

# OPTIMAL MAINTENANCE DECISIONS FOR DIKES

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To protect the Dutch polders against flooding, more than 2500 km of dikes have been constructed. Due to settlement, subsoil consolidation, and relative sea-level rise, these dikes slowly sink “away into the sea” and should therefore be heightened regularly (at present, every 50 years). In this respect, one is interested in safe and cost-optimal dike heightenings for which the sum of the initial cost of investment and the future (discounted) cost of maintenance is minimal.

For optimization purposes, a maintenance model has been developed for dikes subject to uncertain crest-level decline. On the basis of engineering knowledge, crest-level decline has been modeled as a monotone stochastic process with expected decline being either linear or nonlinear (i.e., linear after transformation) in time. For both models and for a particular unit time, the increments are distributed according to mixtures of exponentials.

In a case study, the maintenance decision model has been applied to the problem of heightening the Dutch “Oostmolendijk.”

## 1. INTRODUCTION

To protect the Dutch polders against flooding, a network of dikes, dams, and barriers has been constructed. For the largest part, this network consists of dikes at a total length of more than 2500 km. In order to provide for the long-term safety, these dikes have to be maintained on the basis of 5-yearly inspections as laid down in the Dutch Flood Protection Act. Unfortunately, due to a combination of settlement, subsoil consolidation, and relative sea-level rise (denoted by *crest-level decline*), the dikes slowly sink “away into the sea” and should therefore be heightened and strengthened regularly. In this paper, we present a probabilistic model that enables us to determine safe dike heightenings for which the cost of maintenance is minimal.

Given the acceptable probability of failure of a dike section, its crest height is currently determined on the basis of the design water level plus a safety margin (needed to cope with possible wave run-up, gust and squall oscillations, seiches, and crest-level decline during 50 years). The present Dutch dike design prescribes dikes to be heightened every 50 years, which, however, might not be economical. For details on the Dutch flood protection program, see, for example, Vrijling [18].

In this paper, we present a new probabilistic model for determining safe dike heightenings that optimally balance the initial cost of investment against the future cost of maintenance. The basic idea behind our model comes from van Dantzig [13] and differs from the latter in the sense that we regard crest-level decline as a stochastic process rather than as a deterministic number. Moreover, we consider condition-based preventive maintenance (carried out at times determined by 5-yearly dike inspections) rather than time-based preventive maintenance (carried out at predetermined repair times). Furthermore, in contrast with van Dantzig’s model, our model only includes the cost of maintenance, not the cost of possible flooding. van Dantzig’s model was extended by Vrijling and van Beurden [19], who assumed the average rate of crest-level decline to be uncertain.

In essence, van Dantzig optimizes design water levels, whereas we optimize dike heightenings to cope with uncertain crest-level decline when these design water levels are given. The advantage of determining design water levels and dike heightenings separately is that design water levels do not solely result from economic considerations, but also from political considerations.

By combining van Dantzig’s flood model and our maintenance model, determining an optimal dike height on the basis of economic (and/or political) considerations can proceed as follows:

- First, the optimal design water level of a dike section is obtained on the basis of an economic optimization using van Dantzig’s model and/or political considerations. This design water level is optimal in the sense that it balances the initial cost of heightening all dike sections up to the design water level against the future cost of flooding. Note that settlement, subsoil consolidation, and relative sea-level rise are *not included* in this optimal design water level.
- Second, the optimal dike heightening to cope with crest-level decline is obtained on the basis of an economic optimization using the maintenance model

proposed in this paper. This dike heightening is optimal in the sense that it balances the initial cost of heightening a dike section from the design water level up to the desired dike height against the future cost of maintenance. Note that settlement, subsoil consolidation, and relative sea-level rise are *included* in the optimal dike heightening.

- Third, the optimal dike height equals the sum of the optimal design water level and the optimal dike heightening.

To account for possible crest-level decline in a period of 50 years, dike heights are nowadays designed on the basis of average rates of crest-level decline. In order for the stochastic process of crest-level decline to be based on its (uncertain) limiting average rate, we consider it as a generalized gamma process.

A gamma process is a stochastic process with independent nonnegative increments having gamma distributions with given scale parameters and shape parameters proportional to the length of the time interval over which the increments are taken. A generalized gamma process is then defined as a scale mixture of gamma processes, where the scale parameter can be interpreted as the unknown limiting average rate of crest-level decline. Note that the Brownian motion with drift (a stochastic process with stationary independent decrements and increments having a Normal distribution) is not applicable in this context, since we must require that the increments are nonnegative.

A great advantage of modeling deterioration as a generalized gamma process is that we can always find units of time of equal length for which the increments are distributed according to a mixture of exponential probability densities (which facilitates the algebraic manipulations considerably). On the basis of generalized gamma processes, tailor-made decision models have been built and implemented to optimize maintenance of the seabed protection of the Eastern Scheldt barrier, beaches, and berm breakwaters (see van Noortwijk and colleagues [14–17]).

In addition to the generalized gamma process, with expected decline being linear in time, we study a monotone stochastic process with expected decline being nonlinear in time. The latter process has been derived from a physical law which is well accepted by engineers in soil mechanics: the law of settlement and subsoil consolidation of Terzaghi and Koppejan [7]. In doing so, we base our probabilistic model on well-known engineering knowledge, an approach which has recently been proposed by Barlow [1] and Mendel [10].

For the purpose of finding an optimum balance between the initial cost and the future cost, which is the area of life-cycle costing, we can best use the criterion of the expected discounted costs over an unbounded time horizon (for a discussion, see van Noortwijk and Peerbolte [16]). These costs can be determined by applying the discrete renewal theorem, where the renewals are the events at which a dike is heightened.

The paper is organized as follows. The cost of one dike heightening, proportional to the increase in dike volume, is presented in Section 2. In Sections 3 and 4, analytic expressions are derived for the expected discounted costs over an unbounded horizon under linear and nonlinear crest-level decline, respectively. The

maintenance model is applied to the Dutch “Oostmolendijk” in Section 5. Section 6 ends with some conclusions. Necessary theorems can be found in the Appendix.

## 2. THE COST OF ONE DIKE HEIGHTENING

In modeling the maintenance of dikes, we make a distinction between the initial dike heightening and the future dike heightenings due to crest-level decline. The initial dike heightening entails heightening the crest level *and* broadening the base, whereas the future dike heightenings leave the base unchanged (see Fig. 1). Determining the cost of heightening dikes is the subject of study in this section.

For each dike heightening, the cost can be subdivided into the fixed cost  $c_f$  (cost of mobilization and road reconstruction) and the variable cost  $c_v$  (cost per cubic meter of dike volume). By using the schematized cross section in Figure 1, the dike volume is a quadratic function of the dike height  $h$  (in meters) in the following way:

$$v(h) = wlh + \frac{l}{2} \left( \frac{1}{\tan(\varphi)} + \frac{1}{\tan(\omega)} \right) h^2 = v_1 h + v_2 h^2 \quad (1)$$

in cubic meters ( $\text{m}^3$ ), where  $h$  is the crest level,  $w$  is the crest width,  $l$  is the length of the dike section, and  $\varphi$  and  $\omega$  are the angles of the inner slope and the outer slope, respectively.

The costs of initially heightening the dike from  $h_0$  up to  $h$  m and changing the base width accordingly, where  $h_0 < h$ , are simply

$$c_0(h) = c_f + c_v[v(h) - v(h_0)]. \quad (2)$$

van Dantzig [13] approximated Eq. (1) by a linear function of  $h$ , although he acknowledged that his approximation is not valid for large  $h$ : Indeed, the higher the dike, the broader the base that is required.

Similarly, the costs of future heightenings to annul a crest-level decline of  $x$  m, while keeping the base width unchanged, can be written as a linear function of  $x$  (see Fig. 1):

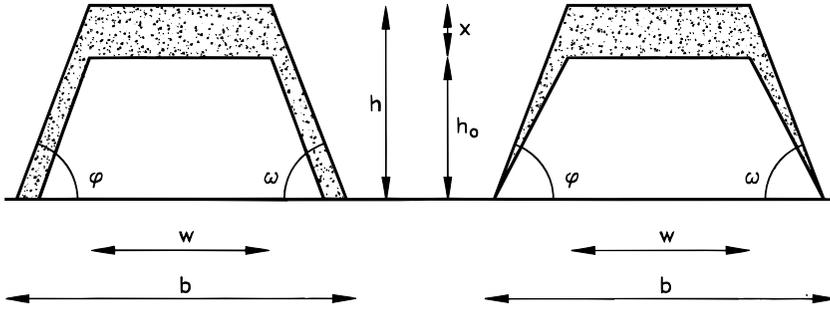
$$c_f + c_v \tilde{v}(h, x) = c_f + c_v \left[ wl + \frac{hl}{2} \left( \frac{1}{\tan(\varphi)} + \frac{1}{\tan(\omega)} \right) \right] x = c_f + c_v \tilde{v}_1(h)x, \quad (3)$$

where  $h$ ,  $w$ ,  $l$ ,  $\varphi$ , and  $\omega$  are defined as in Eq. (1).

## 3. LINEAR CREST-LEVEL DECLINE

### 3.1. The Stochastic Process of Crest-Level Decline

In this subsection, a probabilistic model for the process of crest-level decline is derived based on the unknown limiting average rate. The nondecreasing stochastic process is denoted by  $\{X(t) : t \geq 0\}$ , where  $X(t)$  represents the cumulative maximum amount of crest-level decline at time  $t$  and  $X(0) = 0$  with probability 1. For every uniform time partition in time intervals of length  $\tau > 0$ ,  $D_i(\tau) = X(i\tau) - X([i - 1]\tau) \geq 0$ ,  $i \in \mathbb{N}$ .



**FIGURE 1.** The cross section of a dike before and after heightening: the initial dike heightening from  $h_0$  up to  $h$  m (left) and a future dike heightening of  $x$  m (right).

Furthermore, due to the lack of data, we judge the infinite sequence of increments  $\{D_i(\tau) : i \in \mathbb{N}\}$  to be *exchangeable* (i.e., the order in which the increments occur is irrelevant). In mathematical terms, exchangeability means that the probability density function of the random vector  $(D_1(\tau), \dots, D_n(\tau))$  is invariant under all  $n!$  permutations of the coordinates; that is,

$$p_{D_1(\tau), \dots, D_n(\tau)}(\delta_1, \dots, \delta_n) = p_{D_1(\tau), \dots, D_n(\tau)}(\delta_{\pi(1)}, \dots, \delta_{\pi(n)}), \quad (4)$$

where  $\pi$  is any permutation of  $1, \dots, n$ , for all  $n \in \mathbb{N}$  and  $\tau > 0$ . The notion of exchangeability is weaker than the notion of independence.

In order for a stochastic deterioration process with nonnegative exchangeable increments to be based on the unknown limiting average rate, van Noortwijk et al. [15] have argued that it can best be regarded as a *generalized gamma process*. In fact, the generalized gamma process can be characterized by conditioning on sums of increments, while achieving consistency in the sense that probability distributions of increments and sums of increments belong to the same family of distributions, and by assuming the probability model to be independent of the scale of measurement (i.e., to be a scale mixture).

For the generalized gamma process, the joint probability density function of the increments  $D_1(\tau), \dots, D_n(\tau)$  is given by a mixture of conditionally independent gamma densities:

$$p_{D_1(\tau), \dots, D_n(\tau)}(\delta_1, \dots, \delta_n) = \int_0^\infty \prod_{i=1}^n \frac{\delta_i^{a\tau-1}}{\Gamma(a\tau)} \left(\frac{a\tau}{\theta}\right)^{a\tau} \exp\left(-\frac{a\tau\delta_i}{\theta}\right) dP_{\Theta(\tau)}(\theta) \quad (5)$$

for some constant  $a > 0$  with

$$\begin{aligned} E(X(n\tau)) &= E(n\Theta(\tau)), \\ \text{Var}(X(n\tau)) &= \left(1 + \frac{1}{na\tau}\right) E([n\Theta(\tau)]^2) - [E(n\Theta(\tau))]^2 \end{aligned} \quad (6)$$

for all  $\tau > 0$ , provided the first and the second moment of the probability distribution of  $\Theta(\tau)$  exist. By the strong law of large numbers for exchangeable random quantities, the probability distribution  $P_{\Theta(\tau)}$  of the random quantity  $\Theta(\tau)$  represents the uncertainty in the unknown limiting average amount of crest-level decline per time interval of length  $\tau$ :  $\lim_{n \rightarrow \infty} [(\sum_{i=1}^n D_i(\tau))/n]$ .

A useful property of the generalized gamma process is that the mixture of gamma densities in Eq. (5) transforms into a mixture of exponential densities if  $\tau = a^{-1}$ :

$$p_{D_1(a^{-1}), \dots, D_n(a^{-1})}(\delta_1, \dots, \delta_n) = \int_0^\infty \prod_{i=1}^n \frac{1}{\theta} \exp\left(-\frac{\delta_i}{\theta}\right) dP_\Theta(\theta) = f_n\left(\sum_{i=1}^n \delta_i\right), \quad (7)$$

where  $(\delta_1, \dots, \delta_n) \in \mathbb{R}_+^n$  and zero otherwise, for  $\mathbb{R}_+ = [0, \infty)$ . The infinite sequence of random quantities  $\{D_i(a^{-1}) : i \in \mathbb{N}\}$  is said to be  $l_1$ -isotropic (or  $l_1$ -norm symmetric), since its distribution can be written as a function of the  $l_1$ -norm (see Misiewicz and Cooke [11]). Given the value of the  $l_1$ -norm (or the average deterioration), this means that the decision-maker is indifferent to the way this norm is obtained. In other words, all combinations leading to the same  $l_1$ -norm have the same degree of belief for our decision-maker (for details, see Barlow and Mendel [2] and van Noortwijk et al. [14]).

The unit time for which the increments of crest-level decline are  $l_1$ -isotropic can be obtained, for example, by specifying the variance of the generalized gamma process in Eq. (6). For fixed  $\tau > 0$ , the smaller the unit-time length for which the increments are  $l_1$ -isotropic; that is, the smaller  $\Delta = a^{-1}$ , the more deterministic the deterioration process. As we shall see in Section 3.2, for this unit-time length, denoted by  $\Delta = a^{-1}$ , many probabilistic properties of the stochastic deterioration process, like the probability of exceedence of a failure level, can be expressed in explicit form conditional on the limiting average.

In conclusion, we advocate regarding the stochastic process of crest-level decline as a generalized gamma process with probability distribution on the limiting average rate of crest-level decline. To keep the mathematics of the decision model tractable, we impose the property of *posterior linearity* introduced by Diaconis and Ylvisaker [4] [i.e.,  $E(X(2\tau)|D_1(\tau) = \delta_1) = c_1\delta_1 + c_2$  for some constants  $c_1, c_2 > 0$  and  $\tau > 0$ ]. Note that, due to exchangeability, before observing  $D_1$ ,  $E(D_2) = E(D_1)$ . If posterior linearity holds, then the mixing distribution in Eq. (5) is an inverted gamma distribution (see Diaconis and Ylvisaker [4]). The mathematical tractability is especially useful if one wants to update the prior distribution of the limiting average rate of crest-level decline with actual observations. In fact, using Bayes' theorem, the posterior distribution is also an inverted gamma distribution.

From now on, we consider increments of crest-level decline that are  $l_1$ -isotropic with respect to the units of time  $\{([i-1]\Delta, i\Delta) : i \in \mathbb{N}\}$ . For notational convenience, let  $D_i = D_i(\Delta)$  and  $X_n = \sum_{i=1}^n D_i$  for all  $n \in \mathbb{N}$ , and let  $\Theta$  represent the uncertainty in the limiting average rate of crest-level decline,  $\lim_{n \rightarrow \infty} [(\sum_{i=1}^n D_i)/n]$ , with probability density function

$$\text{Ig}(\theta|\nu, \mu) = [\mu^\nu/\Gamma(\nu)]\theta^{-(\nu+1)}\exp(-\mu/\theta)I_{(0, \infty)}(\theta).$$

The statistic  $[n, \sum_{i=1}^n D_i]$  is sufficient for  $\Theta$ . The mean and the variance of  $X_n$  are  $E(X_n) = nE(\Theta)$  and  $\text{Var}(X_n) = E(n\Theta^2) + \text{Var}(n\Theta)$ , respectively.

### 3.2. The Expected Discounted Cost of Dike Heightening

The 1995 Dutch Flood Protection Act prescribes the dikes be inspected every 5 years. For this reason, we assume the dike section to be periodically inspected at times  $\{jk\Delta : j \in \mathbb{N}\}$  for fixed  $k \in \mathbb{N}$ , where  $k\Delta = 5$  years. Each heightening brings the dike section back into its ‘‘as good as new state.’’ Therefore, we may consider the maintenance process as a renewal process, where each renewal cycle ends at an inspection time  $jk\Delta$  when the inspection reveals that the dike section should be heightened (for some  $j \in \mathbb{N}$ ). We assume that inspection of the dike takes negligible time and does not degrade the dike.

In this paper, the failure level  $s$  is defined as the design water level plus a small safety margin needed to cope with wave run-up, oscillations, seiches, and crest-level decline during an inspection interval of 5 years (not 50 years as in the present dike design). Failure is then defined as the event in which a dike height drops below the failure level  $s$ ; it can only be noted through inspection. When inspection reveals that the crest level of a dike section is lower than the failure level  $s$ , it should be heightened. Note that a failure need not imply a collapse: A failed dike section only collapses when the applied water level exceeds the actual dike height.

Let  $y = h - s$  represent the decision to choose the dike to be  $y$  m higher than its failure level  $s$  and let the times at which the failure level is first crossed be conditionally independent random quantities having a discrete probability function  $p_i(\theta, y)$  and associated repair cost  $c_i(\theta, y)$ , with respect to the units of time  $\{([i - 1]\Delta, i\Delta] : i \in \mathbb{N}\}$ , when the limiting average rate of crest-level decline is  $\theta$  and the decision-maker chooses the dike to be  $y$  m higher than its failure level  $s$ .

To obtain optimal maintenance decisions in uncertainty, we use statistical decision theory (see DeGroot [3, Chap. 8]). Let  $L_\alpha(\theta, y)$  be the (monetary) loss when the decision-maker chooses the dike to be  $y$  m higher than its failure level  $s$  and the limiting average rate of crest-level decline is given by  $\theta$ . The decision-maker can best choose decision  $y^*$  for which the expected loss is minimal. A decision  $y^*$  is called an *optimal decision* or a *Bayes optimal decision* when

$$E(L_\alpha(\Theta, y^*)) = \min_{y \in (h_0 - s, \infty)} E(L_\alpha(\Theta, y)). \quad (8)$$

Such optimal maintenance decisions are also known as Bayes adaptive maintenance policies (see, e.g., McCall [8]): They can be revised in the light of new observations by replacing the prior distribution of the limiting average rate of crest-level decline with the posterior distribution.

Because determining optimal dike heightenings actually means balancing the initial cost against the future cost, the criterion of discounted costs can best serve as a loss function (for a discussion, see van Noortwijk and Peerbolte [16]). The expected *discounted costs over an unbounded horizon* can be determined by summing

the expected discounted values of the costs over an unbounded horizon, where the discounted value of the cost  $c_n$  in unit time  $n$  is defined to be  $\alpha^n c_n$  with discount factor  $\alpha = [1 + (r/100)]^{-1}$  and discount rate  $r\%$  ( $r > 0$ ):

$$L_\alpha(\theta, y) = c_0(s + y) + \frac{\sum_{j=1}^{\infty} \alpha^{jk} \sum_{i=(j-1)k+1}^{jk} c_i(\theta, y) p_i(\theta, y)}{1 - \sum_{j=1}^{\infty} \alpha^{jk} \sum_{i=(j-1)k+1}^{jk} p_i(\theta, y)}, \quad (9)$$

where the initial cost  $c_0(s + y)$  stems from Eq. (2). Equation (9) follows from the discrete renewal theorem (see van Noortwijk and Peerbolte [16]). Note that we cannot use the criterion of the expected average costs per unit time, because the contribution of the initial cost to the average costs would be completely ignored [i.e.,  $\lim_{n \rightarrow \infty} c_0(s + y)/n = 0$ ]. The expected nondiscounted costs over a bounded horizon  $(0, n]$  cannot be applied either, because the dike-heightening decisions would then depend on the length of the time horizon  $n$ . The criterion of the expected discounted costs over an unbounded horizon has also been used successfully by van Dantzig [13].

The mixture of exponential densities of Eq. (7) enables us to express various probabilistic properties in explicit form when  $\theta$  is given. For the purpose of optimal dike heightening, two probabilistic properties are needed: (i) the probability of exceedence of the failure level in unit time  $i$  and (ii) the expected cost of dike heightening due to exceedence of the failure level in unit time  $i$ . These two properties are derived in Theorem 1 (see the Appendix).

First, the conditional probability of exceedence of the failure level in unit time  $i$ , when the limiting average crest-level decline is  $\theta$  and when the decision-maker chooses the dike to be  $y$  m higher than its failure level  $s$ , can be written as

$$p_i(\theta, y) = \Pr\{X_{i-1} \leq y, X_i > y | \theta\} = \frac{1}{(i-1)!} \left(\frac{y}{\theta}\right)^{i-1} \exp\left(-\frac{y}{\theta}\right) \quad (10)$$

for  $i = 1, 2, \dots$ , and  $\theta, y > 0$ . This discrete probability function can be recognized as the Poisson distribution with mean lifetime  $1 + (y/\theta)$  and variance  $y/\theta$ .

Second, the expected costs of dike heightening due to exceedence of the failure level in unit time  $i$ , when the limiting average crest-level decline is  $\theta$  and when the decision-maker chooses the dike to be  $y$  m higher than its failure level  $s$ , can be written [using Eq. (3)] as

$$\begin{aligned} E([c_f + c_v \tilde{v}_1(s + y) X_{jk}] I_{[0, y]}(X_{i-1}) I_{(y, \infty)}(X_i) | \theta) \\ = \{c_f + c_v \tilde{v}_1(s + y)[y + (jk - i + 1)\theta]\} p_i(\theta, y) = c_i(\theta, y) p_i(\theta, y) \end{aligned} \quad (11)$$

for  $i = (j - 1)k + 1, \dots, jk$ , where  $j, k = 1, 2, \dots$ , and  $\theta, y > 0$ .

In conclusion, the expected discounted costs over an unbounded horizon can be obtained by substituting Eqs. (10) and (11) into Eq. (9) and by taking the expectation

with respect to the probability distribution of  $\Theta$ . The optimal dike heightening follows from Eq. (8) (for an example, see Section 5).

## 4. NONLINEAR CREST-LEVEL DECLINE

### 4.1. The Stochastic Process of Crest-Level Decline

Although the assumption of expected crest-level decline being linear in time is quite reasonable when data are lacking, the question arises of how to proceed when data give evidence of an expected decline being nonlinear in time. In order to investigate the sensitivity of the optimal dike heightening to different rates of crest-level decline, we also consider stochastic processes with nonnegative, but nonexchangeable, increments.

Engineering knowledge suggests the expected crest-level decline to be a logarithmic function of time. Recall that the process of crest-level decline is a combination of settlement, subsoil consolidation, and relative sea-level rise.

Settlement and subsoil consolidation has thoroughly been studied by Koppejan [7]. With empirical experiments, he showed that a large time  $t$  after increasing the stress from  $p'_1$  to  $p'_2$ , the thickness of a compressed layer of sand or clay behaves according to the so-called formula of Terzaghi and Koppejan:

$$z(t) = z_0 \left( \frac{1}{C_p} + \frac{1}{C_s} \frac{\ln(t)}{\ln(10)} \right) \ln \left( \frac{p'_2}{p'_1} \right), \quad (12)$$

where

$z_0$  = the initial thickness of the layer (m),

$z(t)$  = the thickness of the compressed layer at time  $t$  (m),

$C_p$  = primary compression constant,

$C_s$  = secondary compression constant,

$t$  = time (s),

$p'_1$  = initial stress (N/m<sup>2</sup>),

$p'_2$  = increased stress (N/m<sup>2</sup>).

Relative sea-level rise has probably the following causes: melting of glaciers, changes in the Greenland and the Antarctic ice caps, thermal expansion of the oceans, and, for The Netherlands, readjustment of the earth crust due to the melting away of the Fennoscandian ice cap about 10,000 years ago. The estimates of the relative sea-level rise for the next century vary between 20 and 120 cm, with a best estimate of 60 cm (see Houghton et al. [5], van Dantzig [13], and Vrijling and van Beurden [19]).

In order to preserve the mathematical tractability in determining the expected discounted costs, when transforming linear decline into nonlinear decline, we link

up with a stochastic process having  $l_1$ -isotropic increments in the following way. Let us consider an infinite sequence of random quantities  $\{D_i : i \in \mathbb{N}\}$  that is transformed  $l_1$ -isotropic in the sense that the probability density function of the random vector  $(D_1, \dots, D_n)$  can be written as a function of the statistic  $\sum_{i=1}^n D_i/\beta_i$  for all  $n \in \mathbb{N}$ :

$$p_{D_1, \dots, D_n}(\delta_1, \dots, \delta_n) = \int_0^\infty \prod_{i=1}^n \frac{1}{\beta_i \theta} \exp\left(-\frac{\delta_i}{\beta_i \theta}\right) dP_\Theta(\theta) = f_n\left(\sum_{i=1}^n \delta_i/\beta_i\right) \quad (13)$$

for  $(\delta_1, \dots, \delta_n) \in \mathbb{R}_+^n$  and zero otherwise, where  $\beta_i > 0$  for  $i = 1, \dots, n$  and  $\beta_i \neq \beta_j$  unless  $i = j$ . The conditional cumulative distribution function of the sum  $X_n = \sum_{i=1}^n D_i$ , when  $\theta$  is given, is called the general Erlang or general gamma distribution:

$$\Pr\{X_n \leq x|\theta\} = 1 - \sum_{i=1}^n \frac{1}{\prod_{j=1, j \neq i}^n [1 - \beta_j/\beta_i]} \exp\left(-\frac{x}{\beta_i \theta}\right). \quad (14)$$

This probability distribution has been used in theories of radioactive decay, queuing, reliability, and psychology (see, e.g., Jensen [6] and McGill and Gibbon [9]).

In order that the expected deterioration be a logarithmic function of time (for large values of time), we set  $\beta_i = a/(b + i - 1)$  for all  $i \in \mathbb{N}$ , where  $a, b > 0$ . For large  $n$  [viz. for large time  $t$  in Eq. (12)], the expected crest-level decline conditional on  $\theta$  can then be written as

$$E(X_n|\theta) = \left[ \sum_{i=1}^n a/(b + i - 1) \right] \theta \approx a[\ln(n) - \psi(b)]\theta \quad (15)$$

as  $n \rightarrow \infty$ . The so-called *digamma function* or *Euler's psi function*  $\psi(b)$  can be negative<sup>1</sup> (see Nielsen [12, Sect. 5]). In specifying the parameters  $a$  and  $b$ , settlement, subsoil consolidation, and relative sea-level rise must be incorporated.

The mixture of conditionally independent exponentials with different means, Eq. (13), converges to the mixture of conditionally independent exponentials with equal means, Eq. (7), as  $\beta_i \rightarrow 1$  for all  $i = 1, \dots, n$ . Therefore, it is convenient that the increments  $D_i$ ,  $i \in \mathbb{N}$ , are defined with respect to the same units of time (of length  $\Delta$ ) as for the  $l_1$ -isotropic increments in Section 3.1. The mean and the variance of  $X_n$  are  $E(X_n) = \sum_{i=1}^n \beta_i E(\Theta)$  and  $\text{Var}(X_n) = E(\text{Var}(X_n|\Theta)) + \text{Var}(E(X_n|\Theta))$ , respectively.

## 4.2. The Expected Discounted Costs of Dike Heightening

In a similar way as was done for linear expected crest-level decline (Section 3), we can determine two important probabilistic properties for nonlinear expected crest-level decline: (i) the probability of exceedence of the failure level in unit time  $i$  and (ii) the expected costs of dike heightening due to exceedence of the failure level in

<sup>1</sup> For the digamma function, which is defined by  $\psi(x) = \Gamma'(x)/\Gamma(x)$ , we have  $\psi(x) \leq 0$  for  $0 < x \leq x_0$  and  $\psi(x) > 0$  for  $x > x_0$ , where  $x_0 \approx 1.462$ .

unit time  $i$ . Conditional on the limiting weighted average  $\Theta$ , these two properties are derived in Theorem 3 (see the Appendix).

First, the conditional probability of exceedence of the failure level in unit time  $i$ , when the limiting weighted average is  $\theta$  and when the decision-maker chooses the dike to be  $y$  m higher than its failure level  $s$ , can be written as

$$p_i(\theta, y) = \sum_{m=1}^i \frac{\beta_i/\beta_m}{\prod_{h=1, h \neq m}^i [1 - \beta_h/\beta_m]} \exp\left(-\frac{y}{\beta_m \theta}\right) \quad (16)$$

for  $i = 1, 2, \dots$ , and  $\theta, y > 0$ . When  $\beta_i = a/(b + i - 1)$  for all  $i \in \mathbb{N}$ , this discrete probability function simplifies to the negative binomial distribution with parameters  $1 - \exp[-y/(a\theta)]$  and  $b$  (see Jensen [6]):

$$p_i(\theta, y) = \binom{b + i - 1}{i - 1} \left[1 - \exp\left(-\frac{y}{a\theta}\right)\right]^{i-1} \left[\exp\left(-\frac{y}{a\theta}\right)\right]^b, \quad (17)$$

for  $i = 1, 2, \dots$ , and  $\theta, y > 0$ , with mean lifetime  $1 + b\{\exp[y/(a\theta)] - 1\}$  and variance  $b \exp[y/(a\theta)]\{\exp[y/(a\theta)] - 1\}$ .

Second, the expected costs of dike heightening due to exceedence of the failure level in unit time  $i$ , when the limiting weighted average is  $\theta$  and when the decision-maker chooses the dike to be  $y$  m higher than its failure level  $s$ , can be written as [by using Eq. (3), and Theorems 3 and 2 in the Appendix]

$$\begin{aligned} & E\left[c_f + c_v \tilde{v}_1(s + y) X_{jk}\right] I_{[0, y]}(X_{i-1}) I_{(y, \infty)}(X_i) | \theta \\ &= \left[ c_f + c_v \tilde{v}_1(s + y) \left( y + \sum_{h=i}^{jk} \beta_h \theta \right) \right] p_i(\theta, y) = c_i(\theta, y) p_i(\theta, y) \end{aligned} \quad (18)$$

for  $i = (j - 1)k + 1, \dots, jk$ , where  $j, k = 1, 2, \dots$ , and  $\theta, y > 0$ .

In conclusion, the expected discounted costs over an unbounded horizon can be obtained by substituting Eqs. (17) and (18) into Eq. (9) and by taking the expectation with respect to  $\text{Ig}(\theta | \nu, \mu)$ , the inverted gamma distribution. The optimal dike heightening follows from Eq. (8).

## 5. CASE STUDY: THE DUTCH "OOSTMOLENDIJK"

The above decision model for optimal dike heightening has been applied to the "Oostmolendijk," a dike section with a length of 1000 m in the west of The Netherlands. The "Oostmolendijk" is located between the towns of Ridderkerk and Hendrik-Ido-Ambacht, along the river Noord, and belongs to the dike ring IJsselmonde. In the last decades, the "Oostmolendijk" has been subject to extreme settlement and subsoil consolidation: about 0.60 m in the period 1969–1981 and about 0.15 m in the period 1981–1989 (unfortunately, only these two data points are available). In 1969, its crest-level height was about 5.20 m +NAP (normal Amsterdam level), whereas the last dike heightening, in 1991, resulted in a crest-level height of 4.90 m +NAP (the difference is due to reduction of the design water level by the

Maeslant storm-surge barrier in the “Nieuwe Waterweg”). With respect to the ground level being 1 m +NAP, the dike height  $h$  was 3.90 m (see Fig. 1).

For obtaining an optimal dike heightening for the “Oostmolendijk,” we use the parameters in Table 1. The unit time for which the increments of crest-level decline are distributed as mixtures of exponentials ( $\Delta = 5/3$  year) has been determined by specifying the variance of the generalized gamma process in Eq. (6). The probability density function of  $\Theta$ , the limiting average rate of crest-level decline per unit time, has been obtained by assessing its 5th and 95th percentiles (i.e.,  $\theta_{0.05}$  and  $\theta_{0.95}$ , respectively, in Table 1). The expected crest-level decline in a period of 50 years is 1.30 m: 1.00 m is due to settlement and subsoil consolidation and 0.30 m is due to relative sea-level rise.

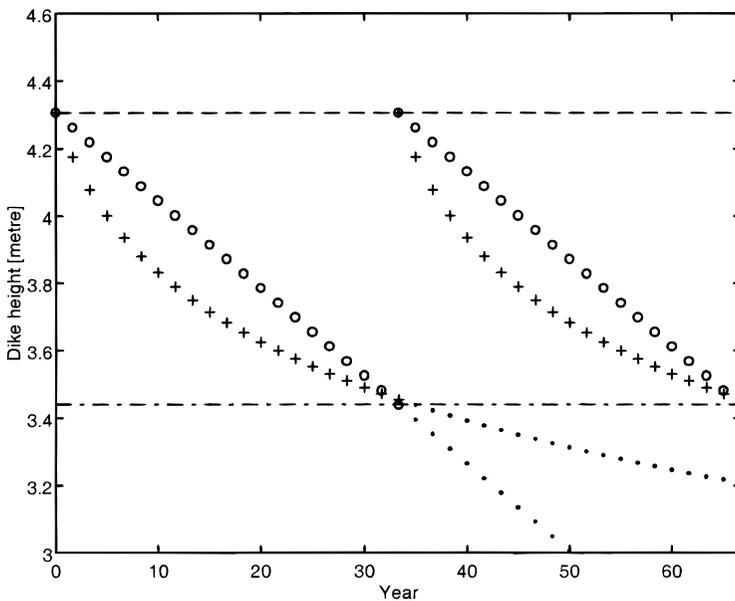
**TABLE 1.** Parameters of the Dike Heightening Model for the Dutch “Oostmolendijk”

Parameter	Description	Value	Dimension
$\Delta$	Unit time	5/3	Year
$k$	Inspection-interval length	3	Unit time
$k\Delta$	Inspection-interval length	5	Year
$r$	Discount rate per year	5	%
$\alpha$	Discount factor per unit time	0.9219	
$c_f$	Fixed cost	$1.8 \times 10^6$	Dfl
$c_v$	Variable cost	30	Dfl/m <sup>3</sup>
$\Theta$	Limiting average crest-level decline	(0, $\infty$ )	m/unit time
$\theta_{0.05}$	5th percentile average crest-level decline	0.033	m/unit time
$\theta_{0.50}$	50th percentile average crest-level decline	0.043	m/unit time
$\theta_{0.95}$	95th percentile average crest-level decline	0.057	m/unit time
$\nu$	Shape parameter of $\text{Ig}(\theta \nu, \mu)$	35.86	
$\mu$	Scale parameter of $\text{Ig}(\theta \nu, \mu)$	1.511	
$E(\Theta)$	Mean of average crest-level decline	0.043	m/unit time
$E(\Theta/\Delta)$	Mean of average crest-level decline	0.026	m/year
$\text{Var}(\Theta)$	Variance of average crest-level decline	$5.5 \times 10^{-5}$	(m/unit time) <sup>2</sup>
$a$	Parameter nonlinear crest-level decline	8.95	
$b$	Parameter nonlinear crest-level decline	3	
$h_0$	Crest-level height before heightening	3.44	m
$h$	Crest-level height of the dike section	( $h_0, \infty$ )	m
$l$	Length of the dike section	1000	m
$w$	Crest width of the dike section	7	m
$s$	Failure level of the dike section	3.44	m
$y$	$h - s$	(0, $\infty$ )	m
—	Ground level (or terrain level)	1	m +NAP
$\varphi$	Angle of the inner slope (1:3)	0.32	radians
$\omega$	Angle of the outer slope (1:3)	0.32	radians

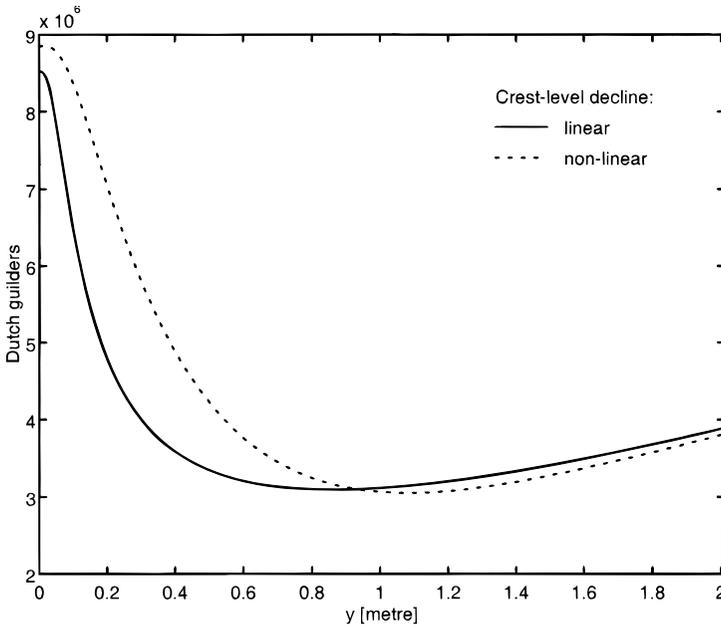
The present Dutch dike design results in a design water level of 3.30 m and a dike heightening to cope with crest-level decline of 1.30 m. Using the formulas in van Noortwijk et al. [14], we choose the failure level such that the probability of exceeding the design water level during an inspection interval of 5 years, when the failure level has not been crossed at the previous inspection, to be at most 0.1 per year; that is, we choose  $s = 3.30 + 0.14 = 3.44$  m. Note that the value of the failure level  $s$  does not affect the overall result of the optimization.

The expected crest-level decline with sums of increments being linear or nonlinear in time under a condition-based maintenance strategy are displayed in Figure 2; when a 5-yearly inspection reveals that the dike section has failed, it is heightened up to 4.30 m. When a larger dike heightening is chosen, the expected deterioration curves in Figure 2 are shifted upward accordingly.

For economic reasons, the decision-maker can best choose a dike heightening  $y$  whose expected discounted costs over an unbounded horizon,  $E(L_\alpha(\Theta, y))$ , are minimal. In Figure 3, the expected discounted costs over an unbounded horizon are shown as a function of  $y$ , for expected crest-level decline being linear and nonlinear in time. The optimal decision, satisfying Eq. (8) under *linear* crest-level decline, is  $y^* = 0.86$  m or, equivalently,  $h^* = 4.30$  m, with expected discounted costs over an unbounded horizon of  $3.09 \times 10^6$  Dutch guilders; the corresponding mean lifetime



**FIGURE 2.** The expected dike height in the event of expected crest-level decline being linear (o) and nonlinear (+) in time; each dike heightening brings the crest level back to  $h = 4.30$  m.

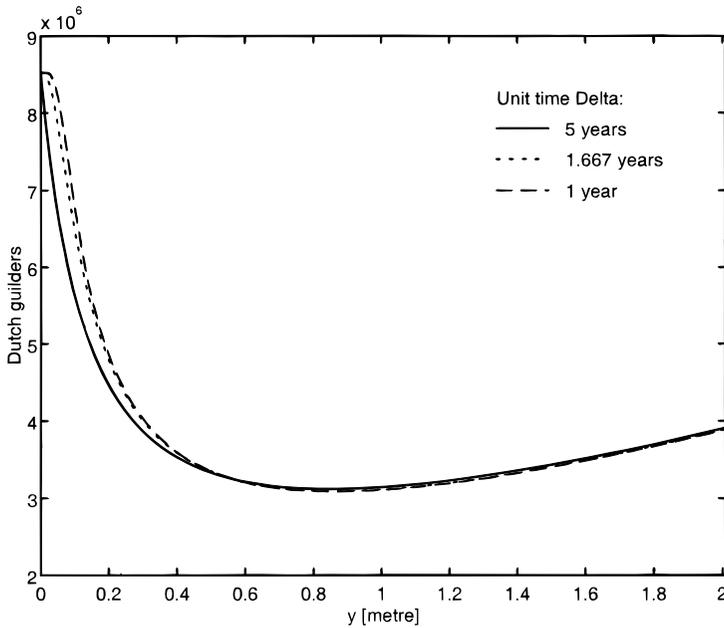


**FIGURE 3.** The expected discounted costs of dike heightening over an unbounded time horizon in the event of expected crest-level decline being linear and nonlinear in time for the Dutch “Oostmolendijk.”

is 36 years. The optimal decision, satisfying Eq. (8) under *nonlinear* crest-level decline, is  $y^* = 1.08$  m or, equivalently,  $h^* = 4.52$  m, with expected discounted costs over an unbounded horizon of  $3.05 \times 10^6$  Dutch guilders; the corresponding mean lifetime is 96 years. The main reason the optimal dike heightening is larger for nonlinear decline than for linear decline is that the variable cost of dike heightening depends on the rate of crest-level decline (which is smaller in the event of nonlinear decline after exceeding the failure level: see Fig. 2). In Figure 3, it can be seen that the larger the dike heightening  $y$ , the higher the initial cost of investment, but the lower the future discounted cost of dike heightening.

As there are only two data points available covering a period of 20 years, it is not recommended to extrapolate the expected crest-level decline over a very large period (such as the mean lifetime of 96 years in the event of optimal dike heightening under nonlinear crest-level decline). Therefore, we suggest choosing the optimal dike heightening decision under linear crest-level decline (i.e.,  $y^* = 0.86$  m) rather than to choose the present heightening decision  $y = 1.30 - 0.14 = 1.16$  m. As soon as more data become available, the parameters of the decision model under nonlinear crest-level decline can be adjusted and the nonlinear model can be used.

The sensitivity of the optimal dike heightening under linear crest-level decline to the choice of the unit time  $\Delta$  is investigated in Figure 4:  $y^*$  hardly depends on  $\Delta$ .

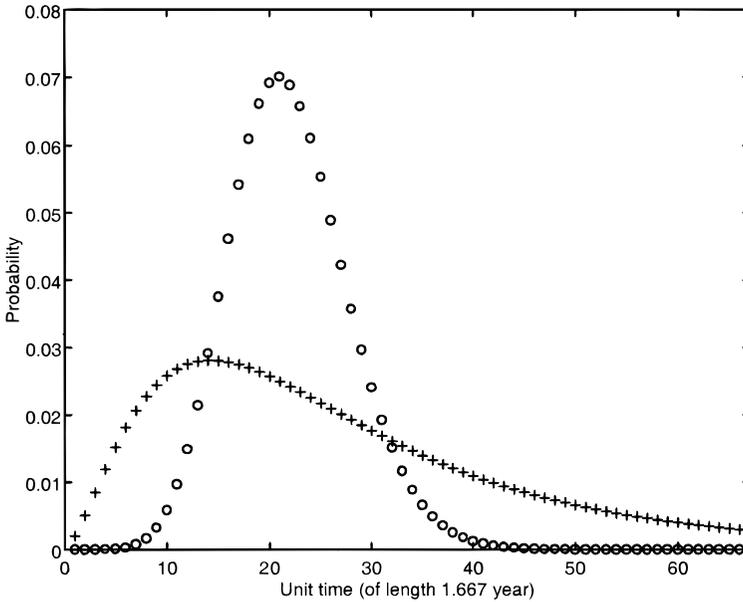


**FIGURE 4.** The expected discounted costs of dike heightening over an unbounded time horizon for different units of time  $\Delta$  in the event of expected crest-level decline being linear in time.

Furthermore, from Table 2, we see that the smaller the expected average rate of linear crest-level decline, the smaller the optimal dike heightening  $y^*$  and the larger the mean time between two dike heightenings. Note that the average rate of crest-level decline in The Netherlands is about 0.5–0.7 cm/year in the lower river area and about 0.3–0.5 cm/year in the upper river area. For the “Oostmolendijk,” this average rate is about 2.6 cm/year.

**TABLE 2.** Optimal Dike Heightenings and the Corresponding Mean Times Between Dike Heightenings for Different Expected Average Rates of Linear Crest-Level Decline

	$E(\Theta/\Delta)$ ( $10^{-2}$ m/year)				
	0.50	1.00	1.50	2.00	2.60
Optimal dike heightening $y^*$ (m)	0.31	0.49	0.63	0.74	0.86
Mean time to dike heightening (years)	84	71	49	42	38



**FIGURE 5.** The probability of failure per unit time in the event of expected crest-level decline being linear (o) and nonlinear (+) in time when  $y = 0.86$  m and  $h = s + y = 4.30$  m at time 0.

When  $y = 0.86$  m at time 0, the expected probabilities of failure per unit time can be determined by integrating Eqs. (10) and (17) over  $\Theta$ . These discrete probability functions are shown in Figure 5. The mean time between two dike heightenings is 23 units of time (38 years) for linear decline and 30 units of time (51 years) for nonlinear decline (while taking into account that dike heightenings can only take place at times of inspection).

## 6. CONCLUSIONS

In this paper, we have presented a decision model for determining dike heightenings that optimally balance the initial cost of investment against the future cost of maintenance. As decision criterion, we have used the expected discounted costs over an unbounded time horizon. An important starting point is the probability distribution of the rate of crest-level decline (a combination of settlement, subsoil consolidation, and relative sea-level rise).

We have investigated two types of monotone crest-level decline: expected decline being linear in time and expected decline being nonlinear in time. For linear decline, we have regarded the deterioration process as a generalized gamma process. For this process, we can always find a uniform time partition such that the joint

probability density function of the increments is a mixture of conditionally independent exponential densities with equal means. A great advantage of exponentially distributed increments is that the expected discounted costs over an unbounded horizon can be expressed in explicit form when the average rate of crest-level decline is given (which facilitates the algebraic manipulations considerably). For nonlinear, strictly monotone decline, we have similarly regarded the joint probability density function of the increments as a mixture of conditionally independent exponential densities with different means. Which model to use depends on the deterioration data.

With respect to the case study on the Dutch “Oostmolendijk,” we can conclude that the value of the optimal dike heightening is sensitive to the rate of crest-level decline (also whether being linear or nonlinear in time), but it is insensitive to the unit time for which the increments are distributed according to a mixture of exponential densities.

The maintenance models that are presented in this paper have the following advantages: They enable optimal dike heightening decisions to be determined under uncertainty, they estimate how much money is needed for the future maintenance of dikes, they do not assume that dikes may rise (as in the case of the Brownian motion with drift model), they are based on random quantities that can be observed (viz. increments of crest-level decline), and they can be expressed in explicit form when the limiting average rate of crest-level decline is given.

### *Acknowledgments*

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## APPENDIX

**THEOREM 1:** *Suppose the infinite sequence of random quantities  $\{D_i; i \in \mathbb{N}\}$  is  $l_1$ -isotropic and  $X_n = \sum_{i=1}^n D_i$  for all  $n \in \mathbb{N}$ , then*

$$\begin{aligned}
 & E([X_j]^m I_{[0,y]}(X_{n-1}) I_{(y,\infty)}(X_n) | \theta) \\
 &= \left\{ \sum_{i=0}^m \frac{m!}{(m-i)!} \binom{j-n+i}{j-n} y^{m-i} \theta^i \right\} \frac{1}{(n-1)!} \left( \frac{y}{\theta} \right)^{n-1} \exp\left(-\frac{y}{\theta}\right), \quad (19)
 \end{aligned}$$

for  $j, n = 1, 2, \dots, j \geq n, m = 0, 1, 2, \dots, y \in (0, \infty)$ , where  $I_A(x) = 1$  if  $x \in A$  and  $I_A(x) = 0$  if  $x \notin A$ .

**PROOF:** Since  $X_n = \sum_{i=1}^n D_i$  for all  $n \in \mathbb{N}$ , it follows that the integration bounds are determined by  $X_j \geq X_{j-1} \geq \dots \geq X_1 \geq 0$ . Moreover,  $X_{n-1} \leq y$  and  $X_n > y$ , where  $n \leq j$ , and the Jacobian equals 1. Hence, we may write

$$\begin{aligned}
 & E([X_j]^m I_{[0,y]}(X_{n-1}) I_{(y,\infty)}(X_n) | \theta) \\
 &= \int_{x_j=y}^{\infty} \int_{x_{j-1}=y}^{x_j} \dots \int_{x_n=y}^{x_{n+1}} \int_{x_{n-1}=0}^{x_n} \dots \int_{x_2=0}^{x_3} \int_{x_1=0}^{x_2} \frac{x_j^m}{\theta^j} \exp\left(-\frac{x_j}{\theta}\right) dx_1 \dots dx_j. \quad (20)
 \end{aligned}$$

This multiple integral can be solved in the following way. The Dirichlet integral gives

$$\int_{x_{n-1}=0}^y \cdots \int_{x_2=0}^{x_3} \int_{x_1=0}^{x_2} 1 \, dx_1 \cdots dx_{n-1} = \frac{y^{n-1}}{(n-1)!} \quad (21)$$

and

$$\int_{x_{j-1}=y}^{x_j} \cdots \int_{x_n=y}^{x_{n+1}} 1 \, dx_n \cdots dx_{j-1} = \frac{[x_j - y]^{j-n}}{(j-n)!}. \quad (22)$$

By applying the transformation  $t = (x_j - y)/\theta$  and using the binomial formula and the gamma function, we obtain

$$\begin{aligned} & \int_{x_j=y}^{\infty} x_j^m [x_j - y]^{j-n} \exp\left(-\frac{x_j}{\theta}\right) dx_j \\ &= \sum_{i=0}^m \binom{m}{i} y^{m-i} \theta^{i+j-n+1} \Gamma(i+j-n+1) \exp\left(-\frac{y}{\theta}\right). \end{aligned} \quad (23)$$

Finally, combining Eqs. (20)–(23) proves the theorem. ■

**THEOREM 2:** Let  $\beta_i > 0$  for  $i = 1, \dots, n$  and  $\beta_i \neq \beta_j$  unless  $i = j$ ,  $n \in \mathbb{N}$ , then

$$\sum_{i=n}^j \frac{\beta_i}{\prod_{h=n, h \neq i}^j [1 - \beta_h/\beta_i]} = \sum_{i=n}^j \beta_i. \quad (24)$$

**PROOF:** Suppose that  $D_n, \dots, D_j$  are independent random quantities and that  $D_i$  has an exponential distribution with mean  $\beta_i\theta$ ,  $i = n, \dots, j$ . On the one hand, we have simply  $\sum_{i=n}^j E(D_i|\theta) = \sum_{i=n}^j \beta_i\theta$ . On the other hand, we can use Eq. (14) to write

$$E\left(\sum_{i=n}^j D_i|\theta\right) = \int_{x=0}^{\infty} \Pr\left\{\sum_{i=n}^j D_i > x|\theta\right\} dx = \sum_{i=n}^j \frac{\beta_i\theta}{\prod_{h=n, h \neq i}^j [1 - \beta_h/\beta_i]}. \quad \blacksquare$$

**THEOREM 3:** Suppose the random quantities  $\{D_i : i \in \mathbb{N}\}$  are exponentially distributed with mean  $\beta_i\theta$ , where  $\beta_i \neq \beta_j$  unless  $i = j$ , and conditionally independent when  $\theta$  is given. Let  $X_n = \sum_{i=1}^n D_i$  for all  $n \in \mathbb{N}$ , then

$$\begin{aligned} & E([X_j]^m I_{[0,y]}(X_{n-1}) I_{(y,\infty)}(X_n) | \theta) \\ &= \sum_{i=n}^j \frac{1}{\prod_{h=n, h \neq i}^j [1 - \beta_h/\beta_i]} \sum_{k=0}^m \binom{m}{k} y^{m-k} (\beta_i\theta)^k \Pr\{X_{n-1} \leq y, X_n > y|\theta\}, \end{aligned}$$

where

$$\Pr\{X_{n-1} \leq y, X_n > y | \theta\} = \sum_{i=1}^n \frac{\beta_n / \beta_i}{\prod_{h=1, h \neq i}^n [1 - \beta_h / \beta_i]} \exp\left(-\frac{y}{\beta_i \theta}\right), \quad (26)$$

for  $j, n = 1, 2, \dots, j \geq n, m = 0, 1, 2, \dots, y \in (0, \infty)$ , where  $I_A(x) = 1$  if  $x \in A$  and  $I_A(x) = 0$  if  $x \notin A$ .

PROOF: Since  $X_n = \sum_{i=1}^n D_i$  for all  $n \in \mathbb{N}$ , it follows that the integration bounds are determined by  $X_j \geq X_{j-1} \geq \dots \geq X_1 \geq X_0 = 0$ . Moreover,  $X_{n-1} \leq y$  and  $X_n > y$ , where  $n \leq j$ , and the Jacobian equals 1. By using

$$\sum_{i=1}^j \frac{D_i}{\beta_i} = \frac{X_j}{\beta_j} + \sum_{i=1}^{j-1} \left( \frac{1}{\beta_i} - \frac{1}{\beta_{i+1}} \right) X_i, \quad (27)$$

we may write

$$\begin{aligned} & E([X_j]^m I_{[0, y]}(X_{n-1}) I_{(y, \infty)}(X_n) | \theta) \\ &= \int_{x_j=y}^{\infty} \int_{x_{j-1}=y}^{x_j} \dots \int_{x_n=y}^{x_{n+1}} \int_{x_{n-1}=0}^{x_n} \dots \int_{x_2=0}^{x_3} \int_{x_1=0}^{x_2} \frac{x_j^m}{\beta_j \theta} \exp\left(-\frac{x_j}{\beta_j \theta}\right) \\ & \quad \times \prod_{i=1}^{j-1} \frac{1}{\beta_i \theta} \exp\left[-\left(\frac{1}{\beta_i} - \frac{1}{\beta_{i+1}}\right) \frac{x_i}{\theta}\right] dx_1 \dots dx_j. \end{aligned} \quad (28)$$

This multiple integral can be solved in four steps.

First, by applying the transformation  $\delta_i = x_i - x_{i-1}$ ,  $i = 1, \dots, n-1$ , and subsequently using Eqs. (27) and (14), we find

$$\begin{aligned} & \int_{x_{n-1}=0}^y \dots \int_{x_2=0}^{x_3} \int_{x_1=0}^{x_2} \prod_{i=1}^{n-1} \frac{1}{\beta_i \theta} \exp\left[-\left(\frac{1}{\beta_i} - \frac{1}{\beta_{i+1}}\right) \frac{x_i}{\theta}\right] dx_1 \dots dx_{n-1} \\ &= \int_{\delta_1=0}^y \dots \int_{\delta_{n-1}=0}^{y - \sum_{h=1}^{n-2} \delta_h} \prod_{i=1}^{n-1} \frac{1}{\beta_i \theta} \exp\left[-\left(\frac{1}{\beta_i} - \frac{1}{\beta_n}\right) \frac{\delta_i}{\theta}\right] d\delta_{n-1} \dots d\delta_1 \\ &= \left( \prod_{i=1}^{n-1} \frac{1}{[1 - \beta_i / \beta_n]} \right) \left\{ 1 - \sum_{i=1}^{n-1} \prod_{h=1, h \neq i}^{n-1} \left( \frac{1 - \beta_h / \beta_n}{1 - \beta_h / \beta_i} \right) \exp\left[-\left(\frac{1}{\beta_i} - \frac{1}{\beta_n}\right) \frac{y}{\theta}\right] \right\} \\ &= \sum_{i=1}^n \frac{\beta_n / \beta_i}{\prod_{h=1, h \neq i}^n [1 - \beta_h / \beta_i]} \exp\left[-\left(\frac{1}{\beta_i} - \frac{1}{\beta_n}\right) \frac{y}{\theta}\right]. \end{aligned} \quad (29)$$

Second, by applying the transformation  $z_i = x_i - y$ ,  $i = n, \dots, j-1$ , and using Eq. (29), we obtain

$$\begin{aligned} & \int_{x_{j-1}=y}^{x_j} \dots \int_{x_n=y}^{x_{n+1}} \prod_{i=n}^{j-1} \frac{1}{\beta_i \theta} \exp\left[-\left(\frac{1}{\beta_i} - \frac{1}{\beta_{i+1}}\right) \frac{x_i}{\theta}\right] dx_n \dots dx_{j-1} \\ &= \sum_{i=n}^j \frac{\beta_j / \beta_i}{\prod_{h=n, h \neq i}^j [1 - \beta_h / \beta_i]} \exp\left[-\left(\frac{1}{\beta_i} - \frac{1}{\beta_j}\right) \frac{x_j}{\theta}\right] \exp\left[-\left(\frac{1}{\beta_n} - \frac{1}{\beta_i}\right) \frac{y}{\theta}\right]. \end{aligned} \quad (30)$$

Third, by applying the transformation  $t_i = (x_i - y)/(\beta_i \theta)$ ,  $i = n, \dots, j$ , and using the binomial formula and the gamma function, the one-dimensional integral over  $x_j$  can be written as

$$\begin{aligned} & \sum_{i=n}^j \frac{\exp\{-y/(\beta_n \theta)\}}{\prod_{h=n, h \neq i}^j [1 - \beta_h/\beta_i]} \int_{x_j=y}^{\infty} \frac{x_j^m}{\beta_i \theta} \exp\left(-\frac{x_j - y}{\beta_i \theta}\right) dx_j \\ &= \sum_{i=n}^j \frac{\exp\{-y/(\beta_n \theta)\}}{\prod_{h=n, h \neq i}^j [1 - \beta_h/\beta_i]} \sum_{k=0}^m \binom{m}{k} y^{m-k} (\beta_i \theta)^k. \end{aligned} \quad (31)$$

Finally, combining Eqs. (28)–(31) proves the theorem. ■