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Reliability Engineering and System Safety 80 (2003) 185–195

RELIABILITY ENGINEERING & SYSTEM SAFETY

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Explicit formulas for the variance of discounted life-cycle cost

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Received 2 May 2002; revised 25 July 2002; accepted 3 January 2003

Abstract

In life-cycle costing analyses, optimal design is usually achieved by minimising the expected value of the discounted costs. As well as the expected value, the corresponding variance may be useful for estimating, for example, the uncertainty bounds of the calculated discounted costs. However, general explicit formulas for calculating the variance of the discounted costs over an unbounded time horizon are not yet available. In this paper, explicit formulas for this variance are presented. They can be easily implemented in software to optimise structural design and maintenance management. The use of the mathematical results is illustrated with some examples.

Keywords: Renewal theory; Discounting; Variance of discounted cost; Maintenance; Life-cycle costing

1. Introduction

In life-cycle costing analyses, optimal design is usually achieved by minimising the expected value of the discounted costs. There are three main cost-based criteria for comparing design and maintenance decisions over an unbounded time horizon: (i) the expected average costs per unit time (which are determined by averaging the costs over an unbounded horizon), (ii) the expected discounted costs over an unbounded horizon (which are determined by summing the present discounted values of the costs over an unbounded horizon), and (iii) the expected equivalent average costs per unit time (which are determined by weighted averaging the discounted costs). From these three criteria, only the last two are appropriate for optimal design decisions, because the contribution of the initial investment cost is ignored for the first one. The notion of equivalent average costs relates to the notions of average costs and discounted costs in the sense that the equivalent average costs per unit time approach the average costs per unit time as the discount rate tends to zero from above. All three cost-based criteria can be obtained in explicit form using the renewal theorem.

Apart from the expected value of the costs, the variance of these costs is also important to know. In this paper, new explicit formulas will be presented for both the variance of the discounted costs over an unbounded horizon and the equivalent long-term average variance of the costs per unit time. As with the expected costs, the formulas have been derived using the renewal theorem. It can be shown that the equivalent average variance of the costs per unit time approaches the average variance of the costs per unit time as the discount rate tends to zero from above. In this manner, the equivalent average variance of the costs per unit time approaches a known result from renewal reward theory.

Use of the new formulas is illustrated by deriving an optimal age replacement strategy for a cylinder. For several age replacement intervals, both the expected value and the variance of the discounted costs are determined. Because the new formulas lead to relatively simple expressions, they can be easily implemented for the purpose of life-cycle costing. Examples of possible applications are the models of Enright and Frangopol [3], Frangopol [8], Frangopol et al. [9–11], Kuschel and Rackwitz [13], Rackwitz [14,15], Speijker et al. [17], and van Noortwijk et al. [21–24]. With these mathematical models, optimal

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doi:10.1016/S0951-8320(03)00023-1

design and maintenance decisions can be made under uncertain deterioration and reliability.

The outline of this paper is as follows. Three cost-based criteria for comparing design and maintenance decisions are listed in Section 2: the average costs per unit time, the discounted costs over an unbounded horizon and the equivalent average costs per unit time. Mathematical formulas for the expected value and the variance of the discounted costs over an unbounded horizon are derived in Sections 3 and 4, respectively. These formulas are based on results of discrete-time renewal theory. In Section 5, similar results are obtained for continuous-time renewal processes. Section 6 contains three illustrations of the theory. Concluding remarks are made in Section 7. Proofs of theorems can be found in Appendix A.

2. Cost-based criteria

The purpose of this paper is to derive both the expected value and the variance of the discounted costs of maintenance for uncertain deterioration. These probabilistic characteristics are useful for the optimisation of design and maintenance. In the design phase, the initial cost of investment has to be balanced against the future cost of maintenance. In the application phase, the cost of inspection and preventive replacement has to be balanced against the cost of corrective replacement and failure. Usually, maintenance is defined as a combination of actions carried out to restore a component or structure, or to 'renew' it to the initial condition. Inspections, repairs, replacements, and lifetime-extending measures are possible maintenance actions. Through lifetime-extending measures, the deterioration can be delayed so that failure is postponed and the component's lifetime is extended.

Roughly, there are two types of maintenance: corrective maintenance (mainly after failure) and preventive maintenance (mainly before failure). Corrective maintenance can best be chosen if the cost arising from failure is low (like for instance replacing a burnt-out light bulb); preventive maintenance if this cost is high (like for instance heightening a dyke). Since the planned lifetime of most structures is very long, maintenance decisions may be compared over an unbounded time horizon.

According to Wagner [25, chapter 11], there are three cost-based criteria that can be used to compare maintenance decisions:

(i) the *expected average costs per unit time*, which are determined by averaging the costs over an unbounded time horizon;
(ii) the *expected discounted costs over an unbounded time horizon*, which are determined by summing the (present) discounted values of the costs over an unbounded time horizon, under the assumption that the value of money decreases with time;
(iii) the *expected equivalent average costs per unit time*, which are determined by weighted averaging the discounted costs over an unbounded time horizon, where the weighted average follows from the use of the discount rate.

The notion of equivalent average costs relates to the notions of average costs and discounted costs in the sense that the equivalent average costs per unit time approach the average costs per unit time as the discount rate tends to zero from above. The cost-based criteria of discounted costs and equivalent average costs are most suitable for balancing the initial building cost optimally against the future maintenance cost. The criterion of average costs can be used in situations in which no large investments are made (like inspections) and in which the time value of money is of no consequence. Often, it is preferable to spread the costs of maintenance over time and to use discounting.

3. Expected value of costs

Maintenance can often be modelled as a (discrete) renewal process, whereby the renewals are the maintenance actions that bring a component back into its original condition or 'good as new state'. After each renewal we start (in a statistical sense) all over again. A discrete renewal process $\{N(n), n = 1, 2, 3, \dots\}$ is a non-negative integer-valued stochastic process that registers the successive renewals in the time interval $(0, n]$. Let the renewal times $T_1, T_2, T_3, \dots$ be non-negative, independent, identically distributed, random quantities having the discrete probability function

$$\Pr\{T_k = i\} = p_i, \qquad i = 1, 2, \dots \quad \sum_{i=1}^{\infty} p_i = 1,$$

where p_i represents the probability of a renewal in unit time i. We denote the cost associated with a renewal in unit time i by c_i, $i = 1, 2, 3, \dots$ The three above-mentioned cost-based criteria will be discussed in more detail in the following sections.

3.1. Expected average costs per unit time

The expected average costs per unit time are determined by simply averaging the costs over an unbounded horizon. They follow from the expected costs over the bounded horizon $(0, n]$, denoted by $E(K(n))$, which is a solution of the recursive equation

$$E(K(n)) = \sum_{i=1}^{n} p_i [c_i + E(K(n-i))], \tag{1}$$

for $n = 1, 2, 3, \dots$ and $K(0) \equiv 0$. To obtain this equation, we condition on the values of the first renewal time T_1 and apply the law of total probability. The costs associated with occurrence of the event $T_1 = i$ are c_i plus the additional expected costs during the interval $(i, n]$, $i = 1, \ldots, n$. Using the discrete renewal theorem (see Feller [5, chapters 12 and 13] and Karlin and Taylor [12, chapter 3]), the *expected average costs per unit time* are

$$\lim_{n\to\infty} \frac{E(K(n))}{n} = \frac{\sum_{i=1}^{\infty} c_i p_i}{\sum_{i=1}^{\infty} i p_i} = \frac{E(c_I)}{E(I)} = \frac{E(\text{cycle cost})}{E(\text{cycle length})}. \quad (2)$$

The random quantity I is the cycle length with cycle cost c_I. Let a renewal cycle be the time period between two renewals, and recognise the numerator of Eq. (2) as the expected cycle cost and the denominator as the expected cycle length (mean lifetime). Eq. (2) is a well-known result from renewal reward theory (see Ref. [16, chapter 3]). The limit in Eq. (2) exists provided that the greatest common divisor of the integers $i = 1, 2, 3, \ldots$ for which $p_i > 0$ is equal to unity. The simplest assumption assuring this is $p_1 > 0$. If $c_i \equiv 1$ for all $i = 1, 2, 3, \ldots$ in Eq. (2), then the expected average number of renewals per unit time is:

$$\lim_{n\to\infty} \frac{E(N(n))}{n} = \frac{1}{\sum_{i=1}^{\infty} i p_i} = \frac{1}{E(I)}$$

being the reciprocal of the mean lifetime.

3.2. Expected discounted costs over an unbounded horizon

Discounting expected costs over an unbounded horizon is based on the assumption that the value of money decreases with time. Since the future cost can be discounted to its present value on the basis of a discount rate, we can compare the value of money at different dates. The discount rate is usually defined as the real interest rate, i.e. the interest rate minus the inflation rate. In mathematical terms, the *(present) discounted value* of the cost c in unit time n is defined as $\alpha^n c$ with $\alpha = [1 + (r/100)]^{-1}$ the *discount factor* per unit time and $r\%$ the *discount rate* per unit time, where $r > 0$. The decision-maker is indifferent to the cost c at time n and the cost $\alpha^n c$ at time 0. Therefore, the higher the discount rate, the more beneficial it is to postpone expensive maintenance measures.

The expected discounted costs over a bounded time horizon can be obtained with a recursive formula similar to that of the expected non-discounted costs in Eq. (1). Again, we condition on the values of the first renewal time T_1 and apply the law of total probability. In this case however, it is desirable to account for the discounted value of the renewal costs c_i plus the additional expected discounted costs in time interval $(i, n]$, $i = 1, \ldots, n$. Hence, the expected discounted costs over the bounded horizon $(0, n]$, denoted by $E(K(n, \alpha))$, can be written as

$$E(K(n, \alpha)) = \sum_{i=1}^{n} \alpha^i p_i [c_i + E(K(n - i, \alpha))], \quad (3)$$

for $n = 1, 2, 3, \ldots$ and $K(0, \alpha) \equiv 0$. By using Ref. [5, chapter 13], the *expected discounted costs over an unbounded horizon* can be written as

$$\lim_{n\to\infty} E(K(n, \alpha)) = \frac{\sum_{i=1}^{\infty} \alpha^i c_i p_i}{1 - \sum_{i=1}^{\infty} \alpha^i p_i} = \frac{E(\alpha^I c_I)}{1 - E(\alpha^I)} = k(\alpha). \quad (4)$$

For a proof, see the first part of Theorem A1. As far as the author knows, expression (4) was first applied to design and maintenance problems by van Noortwijk and van Gelder [24].

We recognise the numerator of Eq. (4) as the expected discounted cycle costs, while the denominator can be interpreted as the probability that the renewal process 'terminates due to discounting'. Such a renewal process is called a *terminating renewal process* since infinite inter-occurrence times can cause the renewals to cease. The inter-occurrence times $Z_1, Z_2, \ldots$ of the imaginary terminating renewal process have the distribution

$$\Pr\{Z_k = i\} = \alpha^i p_i, \qquad i = 1, 2, 3, \ldots$$

and

$$\Pr\{Z_k = \infty\} = 1 - \sum_{i=1}^{\infty} \alpha^i p_i = 1 - P(\alpha).$$

The expected number of 'discounted renewals' over an unbounded time horizon is

$$\lim_{n\to\infty} E(N(n, \alpha)) = \frac{\sum_{i=1}^{\infty} \alpha^i p_i}{1 - \sum_{i=1}^{\infty} \alpha^i p_i} = \frac{P(\alpha)}{1 - P(\alpha)}.$$

This expectation can also be interpreted as the expected value of the total number of renewals in all time, denoted by $N(\infty)$, having a geometric distribution; that is

$$E(N(\infty)) = \sum_{j=0}^{\infty} j[1 - P(\alpha)][P(\alpha)]^j = \frac{P(\alpha)}{1 - P(\alpha)}.$$

3.3. Initial cost of investment

For cost-optimal investment decisions, we are interested in finding an optimum balance between the initial cost of investment and the future cost of maintenance, being the area of life cycle costing. In this situation, the monetary losses over an unbounded horizon are the sum of the initial cost of investment c_0 spent at time zero and the expected discounted future cost $k(\alpha)$:

$$L(\alpha) = c_0 + k(\alpha) = c_0 + \lim_{n\to\infty} E(K(n,\alpha)).$$

For investment decisions, we cannot use the criterion of the expected average costs per unit time

$$L = \lim_{n\to\infty} \frac{c_0 + E(K(n))}{n} = \lim_{n\to\infty} \frac{E(K(n))}{n}$$

because the contribution of the initial cost to the average costs is ignored.

3.4. Expected equivalent average costs per unit time

The expected equivalent average costs per unit time relate to the notions of average costs and discounted costs. To determine this relation, we construct a new infinite stream of identical costs with the same present discounted value as the expected discounted costs over an unbounded time horizon $L(\alpha) = c_0 + k(\alpha)$. This can be achieved by defining the *expected equivalent average costs per unit time* to be

$$\lim_{n\to\infty} \frac{c_0 + E(K(n,\alpha))}{\sum_{i=0}^{n} \alpha^i} = (1-\alpha)\left[c_0 + \lim_{n\to\infty} E(K(n,\alpha))\right] = (1-\alpha)L(\alpha). \quad (5)$$

Hence, we have an infinite stream of costs at times $i = 0, 1, 2, \ldots$ which are all equal to $(1-\alpha)L(\alpha)$. Using the geometric series, we can write the sum of the discounted costs as

$$\sum_{i=0}^{\infty} \alpha^i[(1-\alpha)L(\alpha)] = L(\alpha),$$

for $0 < \alpha < 1$. The expected equivalent average costs per unit time can be interpreted as a weighted average of the non-discounted costs in units of time $i = 0, 1, 2, \ldots$ with non-equal weights $\alpha^i(1-\alpha)$, $i = 0, 1, 2, \ldots$

When the investment cost at time zero is not included, the expected equivalent average costs per unit time (Eq. (5)) should be redefined as

$$\lim_{n\to\infty} \frac{E(K(n,\alpha))}{\sum_{i=1}^{n} \alpha^i} = \frac{1-\alpha}{\alpha}\left[\lim_{n\to\infty} E(K(n,\alpha))\right] = \frac{1-\alpha}{\alpha} k(\alpha). \quad (6)$$

The relation between the equivalent average costs per unit time on the one hand, and the average costs per unit time on the other hand, is now obvious. As α tends to 1 from below, the expected equivalent average costs approach the expected average costs per unit time:

$$\lim_{\alpha\uparrow 1} \frac{1-\alpha}{\alpha} \lim_{n\to\infty} E(K(n,\alpha)) = \lim_{n\to\infty} \frac{E(K(n))}{n} \quad (7)$$

or equivalently

$$\lim_{\alpha\uparrow 1} \frac{1-\alpha}{\alpha} \frac{\sum_{i=1}^{\infty} \alpha^i c_i p_i}{1 - \sum_{i=1}^{\infty} \alpha^i p_i} = \frac{\sum_{i=1}^{\infty} c_i p_i}{\sum_{i=1}^{\infty} i p_i}$$

using L'Hôpital's rule (see the first part of Theorem A2). In deriving the expected average costs per unit time, it is worthwhile to note that the limits of α tending to unity from below and n approaching infinity can be interchanged; that is

$$\lim_{\alpha\uparrow 1}\left\{\lim_{n\to\infty} \frac{E(K(n,\alpha))}{\sum_{i=1}^{n} \alpha^i}\right\} = \lim_{n\to\infty}\left\{\lim_{\alpha\uparrow 1} \frac{E(K(n,\alpha))}{\sum_{i=1}^{n} \alpha^i}\right\} = \lim_{n\to\infty} \frac{E(K(n))}{n}.$$

As a matter of fact, interchanging the above limits leaves us with a straightforward proof for determining the renewal reward theorem.

4. Variance of costs

The aim of this section is to derive the variance of the discounted costs over an unbounded horizon. This variance can be obtained by applying generating functions. As with the expected value of the discounted costs over an unbounded horizon, we can define an equivalent long-term average variance per unit time. This equivalent average variance per unit time approaches the average variance per unit time as the discount factor tends to unity from below (or in other words, the discount rate tends to zero from above). The long-term average variance of the costs per unit time is a known result in renewal reward theory (see Ref. [26, chapter 2] and Ref. [18, chapter 1]). As far as the author is aware, the expression for the variance of the discounted costs over an unbounded horizon is new.

4.1. Average variance of costs per unit time

The average variance of the costs per unit time can be determined by averaging the variance of the costs over an unbounded horizon. It follows from the first and second moment of the costs over the bounded horizon $(0, n]$, denoted by $E(K(n))$ and $E(K^2(n))$, respectively. These moments solve Eq. (1) as well as the recursive equation

$$E(K^2(n)) = \sum_{i=1}^{n} p_i E([c_i + K(n-i)]^2) = \sum_{i=1}^{n} p_i[c_i^2 + 2c_i E(K(n-i)) + E(K^2(n-i))],$$

for $n = 1, 2, 3, \ldots$ This equation is obtained by conditioning on the values of the first renewal time T_1. In a slightly different notation and for continuous renewal times, Wolff [26, chapter 2] proved that the long-term *average variance of the costs per unit time* is

$$\lim_{n\to\infty} \frac{\text{var}(K(n))}{n} = \frac{\text{var}(c_I)E^2(I) + \text{var}(I)E^2(c_I) - 2E(I)E(c_I)\text{cov}(I, c_I)}{[E(I)]^3}. \quad (8)$$

If $c_i \equiv 1$ for all $i = 1, 2, 3, \ldots$ in Eq. (8), then the long-term average variance of the number of renewals per unit time is:

$$\lim_{n\to\infty} \frac{\text{var}(N(n))}{n} = \frac{\text{var}(I)}{[E(I)]^3}.$$

The discrete-time version of this theorem is proved in Ref. [4]. He also showed that, as $n \to \infty$, $N(n)$ is asymptotically normal with mean

$$E(N(n)) \sim \frac{n}{E(I)}$$

and variance

$$\text{var}(N(n)) \sim \frac{n\,\text{var}(I)}{[E(I)]^3}.$$

For continuous-time renewal processes, Wolff [26, chapter 2] showed that $K(n)$ is asymptotically normal as well.

4.2. Variance of discounted costs over an unbounded horizon

The variance of the expected discounted costs over a bounded time horizon can be obtained by the recursive formulas for the first and second moment of the discounted costs. The former can be found in Eq. (3). The latter can be obtained by conditioning on the values of the first renewal time T_1. The expected value of the square of the discounted costs over the bounded horizon $(0, n]$ can be written as

$$\begin{aligned} E(K^2(n,\alpha)) &= \sum_{i=1}^{n} \alpha^{2i} p_i E([c_i + K(n-i,\alpha)]^2) \\ &= \sum_{i=1}^{n} \alpha^{2i} p_i [c_i^2 + 2c_i E(K(n-i,\alpha) \\ &\quad + E(K^2(n-i,\alpha))], \end{aligned} \quad (9)$$

for $n = 1, 2, 3, \ldots$ After some algebra, the second moment of the discounted costs over an unbounded horizon has the form

$$\begin{aligned} \lim_{n\to\infty} E(K^2(n,\alpha)) &= 2\frac{\sum_{i=1}^{\infty} \alpha^i c_i p_i}{1 - \sum_{i=1}^{\infty} \alpha^i p_i} \frac{\sum_{i=1}^{\infty} \alpha^{2i} c_i p_i}{1 - \sum_{i=1}^{\infty} \alpha^{2i} p_i} + \frac{\sum_{i=1}^{\infty} \alpha^{2i} c_i^2 p_i}{1 - \sum_{i=1}^{\infty} \alpha^{2i} p_i} \\ &= 2\frac{E(\alpha^I c_I)}{1 - E(\alpha^I)} \frac{E(\alpha^{2I} c_I)}{1 - E(\alpha^{2I})} + \frac{E(\alpha^{2I} c_I^2)}{1 - E(\alpha^{2I})}. \end{aligned} \quad (10)$$

The proof is presented in the second part of Theorem A1.

The *variance of the discounted costs over an unbounded horizon* can now be easily derived by combining Eqs. (4) and (10); that is

$$\lim_{n\to\infty} \text{var}(K(n,\alpha)) = \lim_{n\to\infty} [E(K^2(n,\alpha)) - E^2(K(n,\alpha))].$$

An interesting special case arises if $c_i \equiv 1$ for all $i = 1, 2, 3, \ldots$ Using the simplifying notation $P(\alpha) = \sum_{i=1}^{\infty} \alpha^i p_i$, the variance of the number of 'discounted renewals' over an unbounded horizon is

$$\lim_{n\to\infty} \text{var}(N(n,\alpha)) = \frac{P(\alpha^2) - P^2(\alpha)}{[1 - P(\alpha^2)][1 - P(\alpha)]^2}. \quad (11)$$

4.3. Equivalent average variance of costs per unit time

As with the expected value of the costs, the equivalent average variance of the costs per unit time relate to the average variance per unit time and the variance of the discounted costs over an unbounded horizon. To establish this relation, let us define the *equivalent average variance of the costs per unit time* to be

$$\lim_{n\to\infty} \frac{\text{var}(K(n,\alpha))}{\sum_{i=1}^{n} \alpha^{2i}} = \frac{1 - \alpha^2}{\alpha^2} \lim_{n\to\infty} \text{var}(K(n,\alpha)). \quad (12)$$

As α tends to 1 from below, the equivalent average variance of the costs per unit time approaches the average variance of the costs per unit time; that is

$$\lim_{\alpha\uparrow 1} \frac{1 - \alpha^2}{\alpha^2} \lim_{n\to\infty} \text{var}(K(n,\alpha)) = \lim_{n\to\infty} \frac{\text{var}(K(n))}{n}. \quad (13)$$

The proof is presented in Theorem A2. In Section 4.1, it was mentioned that the non-discounted costs over an unbounded horizon are asymptotically normal. Therefore, it is to be expected that the discounted costs over an unbounded horizon are approximately asymptotically normal, as long as the discount factor is close to unity. Note that including the investment cost does not affect the variance of the costs: $\text{var}(c_0 + K(n,\alpha)) = \text{var}(K(n,\alpha))$ for $n = 1, 2, \ldots$ If $c_i \equiv 1$

for all $i = 1, 2, 3, \ldots$ then Eq. (13) reduces to

$$\lim_{\alpha \uparrow 1} \frac{1-\alpha^2}{\alpha^2} \lim_{n \to \infty} \mathrm{var}(N(n,\alpha)) = \lim_{n \to \infty} \frac{\mathrm{var}(N(n))}{n}. \tag{14}$$

A direct proof of Eq. (14) can be obtained by considering the function $P(\alpha)$ in Eq. (11) as a generating function and using L'Hôpital's rule in a similar manner as in the derivation of Eq. (A6); that is

$$\lim_{\alpha \uparrow 1} \frac{1-\alpha^2}{\alpha^2} \lim_{n \to \infty} \mathrm{var}(N(n,\alpha)) = \frac{P''(1) + P'(1) - [P'(1)]^2}{[P'(1)]^3} = \frac{\mathrm{var}(I)}{[E(I)]^3}.$$

5. Continuous-time processes

Up to now, we have studied only discrete-time renewal processes. Similar results can be obtained for continuous-time renewal processes; basically by replacing 'summations' with 'integrals'. The expectation and variance of the discounted costs over an unbounded horizon should now be computed by applying Laplace transforms instead of generating functions.

Let $F(t)$ be the cumulative probability distribution of the continuous renewal time $T \geq 0$ and let $c(t)$ be the cost associated with a renewal at time t. Using a terminating renewal argument (see Ref. [6, chapter 11]) and applying Laplace transforms, the expected discounted costs over an unbounded horizon can then be written as (see Ref. [20])

$$\lim_{t \to \infty} E(K(t,\alpha)) = \frac{\int_0^\infty \alpha^t c(t) \mathrm{d}F(t)}{1 - \int_0^\infty \alpha^t \, \mathrm{d}F(t)}, \tag{15}$$

where $K(t,\alpha)$ represents the expected discounted costs in the bounded time interval $(0,t]$, $t > 0$. A proof can be found in Ref. [15], though he used the capitalisation function $\exp\{-\gamma t\}$ instead of α^t implying that $\gamma = -\log \alpha$. Eq. (15) generalises the work of Berg [2] and Fox [7], who studied age and block replacement policies with discounting (see also Ref. [1]). Rackwitz [14] studied the situation $c(t) \equiv c > 0$ for all $t \geq 0$.

In a similar manner, the second moment of the discounted costs over an unbounded horizon is given by

$$\lim_{t \to \infty} E(K^2(t,\alpha)) = 2 \frac{\int_0^\infty \alpha^t c(t) \mathrm{d}F(t)}{1 - \int_0^\infty \alpha^t \, \mathrm{d}F(t)} \frac{\int_0^\infty \alpha^{2t} c(t) \mathrm{d}F(t)}{1 - \int_0^\infty \alpha^{2t} \, \mathrm{d}F(t)} + \frac{\int_0^\infty \alpha^{2t} c^2(t) \mathrm{d}F(t)}{1 - \int_0^\infty \alpha^{2t} \, \mathrm{d}F(t)}. \tag{16}$$

The continuous-time versions of the discrete-time limit theorems (7) and (13) are as follows. As α tends to 1 from below, the equivalent average costs per unit time approach the average costs per unit time:

$$\lim_{\alpha \uparrow 1} (-\log \alpha) \lim_{t \to \infty} E(K(t,\alpha)) = \lim_{t \to \infty} \frac{E(K(t))}{t} \tag{17}$$

using L'Hôpital's rule. Note that

$$\int_0^\infty \alpha^t \, \mathrm{d}t = -\frac{1}{\log \alpha}.$$

As α tends to 1 from below, the equivalent average variance of the costs per unit time approaches the average variance of the costs per unit time; that is

$$\lim_{\alpha \uparrow 1} (-\log \alpha^2) \lim_{t \to \infty} \mathrm{var}(K(t,\alpha)) = \lim_{t \to \infty} \frac{\mathrm{var}(K(t))}{t}. \tag{18}$$

It should be noted that the limits for t approaching infinity of Eqs. (17) and (18) do not always exist. In order for these limits to exist, the renewal times should have a so-called non-lattice distribution. A random quantity X, and its distribution, are called *lattice* if for some $d > 0$

$$\sum_{i=1}^{\infty} \Pr\{X = id\} = 1.$$

For a detailed discussion on the existence of pointwise limits of functions arising in renewal theory, see Ref. [26, chapter 2].

6. Illustrations

This section gives three examples in which the formulas of the expected value and the variance of the discounted costs over an unbounded horizon can be applied.

6.1. Flood prevention

Let the discrete inter-occurrence times of a flood be distributed as a geometric distribution with parameter p, so that

$$p_i = p(1-p)^{i-1}, \qquad i = 1, 2, 3, \ldots$$

The parameter p can be interpreted as the probability of occurrence of a flood per unit time. Furthermore, assume the cost of flood damage to be $c > 0$. The expected value and the variance of the discounted costs over an unbounded horizon are

$$\lim_{n \to \infty} E(K(n,\alpha)) = \frac{\alpha}{1-\alpha} pc$$

and

$$\lim_{n \to \infty} \mathrm{var}(K(n,\alpha)) = \frac{\alpha^2}{1-\alpha^2} p(1-p)c^2,$$

respectively. When the building cost is included as well, these formulas can be used to design flood defences (see Ref. [19]).

6.2. Poissonian failure process

The continuous-time analogue of the discrete geometric distribution is the exponential distribution. Let failures occur according to a Poisson process with arrival rate λ, then the inter-occurrence failure time is exponentially distributed with mean λ^{-1}. With the failure cost being $c > 0$, the expected discounted costs over an unbounded horizon are

$$\lim_{t\to\infty} E(K(t,\alpha)) = -\frac{1}{\log \alpha}\lambda c.$$

Accordingly, the variance of the discounted costs over an unbounded horizon is

$$\lim_{t\to\infty} \mathrm{var}(K(t,\alpha)) = -\frac{1}{\log \alpha^2}\lambda c^2.$$

In Refs. [13,14], the parameter λ represents the outcrossing rate of a Poissonian failure process.

6.3. Age replacement

A well-known preventive maintenance strategy is the age replacement strategy. Under an age replacement policy, a replacement is carried out at age k (preventive replacement) *or* at failure (corrective replacement), whichever occurs first, where $k = 1,2,3,\ldots$ A preventive replacement entails a cost c_{P}, whereas a corrective replacement entails a cost c_{F}, where $0 < c_{\mathrm{P}} \le c_{\mathrm{F}}$.

As a simplified example, we study the maintenance of a cylinder on a swing bridge (adapted from Ref. [21]). Preventive maintenance of a cylinder mainly consists of replacing the guide bushes and plunger and replacing the packing of the piston rod. In the event of corrective maintenance, the cylinder has to be replaced completely because too much damage has occurred. The cost of preventive maintenance c_{P} is 30 000 Dutch guilders, whereas the cost of corrective maintenance c_{F} is 100 000 Dutch guilders. Both maintenance actions bring the cylinder back to its 'good as new state'. The rate of deterioration is based on periodic lifetime-extending maintenance, in terms of cleaning and sealing the cylinder. The time at which the expected condition equals the failure level is 15 years. On the basis of the stochastic deterioration described by a gamma process, the annual probabilities of failure q_i, $i = 1,2,3,\ldots$ can be easily computed. It follows from Eq. (4) that the expected discounted costs of age replacement over an unbounded horizon are

$$\lim_{n\to\infty} E(K(n,\alpha)) = \frac{\left(\sum_{i=1}^{k}\alpha^i q_i\right)c_{\mathrm{F}} + \alpha^k\left(1-\sum_{i=1}^{k} q_i\right)c_{\mathrm{P}}}{1-\left[\left(\sum_{i=1}^{k}\alpha^i q_i\right)+\alpha^k\left(1-\sum_{i=1}^{k} q_i\right)\right]}.$$

Similarly, according to Eq. (10) the second moment of the discounted costs over an unbounded horizon can be

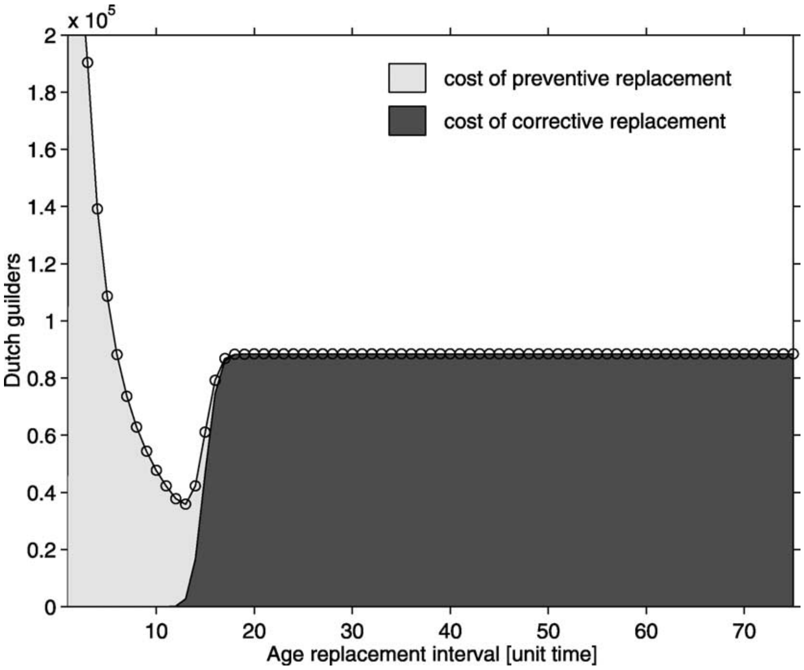

Fig. 1. The expected value of the discounted costs over an unbounded horizon as a function of the age replacement interval k, $k = 1,\ldots,75$.

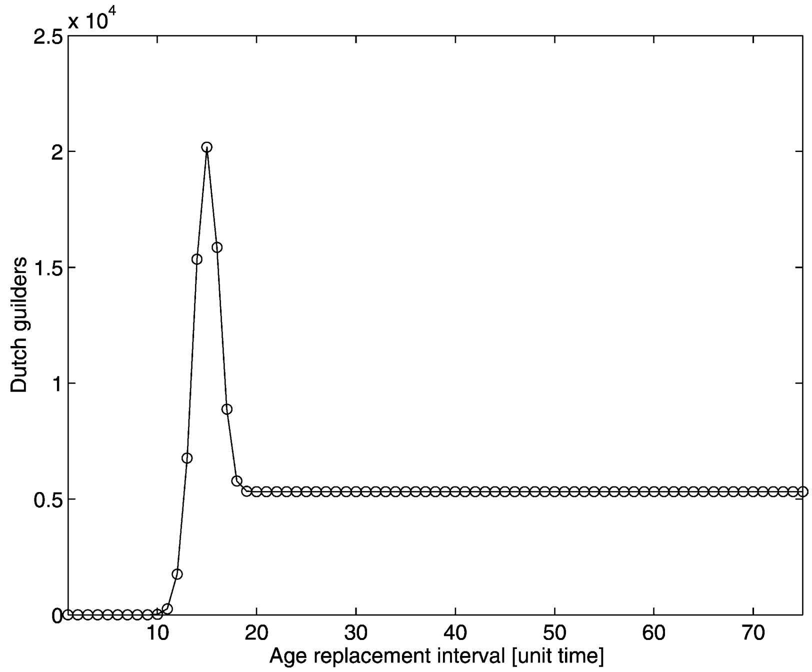

Fig. 2. The standard deviation of the discounted costs over an unbounded horizon as a function of the age replacement interval k, $k = 1, \ldots, 75$.

written as

$$\lim_{n\to\infty} E(K^2(n,\alpha)) = 2\frac{\left(\sum_{i=1}^{k}\alpha^i q_i\right)c_{\mathrm{F}} + \alpha^k\left(1-\sum_{i=1}^{k}q_i\right)c_{\mathrm{P}}}{1-\left[\left(\sum_{i=1}^{k}\alpha^i q_i\right)+\alpha^k\left(1-\sum_{i=1}^{k}q_i\right)\right]} \times \frac{\left(\sum_{i=1}^{k}\alpha^{2i} q_i\right)c_{\mathrm{F}} + \alpha^{2k}\left(1-\sum_{i=1}^{k}q_i\right)c_{\mathrm{P}}}{1-\left[\left(\sum_{i=1}^{k}\alpha^{2i} q_i\right)+\alpha^{2k}\left(1-\sum_{i=1}^{k}q_i\right)\right]} + \frac{\left(\sum_{i=1}^{k}\alpha^{2i} q_i\right)c_F^2 + \alpha^{2k}\left(1-\sum_{i=1}^{k}q_i\right)c_{\mathrm{P}}^2}{1-\left[\left(\sum_{i=1}^{k}\alpha^{2i} q_i\right)+\alpha^{2k}\left(1-\sum_{i=1}^{k}q_i\right)\right]}.$$

The optimal age replacement interval is the interval for which the expected discounted costs over an unbounded horizon are minimal. On the basis of an annual discount rate of 5%, the expected discounted costs of preventive and corrective maintenance are displayed in Fig. 1. The expected discounted costs over an unbounded horizon are minimal for an age replacement interval of 13 years. The standard deviation of the discounted costs over an unbounded horizon is shown in Fig. 2. For an age replacement interval of 15 years, the standard deviation is at a maximum. At this maximum, the variability of a maintenance action being preventive or corrective is maximal. For age replacement intervals smaller than 10 years, the cost of (preventive) maintenance is almost deterministic. For age replacement intervals larger than 20 years, the cost of (corrective) maintenance is to a large degree uncertain due to the variability in the failure times.

Note that the replacement model can also be applied for determining the initial resistance of a structure, which optimally balances the initial cost of investment c_{P} against the future cost of maintenance.

7. Concluding remarks

This paper presents explicit formulas for both the expected value and the variance of the discounted costs over an unbounded horizon. It is shown that there is an interesting connection between the expressions for the discounted costs and well-known results of renewal reward theory (with respect to the long-term average costs per unit time). The variance of the discounted costs over an unbounded horizon is useful to determine (approximate) uncertainty bounds.

The formulas can be applied in situations where regenerative cycles can be identified; that is, after each renewal we start (in a statistical sense) all over again. The advantage of the expressions is that they can be easily computed. Even when we have to rely on Monte Carlo simulation for calculating the probabilistic characteristics of both the renewal cycle length and the renewal cycle cost, the two expressions can be easily used. The reason for this is that both expressions can be reformulated solely in terms of expected values of simple functions of the renewal time. Although renewal times are mainly influenced by failures, they may also depend on inspections, repairs, replacements, and lifetime extensions.

Due to the discount rate, the expressions for the expected value and the variance of the discounted costs over an unbounded horizon may serve as a good approximation in situations with a bounded time horizon larger than 50 years. As an alternative, this paper also presents recursive formulas that can be used to calculate the expected discounted costs over a bounded horizon.

Appendix A. Proofs

Theorem A1. *Let $T_1, T_2, \ldots$ be non-negative, independent, identically distributed, random quantities representing discrete renewal times with probability function $\Pr\{T_k = i\} = p_i$ for $i = 1, 2, \ldots$ where $\sum_{i=1}^{\infty} p_i = 1$. The cost associated with a renewal in unit time i is denoted by c_i, $i = 1, 2, \ldots$ The discounted costs over the bounded horizon $(0, n]$ is denoted by $K(n, \alpha)$ with discount factor α for $n = 1, 2, \ldots$ and $0 < \alpha < 1$. As n approaches infinity, the first and second moment of the discounted costs over an unbounded horizon can then be written as the limits (4) and (10), respectively.*

Proof. The first part of the proof is devoted to deriving the expected discounted costs over an unbounded horizon. For convenience, let us introduce the following simplifying notation: $q_i = \alpha^i p_i$ and $a_i = \alpha^i c_i p_i$ for $i = 1, 2, \ldots$ and $q_0 = a_0 = 0$. The first moment of the discounted cost in unit time i is formed by the difference

$$u_n = E(K(n, \alpha)) - E(K(n-1, \alpha)) \quad \text{for } n = 1, 2, \ldots, \quad u_0 = 0. \tag{A1}$$

By Eq. (A1) as well as the expression for $E(K(n, \alpha))$ in Eq. (3), we obtain the discrete renewal equation

$$u_n = a_n + \sum_{i=0}^{n} q_i u_{n-i} \qquad \text{for } n = 0, 1, 2, \ldots \tag{A2}$$

Consider the generating functions (see Ref. [5, chapter 11])

$$Q(s) = \sum_{n=0}^{\infty} q_n s^n, \qquad A(s) = \sum_{n=0}^{\infty} a_n s^n, \qquad U(s) = \sum_{n=0}^{\infty} u_n s^n,$$

where $0 < s \le 1$ and $U(1) < \infty$, $B(1) < \infty$, and $Q(1) < 1$. The discrete renewal equation (A2) can be rewritten in terms of the corresponding generating functions as follows

$$U(s) = A(s) + Q(s)U(s).$$

Hence, the first moment of the discounted costs over an unbounded horizon is

$$\lim_{n\to\infty} E(K(n, \alpha)) = \sum_{i=0}^{\infty} u_i = U(1) = \frac{A(1)}{1 - Q(1)} = \frac{\sum_{i=0}^{\infty} a_i}{1 - \sum_{i=0}^{\infty} q_i} = \frac{\sum_{i=1}^{\infty} \alpha^i c_i p_i}{1 - \sum_{i=1}^{\infty} \alpha^i p_i},$$

for $0 < \alpha < 1$.

In a similar manner, the second moment of the discounted costs over an unbounded horizon can be obtained. For notational convenience, we define $r_i = \alpha^{2i} p_i$, $b_i = \alpha^{2i} c_i p_i$ and $w_i = \alpha^{2i} c_i^2 p_i$ for $i = 1, 2, \ldots$ and $r_0 = b_0 = w_0 = 0$. Furthermore, let

$$v_n = E(K^2(n, \alpha)) - E(K^2(n-1, \alpha))$$

for $n = 1, 2, \ldots,\ v_0 = 0$.

By substituting the recursive formulas (3) and (9) into this equation, we have

$$v_n = w_n + 2\sum_{i=0}^{n} b_i u_{n-i} + \sum_{i=0}^{n} r_i v_{n-i} \qquad \text{for } n = 0, 1, 2, \ldots$$

This equation can be rewritten in terms of generating functions as follows

$$V(s) = W(s) + 2B(s)U(s) + R(s)V(s),$$

for $0 < s \le 1$, where all generating functions are assumed to be bounded and $Q(1) < 1$ and $R(1) < 1$. After some algebra, the second moment of the discounted costs over an unbounded horizon is

$$\lim_{n\to\infty} E(K^2(n, \alpha)) = \sum_{i=0}^{\infty} v_i = V(1)$$

$$= 2\frac{A(1)}{1 - Q(1)}\frac{B(1)}{1 - R(1)} + \frac{W(1)}{1 - R(1)}$$

$$= 2\frac{\sum_{i=1}^{\infty} \alpha^i c_i p_i}{1 - \sum_{i=1}^{\infty} \alpha^i p_i}\frac{\sum_{i=1}^{\infty} \alpha^{2i} c_i p_i}{1 - \sum_{i=1}^{\infty} \alpha^{2i} p_i} + \frac{\sum_{i=1}^{\infty} \alpha^{2i} c_i^2 p_i}{1 - \sum_{i=1}^{\infty} \alpha^{2i} p_i},$$

for $0 < \alpha < 1$. □

Theorem A2. *Firstly, as α tends to 1 from below, the expected equivalent average costs per unit time (6)* *approach the expected average costs per unit time (2)*:

$$\lim_{\alpha\uparrow 1} \frac{1-\alpha}{\alpha} \lim_{n\to\infty} E(K(n,\alpha)) = \lim_{n\to\infty} \frac{E(K(n))}{n}.$$

Secondly, as α tends to 1 from below, the equivalent average variance of the costs per unit time (12) approaches the average variance of the costs per unit time (8):

$$\lim_{\alpha\uparrow 1} \frac{1-\alpha^2}{\alpha^2} \lim_{n\to\infty} \mathrm{var}(K(n,\alpha)) = \lim_{n\to\infty} \frac{\mathrm{var}(K(n))}{n}.$$

Proof. For notational convenience, let $P(\alpha) = \sum_{i=1}^{\infty} \alpha^i p_i$, $C(\alpha) = \sum_{i=1}^{\infty} \alpha^i c_i p_i$, and $\mathrm{CP}(\alpha) = \sum_{i=1}^{\infty} \alpha^{2i} c_i^2 p_i$. Because the function $P(\alpha)$ can be considered as a generating function, the expectation and variance of the renewal time I can be expressed as

$$E(I) = \sum_{i=1}^{\infty} i p_i = P'(1) \tag{A3}$$

and

$$\mathrm{var}(I) = \left[\sum_{i=1}^{\infty} i^2 p_i\right] - \left(\sum_{i=1}^{\infty} i p_i\right)^2 = P''(1) + P'(1) - [P'(1)]^2, \tag{A4}$$

respectively (see Ref. [5, chapter 11]). By applying L'Hôpital's rule, we have

$$\lim_{\alpha\uparrow 1} \frac{1}{\alpha} \frac{1-\alpha}{1-P(\alpha)} = \frac{1}{P'(1)} = \lim_{\alpha\uparrow 1} \frac{1}{\alpha^2} \frac{1-\alpha^2}{1-P(\alpha^2)}. \tag{A5}$$

As α tends to 1 from below, and using Eqs. (A3) and (A5), the expected equivalent average costs per unit time (Eq. (6)) approach the expected average costs per unit time (Eq. (2))

$$\lim_{\alpha\uparrow 1} \frac{1-\alpha}{\alpha} \lim_{n\to\infty} E(K(n,\alpha)) = \lim_{\alpha\uparrow 1} \frac{C(\alpha)}{\alpha} \frac{1-\alpha}{1-P(\alpha)} = \frac{C(1)}{P'(1)} = \frac{E(c_I)}{E(I)}.$$

As α tends to 1 from below, and using Eqs. (A3)–(A5), the equivalent average variance of the costs per unit time (Eq. (12)) approaches the average variance of the costs per unit time (Eq. (8)):

$$\begin{aligned}
&\lim_{\alpha\uparrow 1} \frac{1-\alpha^2}{\alpha^2} \lim_{n\to\infty} \mathrm{var}(K(n,\alpha)) \\
&= \lim_{\alpha\uparrow 1} \frac{1-\alpha^2}{\alpha^2} \left\{ \frac{2C(\alpha)}{1-P(\alpha)} \frac{C(\alpha^2)}{1-P(\alpha^2)} + \frac{\mathrm{CP}(\alpha)}{1-P(\alpha^2)} - \left[\frac{C(\alpha)}{1-P(\alpha)}\right]^2 \right\} \\
&= \frac{\mathrm{CP}(1)}{P'(1)} + \frac{C(1)}{P'(1)} \lim_{\alpha\uparrow 1} \frac{2C(\alpha^2)[1-P(\alpha)] - C(\alpha)[1-P(\alpha^2)]}{[1-P(\alpha)]^2} \\
&= \frac{\mathrm{CP}(1)}{P'(1)} + \frac{C(1)}{P'(1)} \frac{2C(1)P''(1) + 2C(1)P'(1) - 4C'(1)P'(1)}{2[P'(1)]^2} \\
&= \frac{\mathrm{CP}(1)[P'(1)]^2 + C^2(1)P''(1) + C^2(1)P'(1) - 2C(1)C'(1)P'(1)}{[P'(1)]^3} \\
&= \frac{\{\mathrm{CP}(1) - C^2(1)\}[P'(1)]^2 + \{P''(1) + P'(1) - [P'(1)]^2\}C^2(1)}{[P'(1)]^3} \\
&\quad - \frac{2C(1)P'(1)\{C'(1) - C(1)P'(1)\}}{[P'(1)]^3} \\
&= \frac{\mathrm{var}(c_I)E^2(I) + \mathrm{var}(I)E^2(c_I) - 2E(I)E(c_I)\mathrm{cov}(I,c_I)}{[E(I)]^3}.
\end{aligned} \tag{A6}$$

The third step can be obtained after straightforward, though tedious, algebra; both the numerator and the denominator of the last term of the third equation must be differentiated with respect to α twice in order to be able to successfully apply L'Hôpital's rule. □

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