A Characterisation of Generalised Gamma Processes in Terms of Sufficiency and Isotropy[‡]

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Abstract

To optimise maintenance of deteriorating structures, we need to model the event of failure and the process of deterioration. Due to the common lack of data, there is often only (subjective) information on the limiting average rate of deterioration. Furthermore, most deterioration processes proceed in one direction and in random jumps. In order that stochastic processes with non-negative exchangeable increments be based on the unknown limiting average rate of deterioration, they can best be regarded as generalised gamma processes. In this paper, two new characterisations of generalised gamma processes are given: (i) in terms of conditioning on sums of increments being sufficient statistics and (ii) in terms of isotropy.

Keywords
gamma processes, deterioration processes,
sufficient statistics, isotropy, Brownian motion.

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1 Introduction

In designing hydraulic structures such as dykes, bridges, dams, and other water barriers, a good design ensures that a structure's resistance exceeds the applied stress, at least with a high probability during its service life. Due to uncertain deterioration, the resistance decreases in time creating the need for regular, possibly expensive, maintenance actions. In order to optimise maintenance, we need to model the event of failure and the process of deterioration. Most maintenance models are based on lifetime distributions or Markovian deterioration models. Unfortunately, it is often hard to gather data for estimating either the parameters of a lifetime distribution or the transition probabilities of a Markov chain. Moreover, in case of well-planned preventive maintenance, complete lifetimes will be observed rarely.

In practice, there is often only information in terms of probability distributions on uncertain limiting average rates of deterioration. To give five examples: (i) the limiting average rates of rock displacement for optimising maintenance of the rock dumping of the Eastern-Scheldt barrier (Van Noortwijk, Cooke & Kok [22] and Van Noortwijk, Kok & Cooke [23]); (ii) the limiting average rates of current-induced scour erosion for optimising maintenance of the block mats of the Eastern-Scheldt barrier (Van Noortwijk, Kok & Cooke [23]); (iii) the limiting average rates of ongoing coastal erosion for optimising sand nourishment (Van Noortwijk & Peerbolte [24]); (iv) the limiting average rates of rock displacement for optimising maintenance of berm breakwaters (Van Noortwijk & Van Gelder [25]); and (v) the limiting average rates of settlement for optimising maintenance of dykes (Speijker et al. [21]).

Furthermore, most deterioration processes proceed in one direction and in random jumps. Common practice nowadays is modelling deterioration as a Brownian motion with drift (see e.g. Karlin & Taylor [13, Ch. 7], Hontelez, Burger & Wijnmalen [10], and Pettit [20]). Unfortunately, this process implies the existence of periods in which a structure's resistance actually improves, which does not fit unless the structure undergoes maintenance. Since the Brownian motion has continuous sample paths, it also does not properly model the jumps that occur when the structure is subject to random shocks.

Instead, an adequate deterioration model should have non-negative increments and, due to the lack of data, should have increments that are judged to be exchangeable for every uniform time-partition. In order that stochastic processes with non-negative exchangeable increments be based on the unknown limiting average rate of deterioration, they can best be regarded as generalised gamma processes.

In this paper, two new characterisations of generalised gamma processes are given: (i) in terms of conditioning on sums of increments being sufficient statistics (Sec. 3) and (ii) in terms of isotropy (Sec. 4). The classical characterisation of gamma processes in terms of Poisson processes is briefly reviewed in Sec. 2. The characterisation in terms of sufficiency extends results of Diaconis & Freedman [4, 5] and Küchler &

Lauritzen [15]. The characterisation in terms of isotropy originates from the work of Barlow & Mendel [1] and Misiewicz & Cooke [18]. The proofs of these characterisations can be found in an appendix. The aim of this paper is to present the mathematical framework for modelling deterioration via generalised gamma processes - other papers report on its hydraulic engineering applications (see Van Noortwijk et al. [22, 23, 24, 25] and Speijker et al. [21]).

2 Generalised gamma processes

Before we characterise scale mixtures of gamma processes, called generalised gamma processes, in terms of sufficiency and isotropy, we briefly review the classical characterisation of gamma processes in terms of compound Poisson processes (for an explanation of notation see the appendix).

Definition 1 (Gamma process.) *The gamma process with shape parameter $a > 0$ and scale parameter $b > 0$ is a continuous-time stochastic process $\{Y(t) : t \geq 0\}$ with the following properties:*

1. $Y(0) = 0$ *with probability one;*
2. $Y(\tau) - Y(t)$ *has a gamma distribution* $\mathrm{Ga}(a(\tau - t), b)$ *for all* $\tau > t \geq 0$*;*
3. $Y(t)$ *has independent increments.*

Since the finite-dimensional joint probability density function of the increments is consistent and uniquely defined, Kolmogorov's Extension Theorem assures us that the gamma process exists. By the infinitely divisibility of the the gamma distribution, the gamma process is a Lévy process. Every Lévy process may be written as a sum of a Brownian motion, a deterministic part (linear in time), and an integral of compound Poisson processes, where all the contributing processes are mutually independent. The sample paths of a Lévy process are discontinuous with probability one if the process is monotone, because such a process can be decomposed into a linear part plus an integral of compound Poisson processes. The sample paths of a Brownian motion are continuous with probability one. For details, see Gnedenko & Kolmogorov [9, Ch. 3 & 5], Lévy [16, pp. 173-180], Itô [12, Ch. 1], Ferguson & Klass [8], and de Finetti [3, Ch. 8]. In particular, the characteristic function of the gamma distribution $\mathrm{Ga}(a, b)$, which is given by

$$\phi(u) = [b/(b - iu)]^a = \exp\left\{\int_0^\infty \left(e^{iux} - 1\right) \, dM(x)\right\}$$

where $M(x) = -\,a \int_x^\infty (e^{-by}/y) \, dy$ for $x > 0$, shows us that the gamma process is an integral of compound Poisson processes with jump intensity $M(x)$ (see Gnedenko & Kolmogorov [9, pp. 86-87]). Thus, the gamma process is a pure jump process.

Moran [19] used gamma processes in his theory of the storage of water by dams. Reliability models based on the gamma process have been developed, amongst others, by Dykstra & Laud [6] and Wenocur [26]. Since the Dirichlet distribution can be defined as the probability distribution of a random vector with independent gamma distributed coordinates (with equal scale parameters) divided by their total sum, Ferguson [7] defined the Dirichlet process in terms of the gamma process in a similar way. He used the Dirichlet process for solving Bayesian nonparametric estimation problems.

In structural reliability, it is useful to obtain the cumulative distribution function of the time to failure T, i.e. the time at which a structure's resistance $r_0 - Y(t)$ crosses a fixed stress or failure level s (with r_0 the resistance at time zero):

$$\Pr\{T \leq t\} = \Pr\{Y(t) \geq r_0 - s\} = \int_{r_0 - s}^{\infty} \mathrm{Ga}(x|\, at, b)\, dx = \frac{\Gamma(at, b[r_0 - s])}{\Gamma(at)},$$

where $\Gamma(a, x) = \int_{t=x}^{\infty} t^{a-1} e^{-t}\, dt$ is the incomplete gamma function for $x \geq 0$ and $a > 0$.

3 Characterisation in terms of sufficiency

The purpose of this section is characterising generalised gamma processes in terms of the only (subjective) information that is commonly available: the limiting average rate of deterioration. Let us denote the deterioration process by $\{X(t) : t \geq 0\}$, where $X(t)$ represents the cumulative deterioration at time t and $X(0) = 0$ with probability one. For every uniform time-partition in time-intervals of length $\tau > 0$, we assume $D_i(\tau) = X(i\tau) - X([i-1]\tau) \geq 0$ for $i \in \mathbb{N}$. We derive the generalised gamma process via assumptions on exchangeability and sufficiency by using results of Diaconis & Freedman [5] and Küchler & Lauritzen [15].

The exchangeability assumption means that the order in which the infinite sequence of increments $\{D_i(\tau) : i \in \mathbb{N}\}$ occur is judged to be irrelevant. In mathematical terms, this can be interpreted as that the probability density function of the random vector $(D_1(\tau), \ldots, D_n(\tau))$ is invariant under all $n!$ permutations of the coordinates, i.e.

$$p_{D_1(\tau),\ldots,D_n(\tau)}\left(\delta_1, \ldots, \delta_n\right) = p_{D_1(\tau),\ldots,D_n(\tau)}\left(\delta_{\pi(1)}, \ldots, \delta_{\pi(n)}\right), \tag{1}$$

where π is any permutation of $1, \ldots, n$ for all $n \in \mathbb{N}$ and $\tau > 0$. The notion of exchangeability is weaker than the notion of independence; it can best be utilised in situations with a lack of data.

The sufficiency assumption means that, for every $\tau > 0$, and all $n \geq 2$ and $k < n$, the conditional probability density function of k increments of deterioration, when the sum of n increments is given, can be expressed as

$$p_{D_1(\tau),\ldots,D_k(\tau)|X(n\tau)}(\delta_1, \ldots, \delta_k|\, x) = \frac{\left[\prod_{i=1}^{k} h(\delta_i, \tau)\right] h^{(n-k)}\left(x - \sum_{i=1}^{k} \delta_i, \tau\right)}{h^{(n)}(x, \tau)}, \tag{2}$$

where $h(x,\tau)$ is differentiable and non-negative, $h^{(n)}(x,\tau)$ is the n-fold convolution in x of $h(x,\tau)$ with itself, and $c(\theta)$ is defined by

$$\int_0^\infty h(x,\tau)c(\theta)\exp\{-x/\theta\}\,dx = \int_0^\infty l(x|\theta)\,dx = 1 \tag{3}$$

for $\theta \in (0,\infty)$. In addition, the probability model should be independent of the scale of measurement (e.g. being either inches or centimetres). In other words, the likelihood function $l(x|\theta)$ should be a scale density:

$$l(x|\theta) = f(x/\theta)/\theta \quad \text{for } x,\theta \in (0,\infty). \tag{4}$$

If Eqs. (2-4) are satisfied for all $\tau > 0$, it follows from Theorem 1 (see the appendix) that

$$h(x,\tau) = x^{a\tau-1}/\Gamma(a\tau) \tag{5}$$

for some constant $a > 0$. As a consequence, the joint probability density function of the increments $D_1(\tau),\ldots,D_n(\tau)$ can be written as a mixture of conditionally independent gamma densities:

$$p_{D_1(\tau),\ldots,D_n(\tau)}(\delta_1,\ldots,\delta_n) = \int_0^\infty \prod_{i=1}^n \frac{\delta_i^{a\tau-1}}{\Gamma(a\tau)}\left[\frac{a\tau}{\theta}\right]^{a\tau}\exp\left\{-\frac{a\tau\delta_i}{\theta}\right\}\,dP_{\Theta(\tau)}(\theta) \tag{6}$$

for some constant $a > 0$ with

$$\begin{aligned} E(X(n\tau)) &= E(n\Theta(\tau)), \\ \mathrm{Var}(X(n\tau)) &= \left[1+\frac{1}{na\tau}\right]E([n\Theta(\tau)]^2) - [E(n\Theta(\tau))]^2 \end{aligned} \tag{7}$$

for all $\tau > 0$, provided the first and the second moment of the probability distribution of $\Theta(\tau)$ exist. A stochastic process for which the increments are distributed according to Eq. (6) is called a *generalised gamma process*. By substituting Eq. (5) in Eq. (2), conditioning on sums of increments leads to a transformed Dirichlet distribution:

$$p_{D_1(\tau),\ldots,D_k(\tau)|X(n\tau)}(\delta_1,\ldots,\delta_k|\,x) = \mathrm{Di}_k\left(\frac{\delta_1}{x},\ldots,\frac{\delta_k}{x}\,\middle|\, a\tau,\ldots,a\tau,(n-k)a\tau\right)\left[\frac{1}{x}\right]^k.$$

The generalised gamma process has three useful properties.

First, the probability distribution $P_{\Theta(\tau)}$ on the random quantity $\Theta(\tau)$, with possible values $\theta \in (0,\infty)$, represents the uncertainty in the unknown limiting average amount of deterioration per time-interval of length τ: $\lim_{n\to\infty}[(\sum_{i=1}^n D_i(\tau))/n]$. By the strong law of large numbers for exchangeable random quantities, the average converges with probability one if $E(D_1(\tau)) < \infty$ (see Chow & Teicher [2, p. 227]). In applications of decision theory, the probability distribution of the limiting average rate of deterioration can be the prior distribution, which can be updated in the light of actual data by Bayes' theorem.

Second, the summarisation of the n random quantities $D_1(\tau), \ldots, D_n(\tau)$ in terms of the statistic $[n, \sum_{i=1}^n D_i(\tau)]$ is sufficient for the unknown limiting average rate of deterioration $\Theta(\tau)$. In fact, the characterisation in terms of conditioning on sums of random quantities is motivated by sufficiency ideas, since, by sufficiency, the resulting conditional probability density function does not depend on θ:

$$p_{D_1(\tau),\ldots,D_k(\tau)|X(n\tau),\Theta(\tau)}(\delta_1, \ldots, \delta_k | \, x, \theta) = p_{D_1(\tau),\ldots,D_k(\tau)|X(n\tau)}(\delta_1, \ldots, \delta_k | \, x)$$

for $k < n$. Moreover, since a sum of increments is a single sufficient statistic for the scale parameter, classical results establish, under various regularity conditions, that the scale density in Eq. (4) belongs to the exponential family (see e.g. Koopman [14] and Huzurbazar [11]).

Third, the mixture of gamma's in Eq. (6) transforms into a mixture of exponentials if $\tau = a^{-1}$. The infinite sequence of random quantities $\{D_i(a^{-1}) : i \in \mathbb{N}\}$ is said to be l_1-isotropic (or l_1-norm symmetric), since its distribution can be written as a function of the l_1-norm.

The unit time for which the increments of deterioration are l_1-isotropic can be obtained using the conditional probability density function of the first increment, when the sum of the first and the second increment is given, being a transformed beta distribution with both parameters equal to $a\tau$, i.e.

$$p_{D_1(\tau)|X(2\tau)}(\delta_1 | \, x) = \frac{\Gamma(2a\tau)}{[\Gamma(a\tau)]^2} \frac{\delta_1^{a\tau-1}[x - \delta_1]^{a\tau-1}}{x^{2a\tau-1}} I_{[0,x]}(\delta_1) = \mathrm{Be}\left(\frac{\delta_1}{x}\middle| a\tau, a\tau\right)\frac{1}{x} \qquad (8)$$

for some constant $a > 0$ with

$$\begin{aligned} E(D_1(\tau)|X(2\tau)=x) &= x/2, \\ \mathrm{Var}(D_1(\tau)|X(2\tau)=x) &= [x/2]^2/(2a\tau+1). \end{aligned}$$

Hence, for fixed $\tau > 0$, the smaller the unit-time length for which the increments are l_1-isotropic, i.e. the smaller $\Delta = a^{-1}$, the more deterministic the deterioration process. An alternative way to obtain the unit time for which l_1-isotropy holds is by assessing $\mathrm{Var}(X(n\tau))$ in Eq. (7). This variance approaches $\mathrm{Var}(n\Theta(\tau))$, from above, as $\Delta = a^{-1}$ tends to 0, from above.

For the unit-time length $\Delta = a^{-1}$, many probabilistic properties of the stochastic deterioration process, like the probability of exceedence of a failure level, can be expressed in explicit form conditional on the limiting average deterioration (see e.g. Van Noortwijk, Cooke & Kok [22]). Note that specifying the l_1-isotropic grid of the generalised gamma process is similar to specifying the precision of the Brownian motion with drift.

In conclusion, we advocate regarding stochastic deterioration processes as generalised gamma processes with probability distribution on the limiting average rate of deterioration.

4 Characterisation in terms of isotropy

Even though it is quite reasonable to derive stochastic deterioration processes based on sums of increments that are sufficient statistics for the unknown limiting average rate, a weaker characterisation might be of interest. Weaker conditions can be established by allowing sums of increments to the power, rather than only sums of increments, to serve as sufficient statistics for the scale parameter. To achieve this, we assume the infinite sequence of increments to be isotropic for every uniform time-partition.

The random vector $\mathbf{D}_n = (D_1, \ldots, D_n)$ is said to be l_β-*isotropic* (or l_β-norm symmetric) if its distribution can be written as a function of the statistic $\sum_{i=1}^n D_i^\beta$ where $\beta > 0$; i.e. its distribution is uniform on the l_β-spheres in $\mathbb{R}_+^n$, where $\mathbb{R}_+ = [0, \infty)$. The infinite sequence of random quantities $\{D_i : i \in \mathbb{N}\}$ is l_β-isotropic if $\mathbf{D}_n$ is l_β-isotropic for each $n \in \mathbb{N}$. Mendel [17] and Misiewicz & Cooke [18], amongst others, proved that if the infinite sequence of random quantities $\{D_i : i \in \mathbb{N}\}$ is l_β-isotropic then there exists a probability distribution P_Θ of Θ such that the probability density function of $(D_1, \ldots, D_n)$ is

$$p_{D_1,\ldots,D_n}(\delta_1, \ldots, \delta_n) = \int_0^\infty \prod_{i=1}^n \frac{\beta}{\Gamma(\frac{1}{\beta})} \left[\frac{1}{\theta}\right]^{\frac{1}{\beta}} \exp\left\{-\frac{\delta_i^\beta}{\theta}\right\} dP_\Theta(\theta) = f_n\left(\textstyle\sum_{i=1}^n \delta_i^\beta\right) \tag{9}$$

for $(\delta_1, \ldots, \delta_n) \in \mathbb{R}_+^n$ and zero otherwise. If $\beta = 1$, we have a mixture of n conditionally independent exponentials. If $\beta = 2$, we have a mixture of n conditionally independent normals truncated at zero. Note that isotropy preserves exchangeability and that the statistic $[n, \sum_{i=1}^n D_i^\beta]$ is sufficient for Θ.

The characterisation of generalised gamma processes in terms of isotropy is the following. For every uniform time-partition in time-intervals of length $\tau > 0$, let there be positive continuous functions $\alpha(\tau)$ and $\beta(\tau)$ such that the infinite sequence of powers of increments, $\{D_i(\tau)^{\alpha(\tau)} : i \in \mathbb{N}\}$, is $l_{\beta(\tau)}$-isotropic; that is, such that

$$p_{D_1(\tau),\ldots,D_n(\tau)}(\delta_1, \ldots, \delta_n) = f_n\left(\textstyle\sum_{i=1}^n \delta_i^{\alpha(\tau)\beta(\tau)}\right) \textstyle\prod_{i=1}^n \alpha(\tau)\delta_i^{\alpha(\tau)-1}$$

for all $n \in \mathbb{N}$. If, in addition, the mixing distribution (in Eq. (9)) has finite moments then Theorem 2 from the appendix entails: $\alpha(\tau) = a\tau$ and $\alpha(\tau)\beta(\tau) = 1$ for some constant $a > 0$; Eq. (6) follows accordingly. The theorem has been proved by achieving consistency in the sense that probability distributions of increments and those of sums of increments belong to the same family of distribution.

5 Conclusions

As Barlow & Mendel [1] have argued that appropriate lifetime distributions conditional on the limiting average lifetime are the generalised gamma distributions, we have argued that appropriate deterioration processes conditional on the limiting average rate of deterioration are the generalised gamma processes.

In addition to the classical characterisation of gamma processes in terms of compound Poisson processes, we have presented two new characterisations of generalised gamma processes: (i) in terms of conditioning on sums of increments, serving as sufficient statistics for the unknown limiting average rate, and (ii) in terms of isotropy. A useful property of generalised gamma processes is that we can always find a uniform time-partition for which the joint probability density function of the increments can be written as a mixture of exponentials.

In The Netherlands, generalised gamma processes have been used to model decision problems for optimising maintenance of the sea-bed protection of the Eastern-Scheldt barrier, beaches, berm breakwaters, and dykes (see Van Noortwijk et al. [22, 23, 24, 25] and Speijker et al. [21]).

Appendix

Definition 2 (Gamma distribution.) *A random quantity X has a gamma distribution with shape parameter $a > 0$ and scale parameter $b > 0$ if its probability density function is given by:*

$$\mathrm{Ga}(x|\, a, b) = [b^a/\Gamma(a)]\, x^{a-1} \exp\{-bx\}\, I_{(0,\infty)}(x).$$

Definition 3 (Beta distribution.) *A random quantity X has a beta distribution with parameters $a, b > 0$ if its probability density function is given by:*

$$\mathrm{Be}(x|a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\, x^{a-1}(1-x)^{b-1}\, I_{[0,1]}(x).$$

Definition 4 (Dirichlet distribution.) *A random vector $\mathbf{Y} = (Y_1, \ldots, Y_{n-1})$ has a Dirichlet distribution with parameters $a_1, \ldots, a_n > 0$ if $\mathbf{Y}$ has a probability density function given by:*

$$\mathrm{Di}_{n-1}(\mathbf{y}|\, a_1, \ldots, a_n) = \frac{\Gamma(\sum_{i=1}^{n} a_i)}{\prod_{i=1}^{n} \Gamma(a_i)} \left[1 - \textstyle\sum_{i=1}^{n-1} y_i\right]_+^{a_n - 1} \prod_{i=1}^{n-1} y_i^{a_i - 1}\, I_{[0,1]}(y_i),$$

where $[x]_+ = \max\{0, x\}$.

Theorem 1 *Let $\{X(t) : t \geq 0\}$ be a non-decreasing continuous-time stochastic process with $X(0) = 0$, with probability one, such that for every $\tau > 0$ the infinite sequence of non-negative real-valued increments $D_i(\tau) = X(i\tau) - X([i-1]\tau)$, $i \in \mathbb{N}$, is exchangeable. Moreover, for every $\tau > 0$, and all $n \geq 2$ and $k < n$, the joint conditional probability density function of the increments $D_1(\tau), \ldots, D_k(\tau)$, when $X(n\tau) = x$ is given, can be represented by Eqs. (2-4). Then there exists a constant $a > 0$ such that the joint probability density function of the increments $D_1(\tau), \ldots, D_n(\tau)$ is given by Eq. (6).*

Proof:
By Diaconis & Freedman [5] and Küchler & Lauritzen [15], there exists a probability distribution $P_{\Theta(\tau)}$ such that

$$p_{D_1(\tau),\ldots,D_n(\tau)}(\delta_1, \ldots, \delta_n) = \int_0^\infty \prod_{i=1}^{n} h(\delta_i, \tau) c(\theta) \exp\{-\,\delta_i/\theta\}\, dP_{\Theta(\tau)}(\theta)$$

for every $\tau > 0$ and all $n \in \mathbb{N}$. Since the likelihood function $l(x|\theta)$ is a scale density, the function $h(x, \tau)$ satisfies the functional equation

$$l(x|\theta) = h(x, \tau) c(\theta) \exp\{-\,x/\theta\} = f(x/\theta)/\theta$$

or, with $g(x/\theta) = \exp\{x/\theta\} f(x/\theta)$, $\phi_1(x) = h(x, \tau)$, and $\phi_2(\theta) = \theta c(\theta)$:

$$g(x/\theta) = \phi_1(x)\phi_2(\theta).$$

This functional equation can be recognised as an extension of one of the four well-known Cauchy equations in which $g(x) = \phi_1(x) = \phi_2(1/x)$ (see Huzurbazar [11, p. 204]). Its general solution is $g(x) = c_1 x^{c_2}$, where c_1 and c_2 are arbitrary constants. Hence, using Eq. (3), the functions $h(x, \tau)$ and $c(\theta)$ have the form

$$h(x, \tau) = x^{\alpha(\tau)} / \Gamma(\alpha(\tau) + 1), \qquad c(\theta) = \theta^{-\alpha(\tau)-1},$$

respectively, where $\alpha(\tau) > -1$ is a differentiable function. In turn, $\alpha(\tau)$ satisfies another Cauchy functional equation: $\alpha(n\tau) = n\alpha(\tau)$ for all $\tau > 0$ and $n \in \mathbb{N}$. This functional equation is generated by Eq. (2), i.e.

$$p_{D_1(n\tau)|X(2n\tau)}(\delta|\, x) = p_{D_1(\tau)+\ldots+D_n(\tau)|X(2n\tau)}(\delta|\, x), \tag{10}$$

when dividing both sides of Eq. (10) by $\delta^{n\alpha(\tau)}$ and letting δ approach zero from the right. The general solution is $\alpha(\tau) = a\tau + b$ for some constants $a > 0$ and $b \geq -1$. Since $X(0) = 0$, with probability one, we have $b = -1$.
Eq. (6) follows by replacing θ with $\theta/(a\tau)$, which proves the theorem. □

Theorem 2 *Let $\{X(t) : t \geq 0\}$ be a non-decreasing continuous-time stochastic process with $X(0) = 0$, with probability one, such that for every $\tau > 0$ there are positive continuous functions $\alpha(\tau)$ and $\beta(\tau)$ for which the infinite sequence of random quantities $\{D_i(\tau)^{\alpha(\tau)} : i \in \mathbb{N}\}$ is $l_{\beta(\tau)}$-isotropic with respect to a mixing distribution with finite moments, where $D_i(\tau) = X(i\tau) - X([i-1]\tau)$, $i \in \mathbb{N}$. Then there exists a constant $a > 0$ such that $\alpha(\tau) = a \cdot \tau$ and $\beta(\tau) = (a \cdot \tau)^{-1}$, and the joint probability density function of the increments $D_1(\tau), \ldots, D_n(\tau)$ is given by Eq. (6).*

Proof:
Fix $\tau > 0$. On the one hand, there are functions $\lambda = \alpha(2\tau)$ and $\mu = \beta(2\tau)$ such that the infinite sequence of random quantities $\{X_i^\lambda : i \in \mathbb{N}\}$ is l_μ-isotropic, where $X_i = X(2i\tau) - X(2[i-1]\tau)$, $i \in \mathbb{N}$. By applying Eq. (9) to the probability density function of the random vector $(X_1^\lambda, \ldots, X_n^\lambda)$ and transforming back to $(X_1, \ldots, X_n)$, there exists a probability distribution $\tilde{P}(\varphi)$ such that the joint probability density function of $X_1, \ldots, X_n$ can be written as

$$\tilde{p}(x_1, \ldots, x_n) = \int_{\varphi=0}^{\infty} \prod_{i=1}^{n} \frac{\mu}{\Gamma(\frac{1}{\mu})} \left[\frac{1}{\varphi}\right]^{\frac{1}{\mu}} \lambda x_i^{\lambda-1} \exp\left\{-\frac{x_i^{\lambda\mu}}{\varphi}\right\} d\tilde{P}(\varphi). \tag{11}$$

On the other hand, there are functions $\alpha = \alpha(\tau)$ and $\beta = \beta(\tau)$ such that the infinite sequence of random quantities $\{D_i^\alpha : i \in \mathbb{N}\}$ is l_β-isotropic, where $D_i = X(i\tau) -$

$X([i-1]\tau)$, $i \in \mathbb{N}$. Then, there exists a probability distribution $P(\theta)$ for which the probability density function of the random vector $(D_1, \ldots, D_{2n})$ can be obtained from Eq. (11) by replacing $(\lambda, \mu, n, 2\tau)$ with $(\alpha, \beta, 2n, \tau)$. In turn, we get the joint probability density function of the subsums $X_i = D_{2i-1} + D_{2i}$, $i = 1, \ldots, n$, by applying the one-to-one transformation $\delta_{2i-1} = t_i x_i$, $\delta_{2i} = (1-t_i)x_i$, $i = 1, \ldots, n$, with Jacobian $\prod_{i=1}^n x_i$. Without loss of generality, we shall focus on the case $n = 1$. The probability density function of $X = X_1$ follows by integrating out the variable $t = t_1$:

$$p(x) = \int_{\theta=0}^{\infty} \frac{\alpha^2\beta^2 x^{2\alpha-1}}{\Gamma^2(\frac{1}{\beta})\theta^{2/\beta}} \int_{t=0}^{1} [t(1-t)]^{\alpha-1} \exp\left\{-\frac{x^{\alpha\beta}[t^{\alpha\beta} + (1-t)^{\alpha\beta}]}{\theta}\right\} dt\, dP(\theta).$$

By applying the mean value theorem of the integral calculus and using the beta integral, there exists a constant ξ such that

$$p(x) = \int_{\theta=0}^{\infty} \frac{\alpha^2\beta^2 x^{2\alpha-1}}{\Gamma^2(\frac{1}{\beta})\theta^{2/\beta}}\, \mathrm{B}(\alpha,\alpha) \exp\left\{-\frac{x^{\alpha\beta}\xi}{\theta}\right\} dP(\theta),$$

where $\min\{2^{1-\alpha\beta}, 1\} \le \xi \le \max\{2^{1-\alpha\beta}, 1\}$.

The probability density functions $\tilde{p}(x)$ and $p(x)$ must be equal for all $x > 0$. With the existence of both $E(\Phi^{-r})$ and $E(\Theta^{-r})$ for all $r > 0$, we can prove in two stages that $\lambda = 2\alpha$ and $\lambda\mu = \alpha\beta$.

First, multiply both sides of the equation $\tilde{p}(x) = p(x)$ by $x^{1-2\alpha}$, where $x > 0$. As x approaches zero, from the right, we have

$$a_0 = \lim_{x \downarrow 0} p(x)x^{1-2\alpha} = \lim_{x \downarrow 0} \tilde{p}(x)x^{1-2\alpha} = b_0 \lim_{x \downarrow 0} x^{\lambda-2\alpha},$$

where $a_0, b_0 > 0$. The left-hand side is a constant greater than zero, whereas the limit on the right-hand side equals zero for $\lambda > 2\alpha$ and tends to infinity for $\lambda < 2\alpha$: both leading to a contradiction. Thus $\lambda = 2\alpha$.

Second, we show $\lambda\mu = \alpha\beta$ by subsequently dividing both sides of $\tilde{p}(x) = p(x)$ by $x^{2\alpha-1}$, taking the derivative with respect to x, multiplying by $x^{1-\alpha\beta}$, and letting x tend to zero from above. Then, we have

$$-a_1 = -b_1 \lim_{x \downarrow 0} x^{\lambda\mu-\alpha\beta},$$

where $a_1, b_1 > 0$. The left-hand side is a constant smaller than zero, whereas the limit on the right-hand side equals zero for $\lambda\mu > \alpha\beta$ and tends to minus infinity for $\lambda\mu < \alpha\beta$: both leading to a contradiction. Hence $\lambda\mu = \alpha\beta$.

As a consequence, we may rewrite the equation $\tilde{p}(x) = p(x)$ in the form of two Laplace transforms:

$$\int_0^\infty \tilde{s}^{2/\beta} \exp\left\{-x^{\alpha\beta}\tilde{s}\right\} d\tilde{P}(\tilde{s}) = \int_0^\infty s^{2/\beta} \exp\left\{-x^{\alpha\beta}s\right\} dP(s)\, \frac{\alpha \mathrm{B}(\alpha,\alpha)}{\frac{1}{\beta}\mathrm{B}(\frac{1}{\beta},\frac{1}{\beta})}\, \xi^{-\frac{2}{\beta}},$$

where we have applied the transformations $\tilde{s} = 1/\varphi$ and $s = \xi/\theta$. Hence, we have

$$d\tilde{P}(s) = dP(s)\,\frac{\alpha \mathrm{B}(\alpha,\alpha)}{\frac{1}{\beta}\mathrm{B}(\frac{1}{\beta},\frac{1}{\beta})}\,\xi^{-\frac{2}{\beta}},$$

using the uniqueness of the Laplace transform. By integrating with respect to s on both sides, we can solve for ξ and substitute its value into the equation that was derived from the mean value theorem. Then,

$$\frac{1}{\mathrm{B}(\alpha,\alpha)}\int_{t=0}^{1}[t(1-t)]^{\alpha-1}\exp\left\{-y[t^{\alpha\beta}+(1-t)^{\alpha\beta}]\right\}\,dt = \exp\left\{-y\left[\frac{\alpha \mathrm{B}(\alpha,\alpha)}{\frac{1}{\beta}\mathrm{B}(\frac{1}{\beta},\frac{1}{\beta})}\right]^{\frac{\beta}{2}}\right\},$$

where, for notational convenience, $y = x^{\alpha\beta}/\theta$. By expanding the exponential function with a Taylor series, we are now able to prove that $\alpha\beta = 1$.

The right-hand side results in

$$\tilde{h}(y) = \sum_{j=0}^{\infty}\frac{1}{j!}\left[\frac{\alpha \mathrm{B}(\alpha,\alpha)}{\frac{1}{\beta}\mathrm{B}(\frac{1}{\beta},\frac{1}{\beta})}\right]^{\frac{\beta}{2}j}(-y)^j\,,$$

while the left-hand side can be expanded in a similar way. With the Weierstrass M test, using the upper bound $t^{\alpha\beta}+(1-t)^{\alpha\beta} \le \max\{2^{1-\alpha\beta},1\}$ for all $t \in [0,1]$, we may interchange the order of summation and integration through which the left-hand side can be rewritten as

$$h(y) = \sum_{j=0}^{\infty}\frac{1}{j!}E\left(\left[T^{\alpha\beta}+(1-T)^{\alpha\beta}\right]^j\right)(-y)^j\,,$$

where $T \sim \mathrm{Be}(\alpha,\alpha)$. Since the power series $\tilde{h}(y)$ and $h(y)$ must be equal for all $y > 0$, all their coefficients must coincide. Equating the $j=1$ and $j=2$ terms of both series yields the following relation in $\alpha\beta$:

$$E\left(\left[T^{\alpha\beta}+(1-T)^{\alpha\beta}\right]^2\right) = \left[E\left(T^{\alpha\beta}+(1-T)^{\alpha\beta}\right)\right]^2.$$

Applying Jensen's inequality, we see that $\alpha\beta = 1$.

In conclusion, we have for all $\tau > 0$:

$$\begin{aligned}\alpha(2\tau) &= 2\alpha(\tau),\\ \alpha(2\tau)\beta(2\tau) &= \alpha(\tau)\beta(\tau) = 1.\end{aligned}$$

The solutions of these functional equations are $\alpha(\tau) = a\cdot\tau$ and $\beta(\tau) = (a\cdot\tau)^{-1}$ for some constant $a > 0$.

Note that the above arguments also hold for subdividing $(0,k\tau]$ into k equal time-intervals instead of subdividing $(0,2\tau]$ into 2 equal time-intervals, except that, roughly, k-dimensional Dirichlet integrals replace the 2-dimensional beta integrals.

Eq. (6) follows by replacing θ with $\theta/(a\cdot\tau)$, which proves the theorem. $\square$

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