Continuous-Time Predictive-Maintenance Scheduling for a Deteriorating System

Antoine Grall, Laurence Dieulle, Christophe Bérenguer, and Michel Roussignol

***Abstract*—A predictive-maintenance structure for a gradually deteriorating single-unit system (continuous time/continuous state) is presented in this paper. The proposed decision model enables optimal inspection and replacement decision in order to balance the cost engaged by failure and unavailability on an infinite horizon. Two maintenance decision variables are considered: the preventive replacement threshold and the inspection schedule based on the system state. In order to assess the performance of the proposed maintenance structure, a mathematical model for the maintained system cost is developed using regenerative and semi-regenerative processes theory. Numerical experiments show that the s-expected maintenance cost rate on an infinite horizon can be minimized by a joint optimization of the replacement threshold and the *a periodic* inspection times.**

The proposed maintenance structure performs better than classical preventive maintenance policies which can be treated as particular cases. Using the proposed maintenance structure, a well-adapted strategy can automatically be selected for the maintenance decision-maker depending on the characteristics of the wear process and on the different unit costs. Even limit cases can be reached: for example, in the case of expensive inspection and costly preventive replacement, the optimal policy becomes close to a systematic periodic replacement policy. Most of the classical maintenance strategies (periodic inspection/replacement policy, systematic periodic replacement, corrective policy) can be emulated by adopting some specific inspection scheduling rules and replacement thresholds. In a more general way, the proposed maintenance structure shows its adaptability to different possible characteristics of the maintained single-unit system.

***Index Terms*—Inspection, maintenance cost optimization, predictive maintenance, renewal process, replacement.**

ACRONYMS[1]

CM	corrective maintenance
CR	corrective replacement
MC	Markov chain
PM	preventive maintenance
PR	preventive replacement
rv	random variable.

NOTATION

α, β	[shape, scale] parameter
C_d	cost of "inactivity of the system" per unit of time (all costs are in arbitrary units)
C_i	cost per inspection
C_p	cost per PR
C_c	cost per CR
$C(t)$	cumulative maintenance cost at time t
$d(t)$	time passed in a failed state in $[0, t]$
$\delta_0(\cdot)$	the Dirac mass function on 0
ΔT_i	inter-inspection time between inspections $\#i$ and $\#(i+1)$
E_π	s-expectation with respect to the measure π
EC_∞	long run s-expected cost rate
$f_{\alpha,\beta}$	Gamma pdf of α, β
$\overline{F}_{\alpha,\beta}$	Gamma Sf of α, β
$\Gamma(\cdot)$	Gamma Cdf
$\mathrm{I}_{\{A\}}(x)$	1 if $x \in A$, and 0 otherwise: indicator function
L	failure level (of system state) which corresponds to the CM threshold
$m(\cdot)$	inter-inspection time: inspection scheduling function
M	PR threshold (of system state)
$N_i(t)$	random number of inspections in $[0, t]$
$N_p(t)$	random number of PR in $[0, t]$
$N_c(t)$	random number of CR in $[0, t]$
$\Pr\{dy \mid x\}$	conditional pdf of the MC, $(Y_n)_{n\in\mathbb{N}}$
π	invariant probability measure of $(Y_n)_{n\in\mathbb{N}}$
$(S_i)_{i\in\mathbb{N}}$	replacement times; S_1: first replacement time
$(T_i)_{i\in\mathbb{N}}$	inspection times; T_1: first inspection time
$(X_t)_{t\geq 0}$	process describing the maintained system state; X_t increases with the system deterioration
$(\tilde{X}_t)_{t\geq 0}$	process describing the deteriorating (unmaintained) system state
$(Y_n)_{n\in\mathbb{N}}$	MC describing the system state at inspection times ($Y_n = X_{T_n}$).

I. INTRODUCTION

FOR A gradually deteriorating system, the most advanced preventive-maintenance (PM) strategies rely on the monitoring of a measurable system diagnostic parameter ("system state") and base the maintenance decisions on the level of deterioration of the system. Generally, such a condition-based PM policy is more efficient than a PM policy based only on the system age and on the knowledge of the statistical information on its lifetime [11], [15], [16]. However, as stressed in [11, chapter 5], the price for this higher efficiency is the requirement of a mathematical model for the stochastic deterioration process of the maintained system. There is an economic necessity to quantify and model the deterioration/maintenance

Manuscript received June 3, 2001; revised October 15, 2001.

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Publisher Item Identifier S 0018-9529(02)05622-1.

[1]The singular and plural of an acronym are always spelled the same.

process, because this model can be used by the maintenance decision-maker as a tool to optimize the maintenance decisions and to minimize the total maintenance cost of the system. Usually, the task of deriving such a mathematical model is more complicated than just statistically describing the binary transition from a "good state" to a "failed state."

This paper precisely examines the problem of developing a mathematical maintenance cost model for determining the optimal inspection schedule and the optimal replacement threshold for a 1-unit deteriorating system submitted to a condition-based maintenance policy. It considers a system which undergoes a continuous gradual random deterioration and is subject to nonobvious failures when its state exceeds a level L (the system is not known to have failed unless that is revealed by inspection). The system state is monitored through perfect inspections. When a fault is detected upon inspection, a corrective maintenance (CM) operation replaces the failed system by a new s-identical one. However, both the fault and the CM operation can be very expensive because of, e.g., lower efficiency, production losses, security hazard, and unplanned intervention. In this context a condition-based (or predictive) PM policy can be profitable in order to avoid failure occurrence at the lowest cost and to improve the availability and safety of the maintained system. When the monitored state exceeds a pre-set preventive replacement (PR) threshold M, the device is considered as worn-out and a PR is triggered. A PR or CR can be either a true physical replacement or an overhaul or repair such that the unit is good-as-new after the repair. A replacement is assumed to take negligible time.

The choice of the inter-inspection times and the value M of the PR threshold obviously influence the economic performance of the maintenance policy. For example, in condition-based maintenance practice, the monitoring inspections are usually carried out at fixed-length intervals. However, it might not always be worthwhile to inspect systematically the system, especially if the inspection procedure is costly. The use of a maintenance cost evaluation model is therefore of considerable interest to optimize the maintenance policy. A maintenance cost model is a mathematical model that

- quantifies costs and effects (benefits) of a maintenance strategy; and
- finds the optimum balance between them.

The inspection dates (or the inspection scheduling procedure) and the PR threshold M are the main decision variables for this maintenance problem. Condition-based maintenance modeling has been addressed in several papers [2], [3], [19], [28], but this paper proposes 2 novel developments. Contrary to previous work in which either the PR threshold or the inspection interval is not a decision variable but rather a given number, this paper considers both the PR threshold M and the inspection schedule as decision variables which have to be jointly optimized in order to minimize the total long-term cost of the maintained device. Moreover, this paper does not assume that the system is regularly inspected; irregular inspection dates are allowed: the next inspection date depends on the system state revealed by the current inspection. The maintenance structure in this paper can be considered as a continuous-time extension of the multithreshold maintenance strategy in [12].

The aim of this paper is twofold:

1) Propose a parametric structure for implementing a condition-based maintenance policy; the associated mathematical cost model is derived for a particular deterioration process.
2) Show that the long run s-expected maintenance cost rate can be minimized by an appropriate joint choice of the two considered decision variables (the parameters of the structure).

Section II describes the system characteristics. Section III performs a probabilistic analysis of the maintained system state. Section IV deals with the optimization of the maintenance policy; results from numerical experiments illustrate the behavior of the proposed maintenance policy.

II. System Description and Assumptions

A. Stochastic Deterioration Process

1) Consider a single-unit deteriorating system consisting of 1 component or 1 group of associated (from the maintenance point of view) components. The system-failure behavior might be described by, e.g., a damage accumulation model (for a mechanical system), or the evolution of a defective product rate (for a production line), or a corrosion/erosion level (for a structure). The system-state at time t can be summarized by a random aging variable X_t. In the absence of repair or replacement actions, X_t is an increasing stochastic process, with $X_0 = 0$. The system fails when the aging variable is greater than a fixed L, characteristic of the considered system. For multi-unit systems, this approach can be applied, using a well-chosen 1-D "health index" of the system instead of its complete multidimensional state [10], [11].

2) The deterioration process between two maintenance operations $(\tilde{X}_t)_{t\geq 0}$ is time-homogeneous, and the deterioration between t and u $(t < u)$ is s-independent of the deterioration before t. Moreover, the deterioration process is assumed to evolve on each time interval by means of an infinity of jumps [9]. This leads to the family of positive, increasing jump processes with s-independent stationary increments. The Gamma process belongs to this family and has been used to describe the deterioration and erosion of many civil engineering structures, e.g., dikes, and other structural components [15], [16], [20]–[23], [25]–[27], [29]. It is also a reasonable extension of a deterioration process with exponential jumps [12]. Another interest of this process is the existence of an explicit pdf which permits feasible mathematical developments. The Gamma process is parameterized by α, β which can be estimated from the deterioration data. This paper assumes that $(\tilde{X}_t)_{t\geq 0}$ is a Gamma process and that for all $0 \leq s < t$, the rv, $\tilde{X}_t - \tilde{X}_s$, (the increments of $\tilde{X}_t$ between s and t), has a Gamma pdf with shape-parameter $\alpha \cdot (t-s)$ and scale-parameter β:

$$f_{\alpha\cdot(t-s),\,\beta}(x) = \frac{1}{\Gamma(\alpha\cdot(t-s))} \cdot \frac{1}{\beta^{\alpha\cdot(t-s)}} \cdot x^{\alpha\cdot(t-s)-1} \cdot \exp\left(-\frac{x}{\beta}\right) \cdot \mathrm{I}_{\{x\geq 0\}}. \quad (1)$$

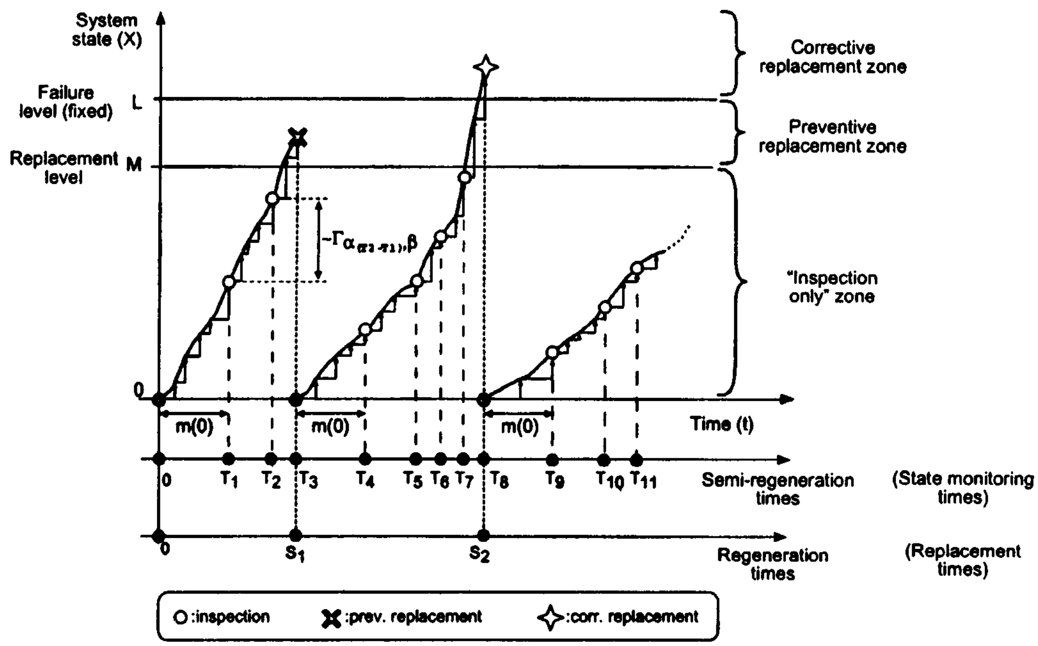

Fig. 1. Schematic evolution of the maintained system state.

The average deterioration speed-rate is $\alpha{\cdot}\beta$; its variance is $\alpha{\cdot}\beta^2$. The choice of α and β allows one to model various deterioration behaviors from almost-deterministic to very-chaotic.

As shown in [24], [25], the deterioration model based on Gamma stochastic processes used in this paper follows directly from minimal hypotheses (stationary, s-independent, nonnegative, and isotropic increments) that can be made on the real deterioration process when detailed deterioration data are lacking.

3) The system is not monitored continuously, but its state can be observed by inspection. Each inspection reveals instantaneously, and without error, the true state of the system.

4) A failure is detected only by inspection. Hence, if the system fails, it remains failed until the next inspection. With this assumption, active alarmed systems are not considered, and the focus is on passive or inaccessible systems or structures whose failures are not obvious to the user and can be difficult to characterize and identify. The system can be declared "failed" as soon as an important defect or deterioration is present (even though the system is still working), a long time before the occurrence of a complete and abrupt failure: a pending failure is considered an actual failure of the system. For example, consider a structure (e.g., dam, dike, bridge, or pipeline) which has to fulfill stipulated safety requirements in order to be in a working condition. Failure is defined by comparing the structure resistance to a given stress (e.g., hydraulic load, traffic weight, or gas pressure). The structure is failed if its resistance drops below a minimum stress threshold because of the deterioration. The resistance of the structure is usually correlated with a physical parameter (e.g., height and cross section of a dike, length of a crack, or depth of a corrosion layer) which can be measured only by inspection. The structure can be in a pending failed state, and this failure (too low safety threshold) can be detected only upon inspection. Underwater or underground pipes with minor (but possibly hazardous) leaks are also examples of systems with nonobvious failures.

B. Maintenance Policy Structure

The maintenance policy is driven by the knowledge of system state after inspection. At every inspection time, $(T_n)_{n\in\mathbb{N}}$, ($T_0 =$ 0), two decisions have to be made. :

1) Determine whether the system should be replaced preventively or correctively, or whether the system should be left as is.
2) Determine the time to the next inspection.

The following maintenance decision frame is adopted (because the maintenance actions are assumed to be instantaneous, T_n^- refers to the time just before the maintenance date), see Fig. 1.

a) If $X_{T_n^-} \geq L$ (system is failed), then a CR is performed and a cost C_c is incurred. An additional cost is incurred by the time $d(t)$ elapsed in the failed state at a cost rate C_d. After the maintenance operation, the system is as good as new ($X_{T_n} = 0$).
b) If $M \leq X_{T_n^-} < L$ (system is still functioning, but too badly deteriorated) then a PR is performed and a cost C_p is incurred ($C_p \leq C_c$). After the maintenance operation, the system is "good as new" ($X_{T_n} = 0$).
c) If $X_{T_n^-} < M$ then the system is left unchanged ($X_{T_n} = X_{T_n^-}$).
d) In all these cases, ΔT_n is chosen by:

$$\Delta T_n = m(X_{T_n}) \tag{2}$$

where $m(\cdot)$ is a decreasing function from $[0, M)$ to $[m_{\min}, m_{\max}]$ with $m_{\min} = m(M) > 0$ and $m_{\max} = m(0) > m_{\min} > 0$; see Fig. 2(a). The time of the next scheduled inspection is then:

$$T_{n+1} = T_n + \Delta T_n. \tag{3}$$

The increments between two inspection-times are always greater than $m_{\min} > 0$; consequently, the sequence of inspec-

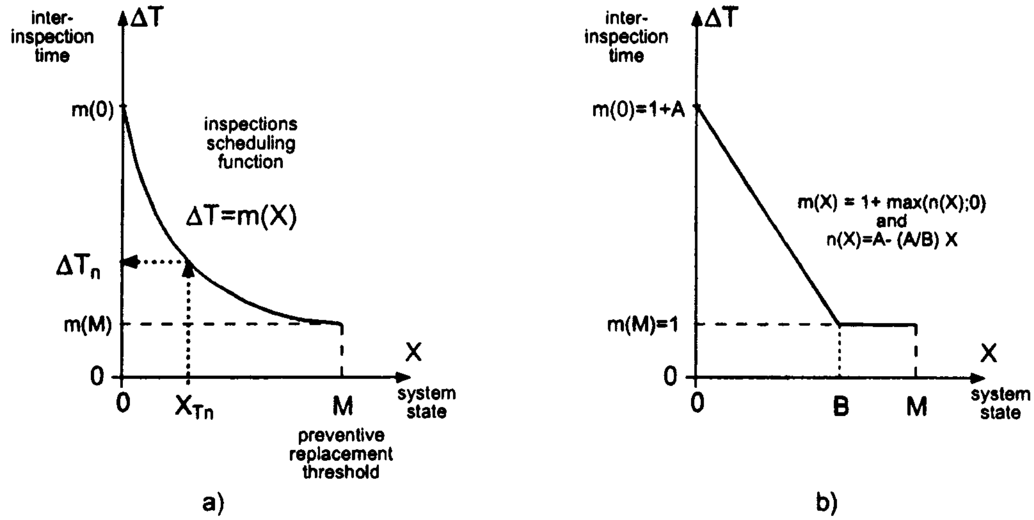

Fig. 2. Inspection scheduling function $\Delta T = m(x)$: (a) General case; (b) choice (linear) with numerical examples.

tion times $(T_i)_{i \in \mathbb{N}}$ is strictly increasing and the possibility of an infinite number of inspections occurring on a finite time interval is avoided. The $m_{\max}$ represents the first inter-inspection time for a system in the initial new state. The assumed shape of $m(\cdot)$ is plausible, because it results in more inspections when it shows a poor system state. For every inspection, a cost C_i is incurred.

For a continuously deteriorating system, the proposed deterioration/maintenance model, although described in mathematical terms, can be applied in practice because the maintenance decisions are based only on the amounts of deterioration which can be measured on the system, and not on abstract quantities like lifetime-distribution parameters or transition probabilities in a finite-state (Markov) model. Moreover, the modeling of the continuous deterioration process by a discrete finite state (Markov) process would require artificially-splitting the system-state range into several discrete states and to estimate a 1-step transition probability matrix from the deterioration data, which can be difficult in practice because of the lack of detailed and reliable deterioration data [7], [13]. A main advantage of this approach is thus to retain the intrinsic continuous nature of the system.

Within this maintenance structure, the 2 decision-variables to be optimized in order to minimize the maintenance cost are the PR threshold M and the "inspection scheduling function," *viz*, the $m(\cdot)$. For this model, the optimization procedure aims at finding a couple (M, m) such that, on an infinite horizon, the cost of inspections and PR balances the cost caused by failures and unavailability. A mathematical model of this maintenance policy and its associated cost is developed in Section III.

C. Examples of 1-Unit Deteriorating Systems Subject to Perfect Repair

Several physical systems (or structures) illustrate the notion of a "1-unit system subject to perfect repair" used throughout this paper:

a) Hydraulic structures for coastal line defense (e.g., dikes or breakwaters) subject to erosion and longshore transport of stones under the action of waves and wind [22], [24]–[26];
b) Cylinder of a swing bridge subject to mechanical damage [27];
c) Induction furnace for copper scrap processing, subject to internal erosion by melted copper [4];
d) Concrete bridge structures subject to development of cracks and to corrosion of steel reinforcement [18].

All these systems gradually deteriorate over their service life, mainly under environmental effects; and Gamma processes have been used to model the stochastic deterioration process in [24], [26], [27]. It is presumed that they experience nonobvious (or pending) failures when they reach a too-low safety level. These failures can be detected only when, considering the accumulated deterioration reported by an inspection, it can be shown that their resistance has dropped below the minimum allowed stress threshold.

These systems are subject to perfect repair or replacement in the sense that they can be considered to be "pulled down" into a good-as-new state after this maintenance action. However, the complete preventive renewal (or PR) for a state-level between M and L and the corrective renewal (or CR) for a state-level beyond L do not necessarily correspond to the same operation in practice. Because it is performed on a more deteriorated system, CM is generally more complex and, hence, more expensive than PM ($C_c > C_p$). Moreover, the cost C_c might also include a part of the cost of the consequences of the system failure on its environment (e.g., breach in a dike, deterioration of the furnace surrounding equipments in case of a leakage of melted copper, hazard or safety costs in the case of bridge structures).

D. Cost-Based Criterion for Maintenance Performance Evaluation

The objective cost function to minimize is the long run s-expected cost rate EC_∞. The cumulative maintenance cost is:

$$C(t) \equiv C_i \cdot N_i(t) + C_p \cdot N_p(t) + C_c \cdot N_c(t) + C_d \cdot d(t). \quad (4)$$

Note that the cost function $C(t)$ depends on M and $m(\cdot)$ through $N_i(t)$, $N_p(t)$, $N_c(t)$, and $d(t)$. Thus

$$\begin{aligned}\mathrm{EC}_\infty &= \lim_{t\to\infty}\left[\frac{\mathrm{E}_0[C(t)]}{t}\right]\\ &= C_i \cdot \lim_{t\to\infty}\left[\frac{\mathrm{E}_0[N_i(t)]}{t}\right] + C_p \cdot \lim_{t\to\infty}\left[\frac{\mathrm{E}_0[N_p(t)]}{t}\right]\\ &\quad + C_c \cdot \lim_{t\to\infty}\left[\frac{\mathrm{E}_0[N_c(t)]}{t}\right] + C_d \cdot \lim_{t\to\infty}\left[\frac{\mathrm{E}_0[d(t)]}{t}\right].\end{aligned} \tag{5}$$

The problem now is to determine the 4 limits in (5). Because all these quantities are a function of $(X_s)_{s\geq 0}$ before t, an expression for the limit of the time average of a function of $(X_s)_{s\geq 0}$ is required. Using elementary renewal theory [1], it is well known that the limit at infinity in (5) can be changed into a ratio of s-expectations on a single renewal cycle:

$$\mathrm{EC}_\infty = \lim_{t\to\infty}\left[\frac{\mathrm{E}_0[C(t)]}{t}\right] = \frac{\mathrm{E}_0[C(S_1)]}{\mathrm{E}_0[S_1]}. \tag{6}$$

However, in the present case, the study of the discrete time process $(X_{S_n})_{n\in\mathbb{N}}$ describing the system state on a renewal-cycle is tedious, and it is more interesting to take advantage of the semi-regenerative properties of $(X_t)_{t\geq 0}$ and the associated Markov chain $(Y_n)_{n\in\mathbb{N}}$.

III. Study of the Maintained System Evolution at Steady State

Evaluation of the maintenance cost—and hence optimization of the maintenance policy—requires the derivation of probabilistic characteristics of the system evolution under the maintenance policy. Several useful regenerative or semi-regenerative properties of $(X_t)_{t\geq 0}$ describing the evolution of the maintained system state can be used to compute efficiently the maintenance cost generated by the proposed policy.

A. (Semi-)Regenerative Properties of the Maintained System-State Process $(X_t)_{t\geq 0}$

1) After each CR or PR, the system is returned to $X_0 = 0$, and its evolution does not depend on the past events. The process $(X_t)_{t\geq 0}$ describing the evolution of the system subject to the previous maintenance actions is then a regenerative process with regeneration times equal to the S_n; see Fig. 1.

2) After each inspection, the evolution of the system depends only on the known system state at this date. The process $(X_t)_{t\geq 0}$ is then a semi-regenerative process with semi-regeneration times equal to the T_n; see Fig. 1. The discrete-time random process describing the system state at each inspection time $(Y_n = X_{T_n})_{n\in\mathbb{N}}$ is a Markov chain (MC) with continuous state space $[0, M)$. Its transition pdf can be written as a combination of a pdf and a Dirac mass function, indicating that after an inspection the system can be restored to the new state with a nonzero probability [14, chapter 3]:

$$\begin{aligned}\Pr\{dy|x\} &= \overline{F}_{\alpha\cdot m(x),\,\beta}(M-x)\delta_0(dy)\\ &\quad + f_{\alpha\cdot m(x),\,\beta}(y-x)\mathrm{I}_{\{x\leq y<M\}}\,dy.\end{aligned} \tag{7}$$

As described in Section IV, these properties allow using the elementary renewal theorems [1]. However, the stationary pdf of the MC $(Y_n)_{n\in\mathbb{N}}$ must be derived.

B. Stationary Law of the MC $(Y_n)_{n\in\mathbb{N}}$

The MC $(Y_n)_{n\in\mathbb{N}}$ takes its values in the continuous space $[0, M)$, and $Y_0 = 0$. As the chain $(Y_n)_{n\in\mathbb{N}}$ comes back to 0 almost surely, that state 0 is then called a regeneration set. This proves the existence of a stationary measure, π, on $[0, M)$ for $(Y_n)_{n\in\mathbb{N}}$ [1, chapter VI, p. 150]. It is a solution of the invariance equation:

$$\pi(\cdot) = \int_{[0,M)} \Pr\{\cdot|x\}\pi(dx). \tag{8}$$

The solution of this state probability is in the form of a convex combination of a pdf and a Dirac mass function:

$$\pi(dx) = a\cdot\delta_0(dx) + (1-a)\cdot b(x)\,dx \tag{9}$$

with $0 < a < 1$ and b a pdf on $[0, M)$. Finding such a solution proves that the chain $(Y_n)_{n\in\mathbb{N}}$ is Harris ergodic [1, chapter VI, p. 154] because it is a probability distribution which is "spreadout," *viz*, not reduced to a pmf [1, chapter VI, p. 140]. Substituting in (8) $\pi(\cdot)$ by its expression from (9) and $\Pr\{\cdot|x\}$ by its expression from (7) leads to the 2 equalities:

$$\begin{aligned}a = a\cdot\overline{F}_{\alpha\cdot m(0),\,\beta}(M) + (1-a)\cdot\int_0^M b(x)\\ \cdot\overline{F}_{\alpha\cdot m(x),\,\beta}(M-x)\,dx\end{aligned} \tag{10}$$

and, for $0 \leq y < M$,

$$b(y) = \frac{a}{1-a}\cdot f_{\alpha\cdot m(0),\,\beta}(y) + \int_0^y b(x)\cdot f_{\alpha\cdot m(x),\,\beta}(y-x)\,dx \quad \text{almost everywhere.} \tag{11}$$

Let $B(y) \equiv ((1-a)/a)\cdot b(y)$; then (11) can be rewritten in the form of a renewal equation:

$$\begin{aligned}B(y) &= f(y) + \int_0^y K(x, y)\cdot B(x)\,dx \quad \text{almost everywhere;}\\ &\qquad K(x, y) \equiv f_{\alpha\cdot m(x),\,\beta}(y-x)\\ f(y) &= f_{\alpha\cdot m(0),\,\beta}(y).\end{aligned} \tag{12}$$

Replace recursively n times, the $B(x)$ in (12) by its expression from (12); then, for $0 \leq y < M$:

$$\begin{aligned}B(y) &= f(y) + \int_0^y \left(\sum_{i=1}^n K^i(x, y)\right)\cdot f(x)\,dx\\ &\quad + \int_0^y K^{n+1}(x, y)\cdot B(x)\,dx;\\ &\quad K^1(x, y) \equiv K(x, y),\\ &\quad K^i(x, y) \equiv \int_{\substack{x<x_1<\cdots\\<x_{i-1}<y}} K(x, x_1)\cdot K(x_1, x_2)\\ &\qquad \cdots K(x_{i-1}, y)\,dx_1\cdots dx_{i-1} \quad \text{for } i>1.\end{aligned} \tag{13}$$

Then Proposition 1 is true.

Proposition 1—The Function:

$$B(y) = f(y) + \int_0^y \left(\sum_{i=1}^{\infty} K^i(x, y) \right) \cdot f(x)\, dx \qquad (14)$$

is the solution of (12); and $\int_0^M B(x)\, dx$ is finite. If g is another solution of (12) such that $\int_0^M g(x)\, dx$ is finite then $B = g$ almost everywhere.

Proof—See [8]: Because $B(y) = ((1-a)/a) \cdot b(y)$, and $b(y)$ is a pdf

$$\int_0^M b(y)\, dy = 1$$

then the real a of (9) is

$$a = \frac{1}{1 + \int_0^M B(x)\, dx}. \qquad (15)$$

IV. Maintenance Policy Optimization

Optimizing the maintenance policy consists of finding both the PR threshold M and the inter-inspection time function $m(\cdot)$ that jointly minimize the long run s-expected maintenance cost rate. Then, derive a computable expression for the s-expected maintenance cost rate on an infinite horizon depending on M and $m(\cdot)$. This suffices for numerical experiments, where no formal optimization is performed.

A. Long-Run s-Expected Maintenance Cost Rate

A preliminary theoretical result on the convergence of functions of semi-regenerative processes is applied to obtain an expression for the long run s-expected maintenance cost rate.

1) Preliminary General Result [5]: Consider a stochastic process $(Z_t)_{t\geq 0}$ with a given state space E which is both semi-regenerative with semi-regeneration times (T_n) and regenerative with regeneration times (S_n).

$V_n \equiv Z_{T_n}$ is the embedded MC.

Consider the stochastic process $(\Phi_t)_{t\geq 0}$ constructed as an additive functional of $(Z_t)_{t\geq 0}$ which has the properties:

- Each rv Φ_t is a function, say Ψ_t, of $(Z_s)_{s\geq 0}$ before time t:

$$\Phi_t = \Psi_t(Z_u,\, 0 \leq u \leq t).$$

- $\Phi_0 = 0$; $\Phi_t \geq 0$ for all $t \geq 0$.
- For all $0 \leq s \leq t$,

$$\Phi_t - \Phi_s = \Psi_{t-s}(Z_u,\, s \leq u \leq t).$$

The random process $(\Phi_t)_{t\geq 0}$ is positive and nondecreasing.

2) Proposition 2: (This proposition follows from Section IV-A-1 [5].)

Let S_1 be the first regeneration point of the random process $(Z_t)_{t\geq 0}$; let:

$$\forall\, t \geq 0,\ \mathrm{E}_0[\Phi_t] < \infty$$
$$\mathrm{E}_0[\Phi_{S_1}] < \infty.$$

Then

$$\lim_{t\to\infty} \left[\frac{\mathrm{E}_0[\Phi_t]}{t} \right] = \frac{\mathrm{E}_\pi[\Phi_{T_1}]}{\mathrm{E}_\pi[T_1]}. \qquad (16)$$

Hence, the limit at infinity can be changed into a ratio of s-expectations on a single semi-renewal cycle.

3) Application of Proposition 2 to the Cost Function: To compute EC_∞, use Proposition 2 with $Z_t = X_t$, $V_n = Y_n$, and successively with:

$$\Phi_t \equiv N_i(t), \quad \Phi_t \equiv N_c(t), \quad \Phi_t \equiv N_p(t), \quad \Phi_t \equiv d(t).$$

All the assumptions of Proposition 2 are verified as follows.

a) Section III proves that

- $(X_t)_{t\geq 0}$ is a regenerative and semi-regenerative process on $[0, M)$;
- 0 is a regeneration set for $(X_t)_{t\geq 0}$ and a recurrent state for $(Y_n)_{n\in\mathbb{N}}$;
- the associated MC $(Y_n)_{n\geq 0}$ is Harris ergodic with invariant probability π.

Note that $T_1 \leq m(0)$. Then $\mathrm{E}_\pi[T_1] \leq m(0) < \infty$. Let $\tau \equiv \inf\{n \geq 1\colon Y_n = 0\}$: the date of the first replacement S_1 is equal to T_τ. Then $\mathrm{E}_0[\tau] < \infty$.

b) $N_p(t) \leq N_i(t)$, $N_c(t) \leq N_i(t)$, $N_i(t) \leq t/m_{\min}$. Then

$$\mathrm{E}_0[N_i(t)] < \infty, \quad \mathrm{E}_0[N_p(t)] < \infty, \quad \mathrm{E}_0[N_c(t)] < \infty.$$

Also $d(t) \leq t$.

Then $\mathrm{E}_0[d(t)] < \infty$ for finite t.

c) $N_i(S_1) = \tau$. Thus $\mathrm{E}_0[N_i(S_1)] < \infty$. At S_1, either a CR or PR occurs. Thus,

$$\mathrm{E}_0[N_c(S_1)] \leq 1 \quad \text{and} \quad \mathrm{E}_0[N_p(S_1)] \leq 1.$$

For the s-expected unavailability duration on a renewal cycle,

$$\mathrm{E}_0[d(S_1)] \leq m(0) \cdot \mathrm{E}_0[\tau] < \infty.$$

Apply Proposition 2 to all the terms of the sum in (5), then:

$$\mathrm{EC}_\infty = \frac{C_i \cdot \mathrm{E}_\pi[N_i(T_1)]}{\mathrm{E}_\pi[T_1]} + \frac{C_p \cdot \mathrm{E}_\pi[N_p(T_1)]}{\mathrm{E}_\pi[T_1]} + \frac{C_c \cdot \mathrm{E}_\pi[N_c(T_1)]}{\mathrm{E}_\pi[T_1]} + \frac{C_d \cdot \mathrm{E}_\pi[d(T_1)]}{\mathrm{E}_\pi[T_1]}. \qquad (17)$$

On a semi-regeneration interval (between two inspections).

$\mathrm{E}_\pi[N_i(T_1)] = 1$. The other quantities, $\mathrm{E}_\pi[N_p(T_1)]$, $\mathrm{E}_\pi[N_c(T_1)]$, $\mathrm{E}_\pi[d(T_1)]$, $\mathrm{E}_\pi[T_1]$ are all computed in a similar way by integration with respect to π (determined in Section III) as follows.

- s-expected number of PR on a semi-regeneration interval:

$$\begin{aligned}\mathrm{E}_\pi[N_p(T_1)] &= \mathrm{P}_\pi(M \leq X_{T_1^-} < L)\\ &= \int_{[0,M)} \left[\overline{F}_{\beta,\alpha m(x)}(M-x) - \overline{F}_{\beta,\alpha m(x)}(L-x) \right] \pi(dx). \end{aligned} \qquad (18)$$

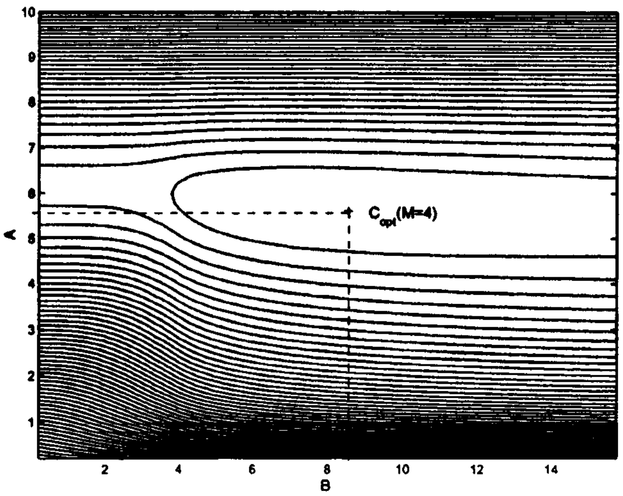

Fig. 3. Iso-level curves of EC_∞ for $C_i = 25, C_p = 50, C_c = 100, C_d = 250$, and $M = 4$—as a function of A and B of the inspection-scheduling rule.

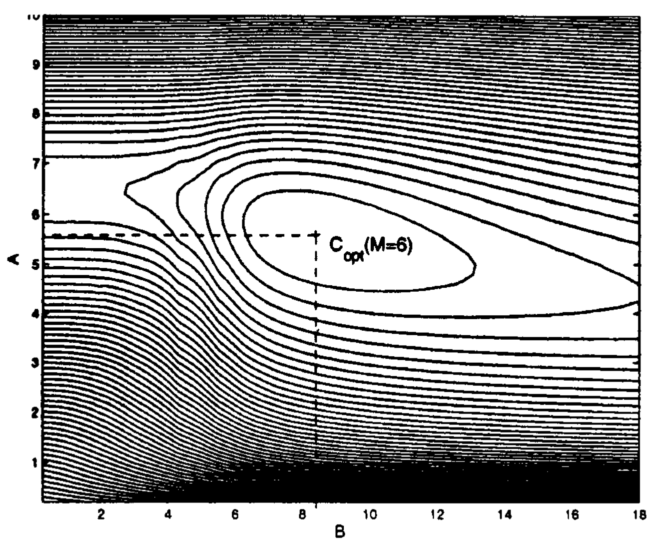

Fig. 4. Iso-level curves of EC_∞ for $C_i = 25, C_p = 50, C_c = 100, C_d = 250$, and $M = 6$—as a function of A and B of the inspection scheduling rule.

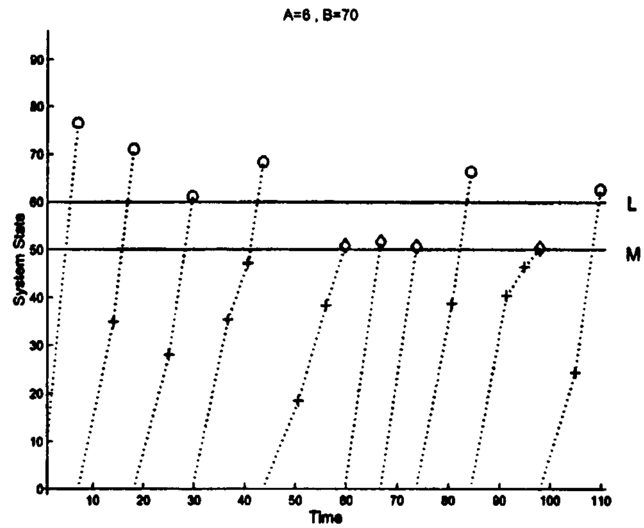

Fig. 5. Simulated evolution of the maintained system state: $\alpha = 1, \beta = 5, L = 60; M = 50, A = 6, B = 70$. $+$: inspection; $\diamond$: PR; $\circ$: CR.

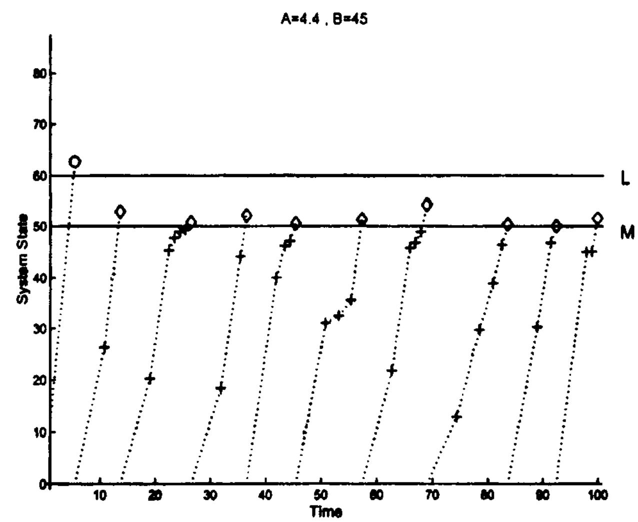

Fig. 6. Simulated evolution of the maintained system state: $\alpha = 1, \beta = 5, L = 60; M = 50, A = 4.4, B = 45$; $+$: inspection; $\diamond$: PR; $\circ$: CR.

- s-expected number of CR on a semi-regeneration interval:

$$\mathrm{E}_\pi[N_c(T_1)] = \mathrm{P}_\pi(X_{T_1^-} \geq L) = \int_{[0,M)} \overline{F}_{\alpha \cdot m(x) \cdot \beta}(L-x)\pi(dx). \quad (19)$$

- s-expected unavailability time on a semi-regeneration interval:

$$\mathrm{E}_\pi[d(T_1)] = \int_{[0,M)} \left(\mathrm{E}_x \int_0^{T_1} \mathrm{I}_{\{X_s \geq L\}}\, ds \right) \pi(dx) = \int_{[0,M)} \left(\int_0^{m(x)} \overline{F}_{\beta,\alpha s}(L-x)\, ds \right) \pi(dx). \quad (20)$$

- s-expected length of a semi-regeneration interval:

$$\mathrm{E}_\pi[T_1] = \int_{[0,M)} m(x)\pi(dx). \quad (21)$$

B. Numerical Experiments

The numerical solution of (12) has been implemented using an integration scheme for integral equations with singular kernels [17, chapter 18]. The pdf $b(\cdot)$, the real a [hence the measure $\pi(\cdot)$] and EC_∞ have been computed by numerical integration using (17)–(21).

For all the examples here, $m(\cdot) = 1 + \max[n(\cdot); 0]$, where $n(\cdot)$ is a linear decreasing function. In this particular case, the decision variables to be optimized are M and the 2 parameters of $n(\cdot)$: A defined by $n(0)$, and B such that $n(B) = 0$, see Fig. 2(b). The value of A controls the time of the first inspection for a system in the initial state, and the value of B determines the evolution of the inter-inspection time as the system deteriorates. The values of the degradation parameters and the unit maintenance costs used in the numerical simulations were chosen arbitrarily to show some important features of this maintenance policy.

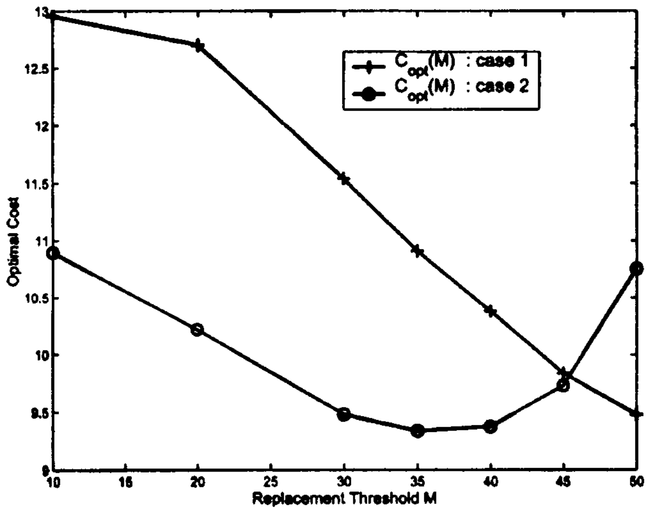

Fig. 7. Evolution of EC_∞ for optimal values of A and B of the inspection scheduling rule, as a function of M for: $C_i = 2$, $C_p = 90$, $C_c = 100$, $C_d = 100$ (for case 1); $C_i = 10$, $C_p = 50$, $C_c = 100$, $C_d = 300$ (for case 2).

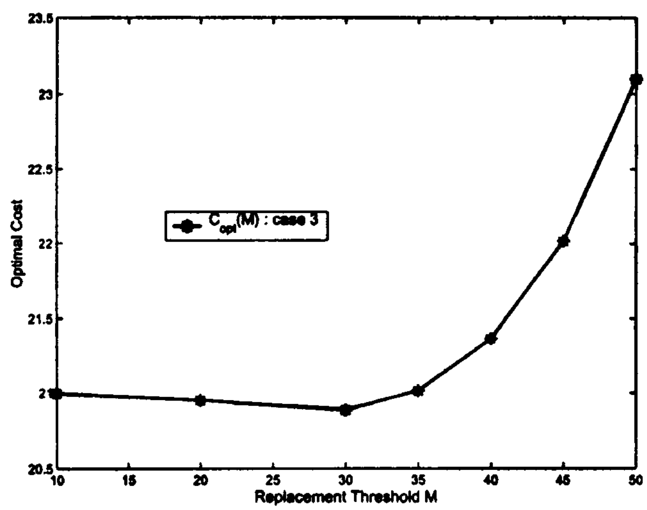

Fig. 8. Evolution of EC_∞ for optimal values of A and B of the inspection scheduling rule, as a function of M for: $C_i = 75$, $C_p = 90$, $C_c = 100$, $C_d = 100$ (for case 3).

1) Optimization of the Maintenance Policy—An Example: This is illustrated with the case of a deteriorating system with $\alpha = 1$ and $\beta = 1$ (which corresponds to an average deterioration speed of $\alpha \cdot \beta = 1$ deterioration rate) and $L = 12$. The numerical results were obtained with

$$C_i = 25, \quad C_p = 50, \quad C_c = 100, \quad C_d = 250.$$

A numerical optimization scheme was used and gives the optimum values of the decision variables $A^* = 5.5$, $B^* = 9$, $M^* = 5.6$ for an optimal cost of 12.2375. To gain more insight in the maintenance policy performance, the maintenance cost was computed for various A and B for a given M. The results for 2 values of M are shown in Figs. 3 and 4 as iso-cost curves in the (A, B) plane.

Fig. 3 represents the iso-level curves of EC_∞ for $M = 4$; the optimum cost is 12.50; it is reached for $A = 5.5$ and it is almost

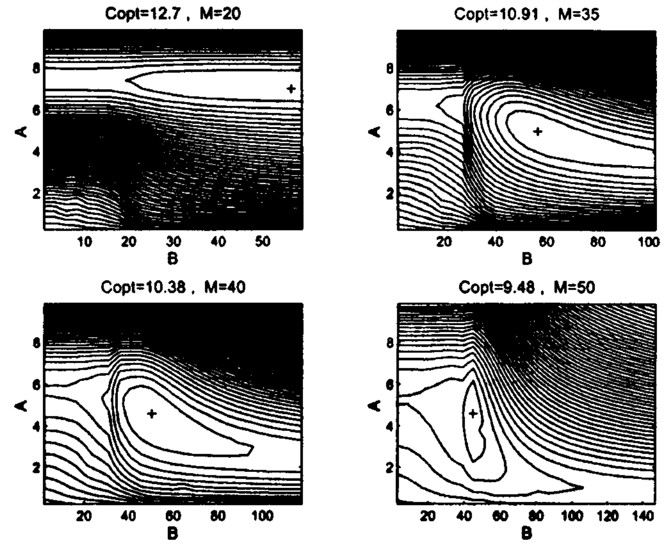

Fig. 9. Iso-level curves of EC_∞ for $C_i = 2, C_p = 90, C_c = 100, C_d = 100$ (for case 1), as a function of A and B of the inspection scheduling rule, for 4 values of M.

independent of the B value. With this optimal policy, $T_1 = 6.5$, and at this time, the probability of exceeding $M \approx 0.9$. Consequently, almost each inspection leads to a replacement of the system and the optimal policy is systematic PR. In this case, the optimization procedure gives information only on the relevant parameter which is the replacement period obtained from A. This case corresponds to $M = 4$, too low with respect to the optimal $M^* = 5.6$. The systematic PR policy is not the best choice for this system.

Fig. 4 represents the iso-level curves of EC_∞ for $M = 6$, a value close to the optimal $M^* = 5.6$. The optimal cost is 12.24. It is reached for A and B close to A^* and B^*. In this case, the proposed policy allows several inspections before replacement. The optimal cost for $M = 6$ ($\mathrm{EC}^*_\infty = 12.24$) is better than the optimal cost for $M = 4$ ($\mathrm{EC}^*_\infty = 12.50$).

2) Simulation With the Maintenance Policy: Consider a system with $\alpha = 1$, $\beta = 5$ (average deterioration speed equal to five deterioration units per unit of time, with a high variance) and $L = 60$ for nonexpensive inspections and costly PR ($C_i = 2$, $C_p = 90$, $C_c = 100$, $C_d = 100$). Figs. 5 and 6 show two simulated histories of the evolution of the maintained system state, for two choices of A, B, M: the maintenance parameters are—

- nonoptimized ($A = 6$, $B = 70$, $M = 50$, $\mathrm{EC}_\infty = 11.89$) in Fig. 5;
- optimized ($A^* = 4.4$, $B^* = 45$, $M^* = 50$, $\mathrm{EC}^*_\infty = 9.48$) in Fig. 6.

In this optimized case, the optimal policy sets a high replacement threshold and involves many inspections in order to monitor precisely the system state so as to trigger a PR only when necessary and to avoid system unavailability.

3) Sensitivity to the Unit Maintenance Costs: For a system with the same deterioration characteristics as in the Section IV-B-2 ($\alpha = 1$, $\beta = 5$, $L = 60$), the influence of the unit maintenance costs on the optimal values of the decision parameters are illustrated. Three cases are considered.

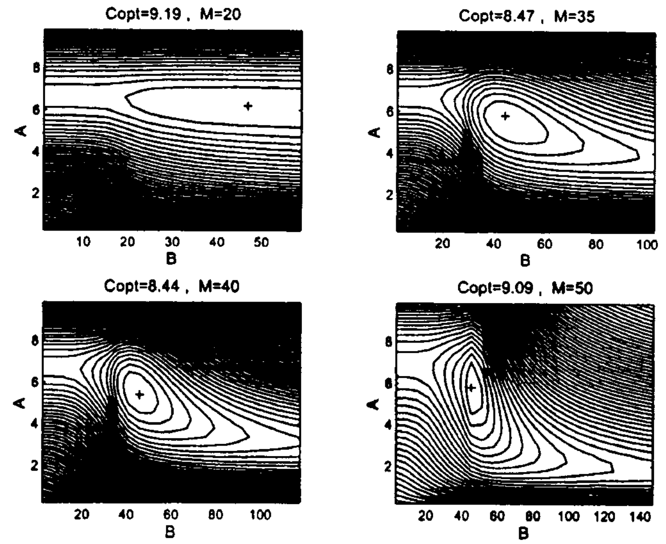

Fig. 10. Iso-level curves of EC_∞ for $C_i = 10$, $C_p = 50$, $C_c = 100$, $C_d = 300$ (for case 2), as a function of A and B of the inspection scheduling rule, for 4 values of M.

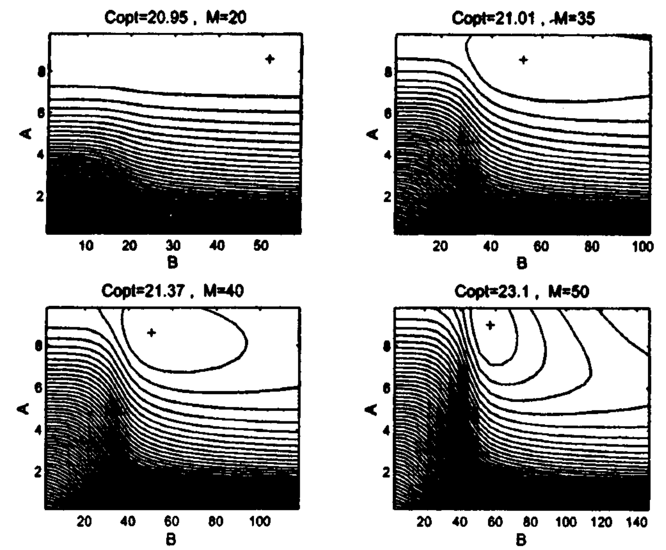

Fig. 11. Iso-level curves of EC_∞ for $C_i = 75$, $C_p = 90$, $C_c = 100$, $C_d = 100$ (for case 3), as a function of A and B of the inspection scheduling rule, for 4 values of M.

1) Nonexpensive inspections and costly PR: $C_i = 2$, $C_p = 90$, $C_c = 100$, $C_d = 100$.
2) Intermediate costs with expensive unavailability: $C_i = 10$, $C_p = 50$, $C_c = 100$, $C_d = 300$.
3) Expensive inspections and costly PR: $C_i = 75$, $C_p = 90$, $C_c = 100$, $C_d = 100$.

Figs. 7 and 8 represent the evolution of EC_∞ as a function of M, for optimal values of A and B. For the three cases and for various choices of M, iso-level curves of the long run s-expected maintenance cost rate are shown in Figs. 9–11.

Whatever the unit maintenance costs are, the greater the value of M, the more the EC_∞ is sensitive to variations of B.

For nonexpensive inspections (case 1, Figs. 7 and 9), the optimal policy combines a high optimal value for M, $M^* \approx 50$, with a small value for A, $A^* \approx 4.4$, and $B^* \approx 45$. Several inspections can be made to monitor precisely the system state before performing a PR; Fig. 6 shows a sample path of the maintained system state in this configuration.

For intermediate inspection and PR costs with expensive unavailability (case 2, Figs. 7 and 10), the optimal values of the maintenance parameters are $A^* \approx 6$, $B^* \approx 45$, $M^* = 35$. A new system is first inspected after $T_1 = A^* + 1 \approx 7$ time units; at this time the average deterioration is $\alpha \cdot \beta \cdot T_1 \approx 35 = M^*$. Thus, a PR is almost always triggered at the first inspection time and the risk of system failure and unavailability is negligible. The optimal maintenance policy is a periodic inspection/PR policy.

For costly inspections and PR (case 3, Figs. 8 and 11) the optimal policy becomes a systematic—PR or CR policy; the optimal values of the maintenance parameters are $A^* \approx 9$ and $M^* = 30$. The value of B does not affect the performance of the policy. The first inspection time is close to $T_1 = 1 + A^* = 10$, and the corresponding mean degradation level reached at T_1 (close to $\alpha \cdot T_1 \cdot \beta = 50$) makes the probability of a replacement occurring at T_1 almost 1. There is almost never a second inspection before the replacement of the system, and the optimal policy is a systematic replacement policy.

References

[1] S. Asmussen, *Applied Probability and Queues*, ser. Wiley Series in Probability and Mathematical Statistics. New York: Wiley, 1987.
[2] A. H. Christer and W. Wang, "A model of condition monitoring of a production plant," *Int. J. Production Res.*, vol. 30, no. 9, pp. 2199–2211, 1992.
[3] ——, "A simple condition monitoring model for a direct monitoring process," *Euro. J. Operational Res.*, vol. 82, pp. 258–269, 1995.
[4] A. H. Christer, W. Wang, and J. M. Sharp, "A state space condition monitoring model for furnace erosion prediction and replacement," *Euro. J. Operational Res.*, vol. 101, pp. 1–14, 1997.
[5] C. Cocozza-Thivent, "Convergence de fonctionnelles de processus semi-régénératifs," (in French), Prépublications de l'Université de Marne la Vallée 02/2000, Mar. 2000.
[6] R. Cooke, M. Mendel, and H. Vrijling, Eds., *Engineering Probabilistic Design and Maintenance for Flood Protection*: Kluwer Academic, 1997.
[7] R. Dekker, "Applications of maintenance optimization models: A review and analysis," *Reliability Engineering and System Safety*, vol. 51, pp. 229–240, 1996.
[8] L. Dieulle, C. Bérenguer, A. Grall, and M. Roussignol, "Continuous time predictive maintenance scheduling for a deteriorating system," in *Proc. Ann. Reliability Maintainability Symp.*, 2001, pp. 150–155.
[9] W. Feller, *An Introduction to Probability Theory and Its Applications*, 2nd ed, ser. Wiley Series in Probability and Mathematical Statistics. New York: Wiley, 1971.
[10] I. Gertsbakh, *Models of Preventive Maintenance*. Amsterdam: North-Holland, 1977.
[11] ——, *Reliability Theory—With Applications to Preventive Maintenance*: Springer, 2000.
[12] A. Grall, C. Bérenguer, and C. Chu, "Optimal dynamic inspection/replacement planning in condition-based maintenance for deteriorating systems," in *Proc. European Conf. Safety and Reliability—ESREL'1998*, S. Lydersen, G. K. Hansen, and H. A. Sandtorv, Eds: A. A. Balkema, pp. 381–388.
[13] J. A. M. Hontelez, H. H. Burger, and J. D. Wijnmalen, "Optimum condition-based maintenance policies for deteriorating systems with partial information," *Reliability Engineering and System Safety*, vol. 51, pp. 267–274, 1996.
[14] J. Medhi, *Stochastic Processes*. New York: Wiley, 1994.
[15] K. S. Park, "Optimal continuous-wear limit replacement under periodic inspections," *IEEE Trans. Reliability*, vol. 37, pp. 97–102, Mar. 1988.
[16] ——, "Optimal wear-limit replacement with wear-dependent failures," *IEEE Trans. Reliability*, vol. 37, pp. 293–294, Mar. 1988.
[17] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in Fortran—The Art of Scientific Computing*: Cambridge University Press, 1992.

[18] D. F. Redmond, A. H. Christer, and S. R. Rigden *et al.*, "O.R. modeling of the deterioration and maintenance of concrete structures," *Euro. J. Operational Res.*, vol. 99, pp. 619–631, 1997.

[19] P. A. Scarf, "On the modeling of condition based maintenance," in *Advances in Safety and Reliability—Proc. ESREL'1997 Int. Conf. Safety and Reliability*, C. Guedes Soares, Ed: Pergamon, 1997, vol. 3, pp. 1701–1708.

[20] N. D. Singpurwalla, "Gamma processes and their generalizations: An overview,", in [6: pp. 67–73].

[21] N. D. Singpurwalla and S. P. Wilson, "Failure models indexed by two scales," *Advances in Applied Probability*, vol. 30, pp. 1058–1072, 1998.

[22] L. J. P. Speijker, J. M. van Noortwijk, M. Kok, and R. M. Cooke, "Optimal maintenance decisions for dikes," *Probability in Engineering and Informational Sciences*, vol. 14, pp. 101–121, 2000.

[23] H. vander Weide, "Gamma processes,", in [6: pp. 77–82].

[24] J. M. van Noortwijk, R. Cooke, and M. Kok, "A Bayesian failure model based on isotropic deterioration," *Euro. J. Operational Res.*, vol. 82, pp. 270–282, 1995.

[25] J. M. van Noortwijk, M. Kok, and R. Cooke, "Optimal maintenance decisions for the sea-bed protection of the eastern-scheldt barrier,", in [6: pp. 25–51].

[26] J. M. van Noortwijk and P. H. A. J. M. van Gelder, "Optimal maintenance decisions for berm breakwaters," *Structural Safety*, vol. 18, no. 4, pp. 293–309, 1996.

[27] J. M. van Noortwijk, "Optimal replacement decisions for structures under stochastic deterioration," in *Proc. 8th IFIP WG 7.5 Working Conf. on Reliability and Optimization of Structural Systems, 1998*, A. S. Nowak, Ed: Univ. of Michigan, 1998, pp. 273–280.

[28] W. Wang, "A model to determine the optimal critical level and the monitoring intervals in condition-based maintenance," *Int. J. Production Research*, vol. 38, no. 6, pp. 1425–1436, 2000.

[29] Y. Yang and G.-A. Klutke, "Lifetime-characteristics and inspections-schemes for Lévy degradation processes," *IEEE Trans. Reliability*, vol. 49, pp. 377–382, Dec. 2000.

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