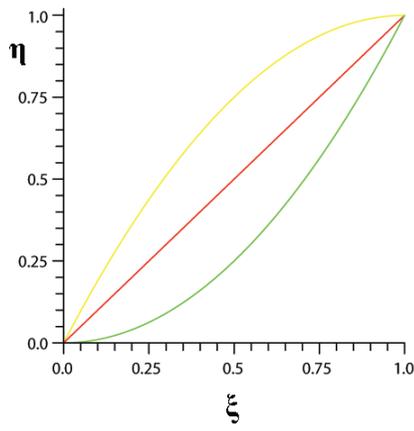
2. Implementing the information of the first derivative

Assuming we have information of the derivatives at the start and end point of the unqualified region, points A and B, a fourth order polynomial response of the function can be assumed in the following form:

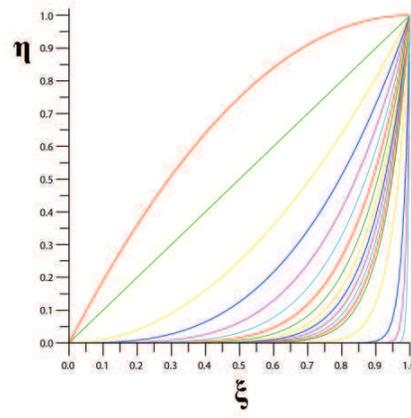
$$\begin{aligned}\eta &= a\xi^4 + b\xi^3 + c\xi^2 + d\xi + e \\ I &= \int_0^m \eta d\xi = \frac{1}{5}am^5 + \frac{1}{4}bm^4 + \frac{1}{3}cm^3 + \frac{1}{2}dm^2 \\ \frac{\partial I}{\partial m} &= 0 \implies a = -\frac{mp - mq + 3n}{m^4}, b = -\frac{-4n - 3mp + mq}{m^3}, c = -3\frac{p}{m}, d = p, e = 0\end{aligned} \quad (16)$$

Then we apply our information about points A and B which means that first those points must be fitted to the function and secondly we assume that the derivation of the function at those points are known and called p and q, respectively.

$$\implies \eta = -\frac{(mp - mq + 3n)x^4}{m^4} - \frac{(-4n - 3mp + mq)x^3}{m^3} - 3\frac{px^2}{m} + px \quad (17)$$



(a) Lower and upper bounds for second order response



(b) Lower bounds for higher order responses

Figure 2. (a) The lower and upper bounds of the response of the limit state function assuming second order response. (b) Lower bounds for the higher order of the response, the very last line in the right presents order of 150.

To clarify the effect of applying the derivation information, a comparison is provided in Figure 3. In this case, Figure 3(a) presents the fourth order upper and lower bounds for the response of the LSF, and Figure 3(b) presents the bounds including the derivatives information. In other words, implementing of derivatives information can effectively reduce the uncertainty of the response of the LSF.

3. Applying extra information

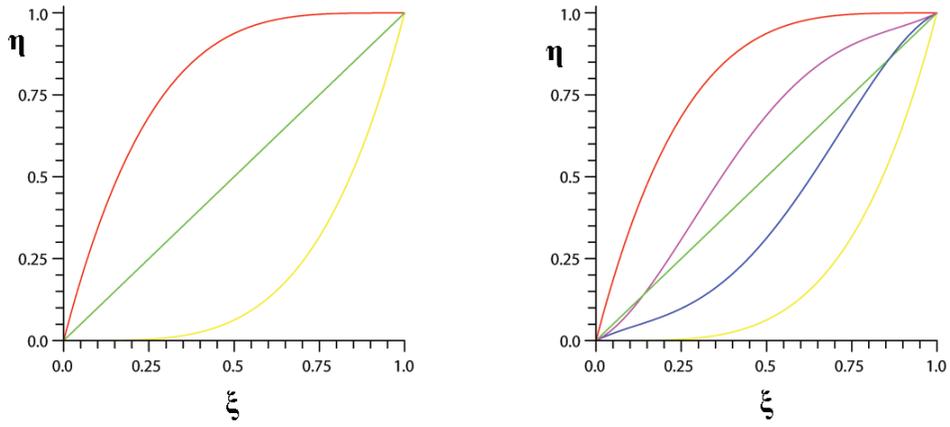
There is still other information which can help us to optimize this approach. This information can be about the second or higher derivatives. For instance, the second order derivative can tell us about the smoothness of the function and the fact that whether we can expect any jitter or not. Moreover, the other points which are in the stable or unstable region bring extra information which can be applied by Bayesian statistics. Implementing of these cases need more research in this field.

B. Two dimensional functions

In a two dimensional limit state equation, the same approximation which was presented can be easily applied assuming triangular approximation. Every closest three points in this method is constructed to a triangular shape. Then, one or two of the joint will be in stable (or safe) and the other(s) in unstable (or unsafe) area. Therefore, the suggested method again can be easily applied. As a result, it divides the triangle into the two parts of stable and unstable. See Figure 4 for an illustration.

IV. Numerical example

One of the important researches in hydraulic engineering focuses on the impact of water waves on walls and other coastal structures, which create velocities and pressure with magnitude much larger than those associated with the propagation of ordinary waves under gravity. The impact of a breaking wave can generate pressures of up to 1000 KN/m^2 which is equal to 100 meters of water head. Although many coastal structures are damaged by breaking waves, very little is known about the mechanism of impacts. Insight into the wave impacts has been gained by investigating the role of entrained and trapped air in wave impacts. In this case, a simplified model of maximum pressure of ocean waves on the coastal structures is presented by Equation 18.



(a) Lower and upper bounds for the fourth order response

(b) Lower and upper bounds for the fourth order response including the derivatives information

Figure 3. Figure (a) presents two limit bounds for the responses of the limit state function assuming the response is fourth order. Figure (b) presents the same bounds applying the information of the first derivatives in points A and B.

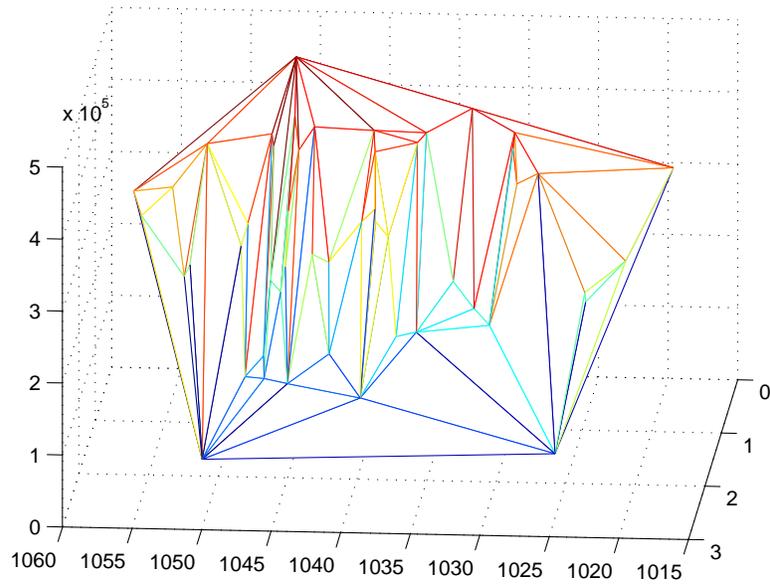


Figure 4. The Delaunay triangulation of a two dimensional limit state equation.

$$P_{max} = C \times \frac{\rho \times k \times u^2}{d}. \quad (18)$$

Where the ρ is density of water, k is the length of hypothetical piston, d is the thickness of air cushion, u is the horizontal velocity of the advancing wave, and C is a constant parameter and equal to $2.7 \text{ s}^2/m$. Having this knowledge, we are willing to find the probability of the event, when the maximum impact pressure exceeds $5 * 10^5 \text{ N/m}^2$ for a specific case. The one dimensional limit state function (LSF) can be defined by Equation 19, where the velocity parameter is assumed to be normally distributed as $N(1.5, 0.45)$.

$$G(u) = 500000 - 98280 \times u^2. \quad (19)$$

It is clear that this limit state function is monotonically decreasing regarding its variable (u), and it is a second order function. This assumption may be obtained by experience or knowledge about the limit state equation when it is implicit, otherwise it can be easily produced. Having this knowledge, we compare the results of Monte Carlo (MC), dynamic bounds (DB), and improved dynamic bounds (IDB) in Table 1. To provide this comparison we accept Equation 14 as our prior knowledge. In other words, the information of derivatives still has not been applied to this comparison. It shows that the IDB is robust and can be used with a good efficiency and a high accuracy.

A comparison between DB and the other level III methods							
MC	N	1E+3	1E+4	1E+5	1E+6	1E+7	1E+8
	P_f	0.0487	0.0478	0.0462	0.0466	0.0466	0.0466
	$V(P_f)$	14.5E-2	4.3E-2	1.44E-2	0.45E-2	0.14E-2	0.045E-2
DB	N	9	16	22	27	33	36
	P_f	0.045	0.0497	0.0462	0.0464	0.0467	0.0466
	$V(P_f)$	14.5E-2	4.3E-2	1.44E-2	0.45E-2	0.14E-2	0.045E-2
IDB	N	4	9	15	16	18	22
	P_f	0.04	0.047	0.0462	0.0466	0.0466	0.0466
	$V(P_f)$	14.5E-2	4.3E-2	1.44E-2	0.45E-2	0.14E-2	0.045E-2

Table 1. A comparison between different level III methods: Monte Carlo (MC), dynamic bounds (DB), Improved dynamic bounds (IDB). P_f stands for the probability of failure and $V(P_f)$ presents its variance. Bold numbers present the values of acceptable engineering level.

V. Conclusions

Dynamic bounds (DB) is a technique suggested for the reliability analysis of engineering problems which have monotonic behavior with a limited number of influential variables. The suggested method is fast and robust; it can be coupled with the complicated limit state functions like finite elements codes to make the probabilistic approaches more applicable in practice. Moreover, this method can be coupled with the importance sampling technique to reduce the number of calculations more efficiently and speed up the whole procedure. This method is efficient when highly accurate results and the consequent enormous number of calculations are needed, the cost of simulations can be highly reduced by DB. In addition, the DB is improved by defining a lower bound for the response surface in indistinct part of the Range of LSF on the base of minimization process. The improved dynamic bounds (IDB) presents more efficiency and accuracy. The numerical results and the effects of the lower bounds on indistinct part of LSF is presented in this paper for a one dimensional case, and the method is described for higher dimensions.

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