

OPTIMAL SAND NOURISHMENT DECISIONS

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ABSTRACT: To maintain the Dutch coastline, every year millions of cubic meters of sand must be supplied at locations subject to permanent erosion. A decision model has been developed to obtain optimal sand nourishment decisions whose expected costs are minimal on the basis of the only information that is available: the probability distribution of the limiting average rate of permanent erosion. This probability distribution is derived on the basis of actual erosion data using Bayes' theorem. In order that the stochastic erosion process be based on this uncertain limiting average, the writers consider it as a generalized gamma process. There are three cost-based criteria for comparing sand nourishment decisions: the average costs per unit time, the discounted costs over an unbounded horizon, and the equivalent average costs per unit time. From these three criteria, only the last two are appropriate to obtain optimal sand nourishment decisions. In a case study, the decision model has been tested to sand nourishment at Zwanenwater, The Netherlands. Although the decision model has been developed for the purpose of sand nourishment, it can be applied to other fields of engineering to solve many problems in the area of life cycle costing.

INTRODUCTION

To prevent the Dutch coast from receding, the government is carrying out sand nourishments at those beach locations where a certain reference line is crossed (Rijkswaterstaat 1990, 1993a). In this paper, we present a probabilistic model that enables us to make decisions for which the expected costs of sand nourishment over an unbounded horizon are minimal. Since the sand nourishment program is based on the observed permanent erosion, we have based our probabilistic model on the uncertain average rates of permanent erosion. To achieve this, the process of permanent erosion has been regarded as a generalized gamma process. In The Netherlands, generalized gamma processes have also been used to model decision problems for optimizing maintenance of the sea-bed protection of the Eastern-Scheldt barrier, berm breakwaters, and dykes (van Noortwijk et al. 1995, 1997, 1996; Speijker et al. 2000).

Using the discrete renewal theorem (Feller 1950, chapters 12 and 13; Karlin and Taylor 1975, chapter 3), three cost-based criteria for comparing decisions over unbounded horizons can be determined: the expected average costs per unit time, the expected discounted costs over an unbounded horizon, and the expected equivalent average costs per unit time. Although the criterion of the average costs is often used for maintenance optimization in mechanical and electrical engineering [e.g., Barlow and Proschan (1996)], it is not useful for nourishment optimization. Instead, the criteria of the discounted costs and the equivalent average costs should be used to find an optimum balance between initial costs and future costs, this being the aim of life cycle costing [e.g., Flanagan et al. (1989)]. Even though the discrete renewal theorem is well known, it has not been often applied to solve problems in life cycle costing.

The novelty of the proposed sand nourishment model is that an economic optimization can be carried out while taking account of the main uncertainties involved. This probabilistic model can be used in combination with existing, deterministic, models for obtaining optimum procedures for beach nourishment [e.g., Bruun (1991), Bruun and Willekes (1992), and Dette et al. (1994)].

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Note. Discussion open until July 1, 2000. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on August 6, 1997. This paper is part of the *Journal of Waterway, Port, Coastal, and Ocean Engineering*, Vol. 126, No. 1, January/February, 2000. ©ASCE, ISSN 0733-950X/00/0001-0030-0038/\$8.00 + \$.50 per page. Paper No. 16423.

The paper is composed as follows. We describe the problem of sand nourishment in the following section. Using Bayes' theorem in the next section, the probability distribution of the limiting average rate of permanent erosion is derived on the basis of actual erosion data at Zwanenwater, The Netherlands. The next section contains the specification of the stochastic process of permanent erosion when the value of the average rate is given. The costs of carrying out one sand nourishment are obtained in the next section. Then these costs are used to obtain the expected loss of an infinite sequence of sand nourishments using the above three cost-based criteria. The sand nourishment decision model is applied to nourishment at Zwanenwater. Finally, we present some conclusions. A necessary mathematical proof can be found in Appendix I.

SAND NOURISHMENT IN THE NETHERLANDS

The Dutch North Sea coast is made up of dunes, dykes, and barriers (like the Eastern-Scheldt barrier). Together they protect the areas that are below sea level (over one third of The Netherlands) against flooding. The main part of the Dutch coastal defense line consists of dunes (254 out of 353 km), varying in width from less than 100 m to several kilometers. Wind, wave, and tidal forces cause dynamic patterns of accretion and erosion along the sandy coast. Due to these natural forces, together with sea-level rise and human interference, the overall sand balance is negative and regular sand nourishment must be carried out to compensate for these sand losses. Nourishment is necessary to protect valuable dune areas and the hinterland, mainly polders (pump-drained areas—in general below sea level—enclosed by dykes, dunes, barriers, and higher ground) that are densely populated and represent important economic values.

The most important advantages of sand nourishments are that they (1) are flexible and allow for spreading of cost; (2) do not necessarily disturb recreational beaches; (3) are less expensive than dykes; and (4) match the natural character of the Dutch coast. The required sand volumes are normally dredged offshore by trailing suction hopper dredgers and pumped towards the beach, against the dune face or even directly dumped on the foreshore. For further details, we refer to Rijkswaterstaat (1990, 1993a) and Verhagen (1993).

When planning sand nourishments, it is important to discriminate between long-term permanent erosion and short-term incidental erosion. Permanent erosion implies a persistent negative sand balance in the coastal profile due to gradients in longshore and/or onshore-offshore sand transport. This type of erosion often results in a gradual, more or less irreversible, loss of sand in the dynamic coastal profile. Incidental erosion, on the contrary, refers to short-term erosion events, for ex-

ample due to incidental storms resulting in a redistribution of sand in the coastal profile. During subsequent calm weather conditions, this sand is partially transported backward in on-shore direction by the actions of waves, currents, and wind.

The management guidelines for the Dutch sandy coast focus on combating permanent erosion. This is concretized by defining the average position of a reference coastline, the so-called basal coastline, which has to be preserved. In each (sandy) coastal profile this basal coastline is defined by the average sand volume, seaward of the dune foot between two horizontal planes above and below the average low waterline (Fig. 1). The dune foot is defined for each cross section separately, in general by the intersection of the average front slope of the dune with the average slope of the adjacent beach. The average low waterline is known from standard tide tables. For the assessment of the basal coastline, the average sand volume is obtained by considering a period of 10 years, from 1980 to 1989, and extrapolating to 1990. The associated basal coastline has been adopted as the (1990) reference for future decisions for sand nourishment. The average sand volume in a coastal profile in a specific year determines the transient coastline.

Every year, the position of the transient coastline is determined and compared with the basal coastline. If this position is landward of the above basal coastline, sand nourishment is required (Fig. 2). The monitoring system includes depth sounding, leveling and stereo photogrammetry. The coastal measurements are stored in the so-called JARKUS database. These observed rates of average erosion are used to plan the annual sand nourishment program: per year about 7 million m^3 of sand has to be supplied against costs of about 70 million Dutch guilders (one U.S. dollar is equivalent to two Dutch guilders).

In addition to the above practice of sand nourishment, eroded dunes need sometimes to be restored irrespective of the average position of the transient coastline. In general, local corrective measures, often on a small scale, are taken to stim-

ulate natural accretion of the dune front or to compensate the dune erosion.

Thus planning of sand nourishments in The Netherlands is mainly based on the average rates of permanent erosion. These rates are uncertain because of natural and/or man-induced changes, as well as because of measurement errors. For example, sediment-trapping structures along the coast or in rivers may cause variations in the average rates of permanent erosion.

Since the lifetime of a sand nourishment depends on the above uncertain average rate of permanent erosion, a probabilistic sand nourishment model should be based on this average rate. Furthermore, we assume that this rate is not influenced by the nourishment itself (Verhagen 1993). According to Verhagen, this assumption is true when the beach nourishment is relatively long and the seaward displacement of the waterline (nourishment width) is not too great. Provided the ratio between nourishment length l and nourishment width w is on the order of 20 or more (which is true for the Zwanenwater nourishment), this assumption is acceptable and the sand nourishment model can be applied. Even though the erosion has a tendency to accelerate immediately after a nourishment (due to end losses and offshore transport of the finest fractions of sand), we consider this as a sort of reshaping effect on the nourished sand after which the original rate of erosion will be restored. The above assumption does not always hold in nature: the postfill erosion rate can be larger than the prefill erosion rate. In this situation, a new stochastic erosion model must be developed in order to account for expected erosion being nonlinear over time.

PROBABILITY DISTRIBUTION OF AVERAGE EROSION

The sand nourishment decision model proposed in this paper has been applied to Zwanenwater, a beach section with a length of 4,300 m in the northwest of The Netherlands. The deep-water wave conditions are characterized by a significant wave height of approximately 1.2 m, which is exceeded during 50% of the time. The 10% and 1% exceedance significant wave heights are about 2.7 and 4.4 m, respectively. Corresponding wave periods range from about 3.5 to 7 s with predominating directions from southwest to northwest. These figures may serve as a rough indication of the wave conditions.

In 1987 a sand nourishment of about $1.00 \times 10^6 m^3$ was carried out, including a dune strengthening of $0.16 \times 10^6 m^3$. This sand nourishment was intended to assure that the basal coastline would not be crossed for a period of more than 20 years by widening the coastal profile with about 20 m. Hence, the ratio between nourishment length and nourishment width is greater than 20 (i.e., 4,300/20). The prenourishment beach section is assumed to be in equilibrium. The data on the beach

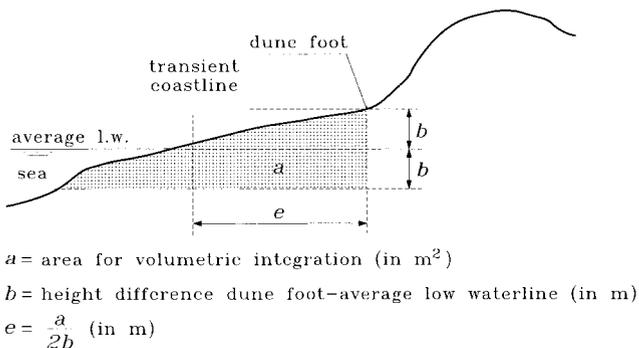


FIG. 1. Definition of Transient Coastline in Terms of Position of Dune Foot and Average Low Waterline

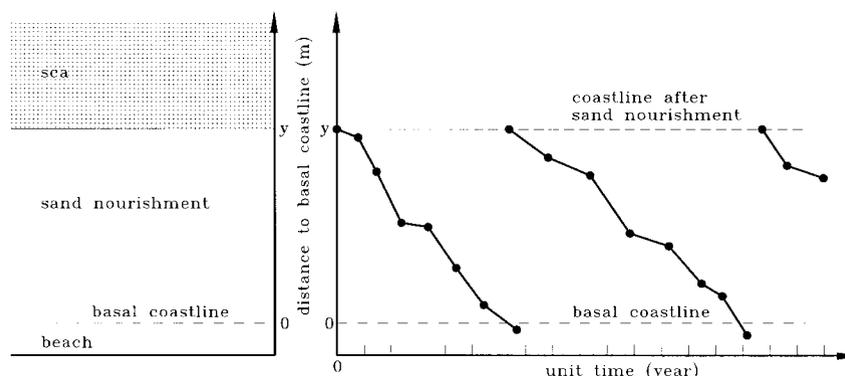


FIG. 2. View from Above of Sand Nourishment up to Distance of y m to Basal Coastline (Left) and Idealized Process of Permanent Erosion Regarded as Renewal Process (Right)

TABLE 1. Parameters of Sand Nourishment Model for Zwanenwater, The Netherlands

Parameter (1)	Description (2)	Value (3)	Dimension (4)
Δ	Unit time	1	year
n	Time-horizon	∞	year
r	Discount rate per year	5	%
α	Discount factor per year	0.9524	—
Θ	Uncertain limiting average rate of erosion	$(-\infty, \infty)$	m/year
$\theta_{0.05}$	5th percentile average rate of erosion	-4.5	m/year
$\theta_{0.50}$	50th percentile average rate of erosion	1.0	m/year
$\theta_{0.95}$	95th percentile average rate of erosion	6.5	m/year
μ'	Posterior parameter average rate of erosion	1	m/year
τ'	Posterior parameter average rate of erosion	23	—
a'	Posterior parameter average rate of erosion	8	—
b'	Posterior parameter average rate of erosion	1,835.1	(m/year) ²
$E(\Theta)$	Posterior mean average rate of erosion	1.0	m/year
c_f	Fixed cost	0.92×10^6	Dfl
c_v	Variable cost	9	Dfl/m ³
l	Sand nourishment length	4,300	m
h	Sand nourishment height	11	m
w	Sand nourishment width	$[0, \infty)$	m
—	Location dune foot	6	m + NAP
ϕ	Angle of the beach-profile slope	3.8×10^{-3}	radians
ψ	Angle of the sea floor	4.0×10^{-3}	radians
y	Distance to the basal coast line	$[0, \infty)$	m
y^*	Optimal distance to the basal coast line	6.5	m
$v(y^*)$	Optimal sand nourishment volume	0.31×10^6	m ³

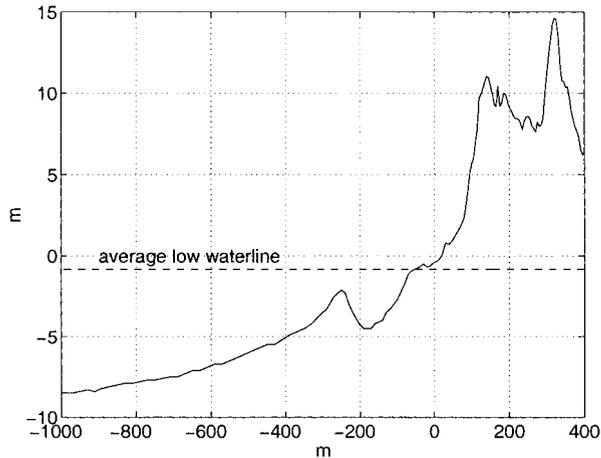


FIG. 3. Cross Section of Beach Profile at Zwanenwater, The Netherlands, from JARKUS File before Sand Nourishment Was Carried Out; Average Low Waterline Is Located at -0.84 m

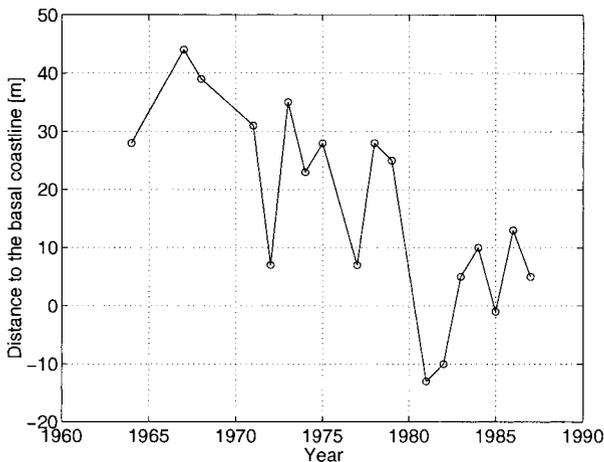


FIG. 4. Cumulative Sums of Permanent and Incidental Erosion during 1964–1986 at Zwanenwater, The Netherlands

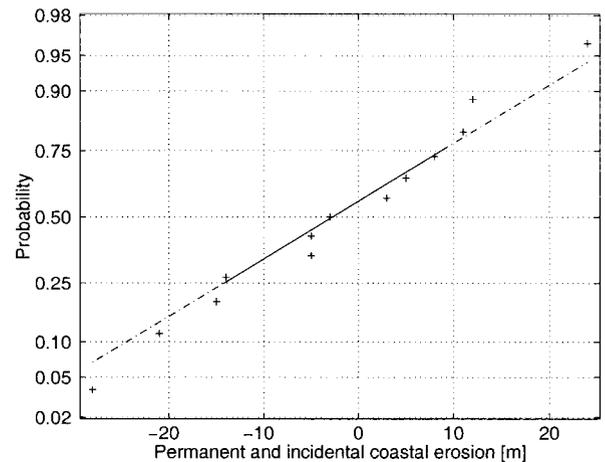


FIG. 5. Normal Probability Plot of Sum of Permanent and Incidental Erosion per Year

profile and the costs of sand nourishment are given in Table 1 [using Rijkswaterstaat (1993b) and the 1986, 1987, and 1988 JARKUS measurements]. To illustrate, one beach profile at Zwanenwater is shown in Fig. 3.

On the basis of the erosion data in Fig. 4, the prior distribution of the average rate of permanent erosion can be updated to the posterior distribution using Bayes' theorem. The erosion data represent the cumulative permanent and incidental erosion during the period 1964–1986, taken from Rijkswaterstaat (1993b). A difficulty in analyzing these data is that there are no measurements available from the years 1965, 1966, 1969, 1970, 1976, and 1980. A normal probability plot of 13 annual measurements (because the years prior to the missing data in Fig. 4 do not represent annual data, they are excluded), gives evidence of normality for the sum of permanent and incidental erosion (Fig. 5).

Suppose the annual sums of the permanent and incidental erosion are conditionally independent, normally distributed random quantities with an unknown value of the mean Θ with $-\infty < \Theta < \infty$ and an unknown value of the precision Λ with $\Lambda > 0$, where the precision is the reciprocal of the variance. Let Z_j denote the sum of the permanent and incidental erosion in time period j consisting of n_j years, $j = 1, 2, \dots$. When $\Theta = \theta$ and $\Lambda = \lambda$, the random quantity Z_j has a normal distribution with mean $n_j\theta$ and precision λ/n_j , $j = 1, 2, \dots$. The likelihood function l of \bar{n} independent observations $z_1, \dots, z_{\bar{n}}$ can thus be written as

$$\begin{aligned}
 l(z_1, \dots, z_{\bar{n}} | \theta, \lambda) &= \prod_{j=1}^{\bar{n}} l(z_j | \theta, \lambda) \\
 &= \prod_{j=1}^{\bar{n}} \frac{\sqrt{\lambda/n_j}}{\sqrt{2\pi}} \exp \left\{ -\frac{\lambda/n_j}{2} (z_j - n_j\theta)^2 \right\} \quad (1)
 \end{aligned}$$

Suppose further that prior information is available in terms of a prior joint distribution of Θ and Λ . In order that the posterior joint probability density function of Θ and Λ can be expressed in explicit form, we assume the random quantities Θ and Λ to have a bivariate normal-gamma density denoted by $\pi(\theta, \lambda)$. In this situation, both the prior joint density and the posterior joint density of Θ and Λ is a normal-gamma density. The normal-gamma density is then said to be a conjugate family of distributions for observations from a normal distribution with unknown mean and unknown precision.

Since the limiting average amount of incidental erosion per year equals zero, the random quantity Θ represents the unknown limiting average amount of permanent erosion per year, i.e.,

$$\lim_{\tilde{n} \rightarrow \infty} \left[\frac{\left(\sum_{j=1}^{\tilde{n}} Z_j \right)}{\left(\sum_{j=1}^{\tilde{n}} n_j \right)} \right]$$

By applying Bayes' theorem and (1), the posterior density function π of Θ when the observations $z_1, \dots, z_{\tilde{n}}$ are given is

$$\pi(\theta|z_1, \dots, z_{\tilde{n}}) = \frac{\int_{\lambda=0}^{\infty} \prod_{j=1}^{\tilde{n}} l(z_j|\theta, \lambda) \pi(\theta, \lambda) d\lambda}{\int_{\theta=-\infty}^{\infty} \int_{\lambda=0}^{\infty} \prod_{j=1}^{\tilde{n}} l(z_j|\theta, \lambda) \pi(\theta, \lambda) d\lambda d\theta} \quad (2)$$

The normal-gamma prior density $\pi(\theta, \lambda)$ can now be constructed in two steps. First, let the marginal density of Λ be a gamma density with shape parameter $a > 0$ and scale parameter $b > 0$, i.e.,

$$\text{Ga}(\lambda|a, b) = [b^a/\Gamma(a)]\lambda^{a-1} \exp\{-b\lambda\}I_{(0,\infty)}(\lambda)$$

where $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$ for every set A . Second, let the conditional density of Θ when $\Lambda = \lambda$ be a normal density with mean $-\infty < \mu < \infty$ and precision $\tau\lambda > 0$, i.e.,

$$N(\theta|\mu, \tau\lambda) = \frac{\sqrt{\tau\lambda}}{\sqrt{2\pi}} \exp\left\{-\frac{\tau\lambda}{2}(\theta - \mu)^2\right\}$$

The prior joint probability density $\pi(\theta, \lambda) = N(\theta|\mu, \tau\lambda)\text{Ga}(\lambda|a, b)$ transforms into the posterior joint probability density [Eqs. (3)–(4) extend the proof of DeGroot (1970, chapter 9) in which $n_j = 1$ for all j]

$$\pi(\theta, \lambda|z_1, \dots, z_{\tilde{n}}) = N(\theta|\mu', \tau'\lambda)\text{Ga}(\lambda|a', b')$$

where

$$\mu' = \frac{\tau\mu + n\bar{z}}{\tau + n}; \quad \tau' = \tau + n; \quad n = \sum_{j=1}^{\tilde{n}} n_j; \quad \bar{z} = \frac{1}{n} \sum_{j=1}^{\tilde{n}} z_j \quad (3a-d)$$

and

$$a' = a + \frac{\tilde{n}}{2}; \quad b' = b + \frac{1}{2} \sum_{j=1}^{\tilde{n}} n_j \left(\frac{z_j}{n_j} - \bar{z} \right)^2 + \frac{\tau n (\bar{z} - \mu)^2}{2(\tau + n)} \quad (4a,b)$$

when the observations $z_1, \dots, z_{\tilde{n}}$ are given.

Since the posterior joint density of Θ and Λ is a normal-gamma distribution, the marginal posterior distribution of Θ is not a normal distribution but a t distribution with $2a'$ degrees of freedom, location parameter μ' , and precision $a'\tau'/b'$:

$$\pi(\theta|z_1, \dots, z_{\tilde{n}}) = \frac{\sqrt{\frac{a'\tau'}{b'}} \Gamma\left(\frac{2a'+1}{2}\right)}{\sqrt{2a'\pi} \Gamma\left(\frac{2a'}{2}\right)} \left[1 + \frac{1}{2a'} \cdot \frac{a'\tau'}{b'} (\theta - \mu')^2 \right]^{-(2a'+1)/2} \quad (5)$$

where the parameters μ' , τ' , a' , and b' can be found in (3)–(4).

After deriving the posterior probability distribution of Θ , the question arises what prior values of μ , τ , a , and b should be chosen. Since there is no prior information available, the prior parameters must represent very vague knowledge. To achieve this, DeGroot (1970, chapter 10) suggests the assumption: $\tau \rightarrow 0$, $a \rightarrow -1/2$, and $b \rightarrow 0$. Then, the posterior parameters μ' , τ' , a' , and b' reduce to

$$\mu' = \bar{z}; \quad \tau' = n \quad (6a,b)$$

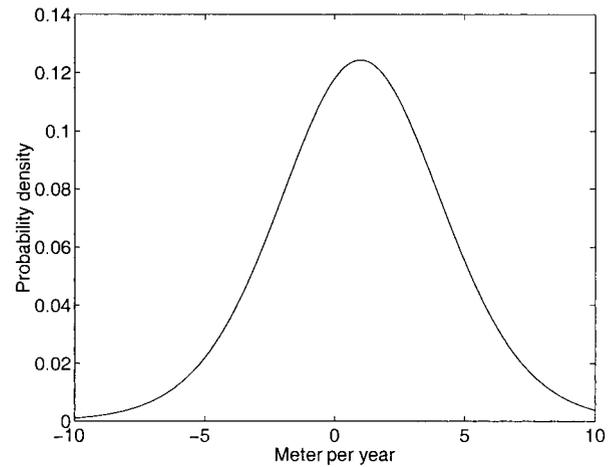


FIG. 6. Probability Density Function of Limiting Average Amount of Permanent Erosion per Year at Zwanenwater, The Netherlands

$$a' = \frac{\tilde{n} - 1}{2}; \quad b' = \frac{1}{2} \left[\left(\sum_{j=1}^{\tilde{n}} \frac{z_j^2}{n_j} \right) - n\bar{z}^2 \right] \quad (6c,d)$$

The measurements shown in Fig. 4 result in the following posterior values: $\mu' = 1$, $\tau' = 23$, $a' = 8$, and $b' = 1,835.1$. The corresponding t distribution is displayed in Fig. 6 with a posterior mean of 1 m/year. The probabilities of permanent coastal erosion and permanent coastal accretion are equal to 0.61 and 0.39, respectively.

STOCHASTIC PROCESS OF PERMANENT COASTAL EROSION

On the basis of the Bayesian analysis described in the previous section, the only information that is available about the stochastic process of permanent coastal erosion concerns the probability distribution of its limiting average rate per year. Since we only have measurements of combinations of permanent and incidental erosion, it is not possible to fit the likelihood function of the underlying stochastic process of permanent erosion representing the inherent fluctuation over time. In other words, it is not possible to subtract the permanent erosion as a function of time out of the survey data in Fig. 4; only its average rate can be obtained. In this situation, we can best choose a likelihood function with weak assumptions. Let us denote the process of permanent erosion by $\{X_i; i \in \mathbb{N}\}$, where X_i represents the cumulative permanent erosion at time $i\Delta$, where $\Delta = 1$ year. We write

$$X_i = \sum_{j=1}^i D_j, \quad i \in \mathbb{N}$$

Figs. 4–5 suggest we may judge the infinite sequence of increments $\{D_i; i \in \mathbb{N}\}$ to be exchangeable, i.e., the order in which the increments occur is irrelevant. The notion of exchangeability is weaker than the notion of independence. Although the assumption of exchangeability is satisfied for Zwanenwater, it may be violated for other locations. If exchangeability does not hold another stochastic erosion model must be developed [see, for example, the stochastic process with expected crest-level decline of a dyke being nonlinear over time in Speijker et al. (2000)].

In order to calculate the expected costs, the probability density function of the limiting average rate of permanent erosion is split up into two parts: (1) the conditional probability density function of the limiting average rate given this rate is negative times the probability that the rate is negative (coastal accretion); and (2) the conditional probability density function

of the limiting average rate given this rate is nonnegative times the probability that the rate is nonnegative (coastal erosion). This section deals with the situation that a nonnegative limiting average rate is given.

Conditional on a nonnegative limiting average amount of permanent erosion, we assume the yearly amounts of permanent erosion to be nonnegative as well. Given a nonnegative value of the average permanent erosion per year, say $\theta > 0$, we further assume that the decision-maker is indifferent to the way this average is obtained. In other words, all combinations leading to the same average have the same degree of belief for our decision-maker. Since we have no other information, we adopt a uniform distribution over all erosion vectors having the same average [for details, see Barlow and Mendel (1992) and Van Noortwijk et al. (1995)]. For a potentially infinite sequence of increments $\{D_i; i \in \mathbb{N}\}$ this means that the joint probability density function of the increments of permanent erosion can be written as a mixture of exponential densities:

$$p_{D_1, \dots, D_n}(\delta_1, \dots, \delta_n) = \int_0^\infty \prod_{i=1}^n \frac{1}{\theta} \exp\left\{-\frac{\delta_i}{\theta}\right\} dP_\Theta(\theta) = f_n\left(\sum_{i=1}^n \delta_i\right) \quad (7)$$

for $(\delta_1, \dots, \delta_n) \in \mathbb{R}_+^n$ and zero otherwise, $\mathbb{R}_+ = [0, \infty)$. The infinite sequence of random quantities $\{D_i; i \in \mathbb{N}\}$ is said to be l_1 -isotropic (or l_1 -norm symmetric), since its distribution can be written as a function of the l_1 -norm (Misiewicz and Cooke 1999). When the value of θ is given, the statistic

$$X_n = \sum_{i=1}^n D_i$$

has a gamma distribution with shape parameter n and scale parameter $1/\theta$. Hence, the nondecreasing stochastic process $\{X_n; n \in \mathbb{N}\}$ is a scale mixture of gamma processes that is also called a generalized gamma process. The probability distribution P_Θ represents the uncertainty in the unknown limiting average rate of permanent erosion per year when the average rate is given to be nonnegative, i.e., by using (5):

$$P_\Theta(\theta) = \Pr\{0 < \Theta \leq \theta | \Theta > 0, z_1, \dots, z_n\} = \frac{\int_{\tilde{\theta}=0}^{\theta} \pi(\theta | z_1, \dots, z_n) d\tilde{\theta}}{\int_{\tilde{\theta}=0}^{\infty} \pi(\theta | z_1, \dots, z_n) d\tilde{\theta}} \quad (8)$$

In summary, the two main uncertainties involved are (1) the uncertainty in the limiting average rate of permanent erosion; and (2) when a nonnegative value of the limiting average rate is given, the uncertainty in the inherent fluctuation of the increments of permanent erosion over time. From now on, when we use the word "erosion," we shall mean permanent erosion.

To make optimal sand nourishment decisions under uncertainty, we can now use statistical decision theory (DeGroot 1970, chapter 8). At beach locations subject to erosion, the decision problem is to choose a sand nourishment decision y , from $[0, \infty)$, such that the nourished coastline is located y m seaward of the basal coastline, where the consequences of the decision depend on the unknown value of the limiting average erosion per year Θ . Let $L(\theta, y)$ be the loss when the decision-maker chooses decision y and when the limiting average erosion Θ has the value θ , where the loss represents the monetary losses due to sand nourishment. The decision-maker can best choose, if possible, a nourishment decision y^* whose expected loss with respect to the probability distribution of Θ is minimal. A decision y^* is called an optimal decision when $E(L(\Theta, y^*)) = \min_{y \in [0, \infty)} E(L(\Theta, y))$. On the basis of three types of cost-based criteria, loss functions are derived in the next two sections.

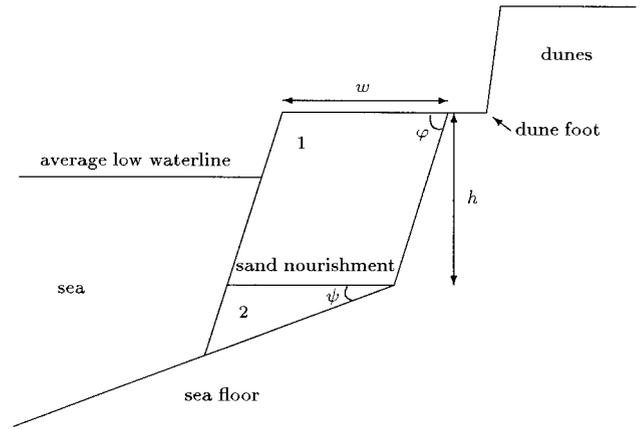


FIG. 7. Compressed Cross Section of Sand Nourishment (1 = Area $v_1 w$; 2 = Area $v_2 w^2$)

COSTS OF ONE SAND NOURISHMENT

For each sand nourishment, the associated costs can be subdivided into the fixed cost c_f (cost of mobilization) and the variable cost c_v (cost per cubic meter sand). Furthermore, we assume that the volume of sand to be supplied is a function of the sand nourishment width w (in meters) in the following way (Fig. 7):

$$v(w) = lh \cdot w + \frac{l \sin(\psi)}{2} \left[\cos(\psi) + \frac{\sin(\psi)}{\tan(\varphi - \psi)} \right] \cdot w^2 = v_1 \cdot w + v_2 \cdot w^2 \quad (9)$$

in cubic meters (m^3), where l is the sand nourishment length, h is the sand nourishment height, φ is the angle of the beach-profile slope, and ψ is the angle of the sea floor. The sand nourishment width w is regarded as the equilibrium fill width that is expected within a couple of months after fill placement. Eq. (9) approximates the formula for a nourished volume after equilibrium which was proposed by Houston (1996) (in fact, he used $w^{5/3}$ instead of w^2). Although we restrict ourselves to a polynomial of degree 2, the decision model can deal with polynomials of any other degree (see Theorem 1 from Appendix I).

Due to the mixture of exponentials in (7), we can express various probabilistic properties in explicit form when a nonnegative value of the limiting average erosion θ is given [for other illustrations, see Van Noortwijk et al. (1995)]. For the purpose of optimal sand nourishment, two probabilistic properties are useful: (1) the probability of exceedence of the basal coastline in unit time i ; and (2) the expected costs of sand nourishment due to exceedence of the basal coastline in unit time i . These two properties are derived in Theorem 1 (Appendix I).

First, the conditional probability of exceedence of the basal coastline in unit time i , when the limiting average erosion is θ and when the decision-maker chooses the coastline after nourishment to be located y m away from the basal coastline, can be written as

$$p_i(\theta, y) = \Pr\{X_{i-1} \leq y, X_i > y | \theta\} = \frac{1}{(i-1)!} \left[\frac{y}{\theta} \right]^{i-1} \exp\left\{-\frac{y}{\theta}\right\} \quad (10)$$

for $i = 1, 2, \dots, \theta > 0$, and $y \geq 0$. This discrete probability function is the Poisson distribution with mean lifetime $1 + (y/\theta)$ and variance y/θ .

Second, the expected costs of sand nourishment due to exceedence of the basal coastline in unit time i , when the limiting average erosion is θ and when the decision-maker chooses the coastline after nourishment to be located y m away from the basal coastline, can be written as [using (9)]:

$$\begin{aligned}
& E\{[c_f + c_v\{v_1X_i + v_2X_i^2\}]I_{[0,y]}(X_{i-1})I_{(y,\infty)}(X_i)|\theta\} \\
& = [c_f + c_v\{v_1(y + \theta) + v_2[(y + \theta)^2 + \theta^2]\}] \\
& \times \frac{1}{(i-1)!} \left[\frac{y}{\theta}\right]^{i-1} \exp\left\{-\frac{y}{\theta}\right\} = c_i(\theta, y)p_i(\theta, y) \quad (10)
\end{aligned}$$

for $i = 1, 2, \dots$, $\theta > 0$, and $y \geq 0$.

Using (10) and (11), the costs $c_i(\theta, y)$ are the expected costs of a sand nourishment resulting in a nourished coastline at a distance of y m to the basal coastline given that the basal coastline is exceeded in unit time i . The costs of nourishment do not depend on the unit time at which the basal coastline is exceeded: i.e., $c_i(\theta, y) \equiv c(\theta, y)$ for all $i \in \mathbb{N}$.

COSTS OF SEQUENCES OF SAND NOURISHMENTS

Until now, we have studied the probability of occurrence and the expected costs of just one sand nourishment. In this section, we derive three cost-based criteria to compare infinite sequences of sand nourishments over unbounded horizons.

Types of Cost-Based Criteria

Wagner (1975, chapter 11) gave two reasons for comparing decisions over unbounded instead of bounded time-horizons. First, in making repeated investment decisions it is better to employ an unbounded horizon model than to simply ignore the future. Second, as we will see later, the mathematical models are less complex, while they still provide reasonable answers in practice. However, since maintenance costs over an unbounded horizon are infinite in most cases, we need models that can handle an infinite accumulation of costs. For this purpose, Wagner (1975, chapter 11) distinguishes three cost-based criteria for comparing decisions over unbounded horizons: the average costs per unit time, the discounted costs over an unbounded horizon, and the equivalent average costs per unit time.

These costs can be determined by formulating the process of sand nourishments as a discrete renewal process. A discrete renewal process $\{N(n):n \in \mathbb{N}\}$ is a nonnegative integer-valued stochastic process that registers the successive renewals in the time-interval $(0, n]$. In coastal management, the renewals are the sand nourishment actions carried out to move the transient coastline y m seaward from the basal coastline (Fig. 2). Conditional on θ , the renewal times T_1, T_2, \dots , are nonnegative, independent and identically distributed, Poisson random quantities having the probability function (10), i.e., $\Pr\{T_k = i|\theta\} = p_i(\theta, y)$, $i \in \mathbb{N}$, when the limiting average erosion is θ and when the decision-maker chooses decision y . The above cost-based criteria will be discussed in more detail in the following subsections.

Average Costs per Unit Time

The expected average costs per unit time are determined by simply averaging the costs over an unbounded horizon. They follow from the expected costs $C(n, \theta, y)$, over the bounded horizon $(0, n]$, that solve the recursive equation

$$C(n, \theta, y) = \sum_{i=1}^n p_i(\theta, y)[c_i(\theta, y) + C(n - i, \theta, y)] \quad (12)$$

for $n \in \mathbb{N}$ and $C(0, \theta, y) \equiv 0$. To obtain this equation, we condition on the values of the first renewal time T_1 and apply the law of total probability. The costs associated with occurrence of the event $T_1 = i$ are $c_i(\theta, y)$ [see (11)] plus the expected additional costs during the interval $(i, n]$, $i = 1, \dots, n$.

Using the discrete renewal theorem (Feller 1950, chapters 12 and 13; Karlin and Taylor 1975, chapter 3), the expected average costs per unit time are

$$\lim_{n \rightarrow \infty} \frac{C(n, \theta, y)}{n} = \frac{\sum_{i=1}^{\infty} c_i(\theta, y)p_i(\theta, y)}{\sum_{i=1}^{\infty} ip_i(\theta, y)} = \frac{c(\theta, y)}{1 + (y/\theta)} = C(\theta, y) \quad (13)$$

Let a renewal cycle be the time-period between two renewals, and we recognize the numerator as the expected cycle costs and the denominator as the expected cycle length. Eq. (13) is a well-known result from renewal reward theory [e.g., Ross (1970, chapter 3)]. If $c(\theta, y) \equiv 1$ in (13), then the expected average number of renewals per unit time is

$$\lim_{n \rightarrow \infty} \frac{M(n, \theta, y)}{n} = \frac{1}{\sum_{i=1}^{\infty} ip_i(\theta, y)} = \frac{1}{1 + (y/\theta)} = \frac{\theta}{y + \theta} \quad (14)$$

being the reciprocal of the mean lifetime.

Discounted Costs over Unbounded Horizon

Discounting expected costs over an unbounded horizon is based on the assumption that the utility of a certain amount of money decreases in time from the standpoint of the present. Using the criterion of the discounted costs, it is possible to compare the value of money at different dates while taking into account the idea that "a dollar today is worth more than a dollar a year from today." In fact, the more money we have available now, the better off we are, for the sooner we can earn more money with it. Formally, the (present) discounted value of the costs c_n in unit time n is defined to be $\alpha^n c_n$ with $\alpha = [1 + (r/100)]^{-1}$ the discount factor per unit time and $r\%$ the real discount rate per unit time, where $r > 0$. The real discount rate equals the nominal rate of interest minus the rate of inflation. The decision-maker is indifferent between the costs c_n at time n and the costs $\alpha^n c_n$ at time 0. Therefore, the higher the discount rate, the better it is to postpone expensive sand nourishment actions. What discount rate is to be taken depends on the decision problem.

The expected discounted costs over a bounded time-horizon can be obtained with a recursive formula similar to that of the expected costs in (12). Again, we condition on the values of the first renewal time T_1 and apply the law of total probability. In this case, however, we want to account for the discounted value of the renewal costs $c_i(\theta, y)$ plus the additional expected discounted costs in time-interval $(i, n]$, $i = 1, \dots, n$. Hence, the expected discounted costs over the bounded horizon $(0, n]$ can be written as

$$C_\alpha(n, \theta, y) = \sum_{i=1}^n \alpha^i p_i(\theta, y)[c_i(\theta, y) + C_\alpha(n - i, \theta, y)] \quad (15)$$

for $n \in \mathbb{N}$, and $C_\alpha(0, \theta, y) \equiv 0$.

By using Feller (1950, chapter 13), the expected discounted costs over an unbounded horizon $C_\alpha(\theta, y)$ can be written as

$$\begin{aligned}
C_\alpha(\theta, y) & = \lim_{n \rightarrow \infty} C_\alpha(n, \theta, y) \\
& = \frac{\sum_{i=1}^{\infty} \alpha^i c_i(\theta, y)p_i(\theta, y)}{1 - \sum_{i=1}^{\infty} \alpha^i p_i(\theta, y)} = \frac{\alpha \exp\{-(1 - \alpha)y/\theta\}}{1 - \alpha \exp\{-(1 - \alpha)y/\theta\}} \cdot c(\theta, y) \quad (16)
\end{aligned}$$

We recognize the numerator of $C_\alpha(\theta, y)$ as the discounted cycle costs, while the denominator can be interpreted as the probability that the renewal process terminates due to discounting. Such a renewal process is called a terminating renewal process since infinite interoccurrence times can cause the renewals to cease. The interoccurrence times $\tilde{T}_1, \tilde{T}_2, \dots$ of our imaginary

terminating renewal process have the distribution $\Pr\{\tilde{T}_k = i\} = \alpha^i p_i(\theta, y)$, $i \in \mathbb{N}$, and

$$\Pr\{\tilde{T}_k = \infty\} = 1 - \sum_{i=1}^{\infty} \alpha^i p_i(\theta, y)$$

The expected number of imaginary “discounted renewals” over an unbounded time-horizon is

$$\lim_{n \rightarrow \infty} M_\alpha(n, \theta, y) = \frac{\sum_{i=1}^{\infty} \alpha^i p_i(\theta, y)}{1 - \sum_{i=1}^{\infty} \alpha^i p_i(\theta, y)} = \frac{\Pr\{\tilde{T}_k < \infty\}}{\Pr\{\tilde{T}_k = \infty\}} \quad (17)$$

Equivalent Average Costs per Unit Time

The expected equivalent average costs per unit time relate to the two notions of average costs and discounted costs. To determine this relation, we construct a new infinite stream of identical costs with the same present discounted value as the discounted costs over an unbounded time-horizon $C_\alpha(\theta, y)$. This can easily be achieved by defining an infinite stream of costs appearing at times $i = 0, 1, 2, \dots$, which are all equal to $(1 - \alpha)C_\alpha(\theta, y)$. Using the geometric series, we can write

$$\sum_{i=0}^{\infty} \alpha^i (1 - \alpha) C_\alpha(\theta, y) = C_\alpha(\theta, y) \quad (18)$$

for $\theta > 0$, $y \geq 0$, and $0 < \alpha < 1$. We call $(1 - \alpha)C_\alpha(\theta, y)$ the equivalent average costs per unit time. As α tends to 1, from below, the equivalent average costs approach the average costs per unit time:

$$\lim_{\alpha \uparrow 1} (1 - \alpha)C_\alpha(\theta, y) = C(\theta, y) \quad (19)$$

for $\theta > 0$, $y \geq 0$, using L'Hôpital's rule.

Choice of Cost-Based Criteria

For a cost-optimal sand nourishment program, we are interested in finding an optimum balance between the initial costs and the future costs, this being the aim of life cycle costing. Let the transient coastline at time 0 be the basal coastline. Then, the first sand nourishment has to be carried out at a cost of

$$c_0(y) = [c_f + c_v \nu(y)] I_{(0, \infty)}(y)$$

On the basis of the following two loss functions, both permanent coastal erosion ($\theta > 0$) and permanent coastal accretion ($\theta \leq 0$) are taken into account. The monetary losses over an unbounded horizon are the sum of the initial costs and the expected discounted future costs,

$$L_\alpha(\theta, y) = \begin{cases} c_0(y) + C_\alpha(\theta, y) & \text{if } \theta > 0 \\ c_0(y) & \text{otherwise} \end{cases} \quad (20)$$

when the decision-maker chooses decision y , the limiting average erosion or accretion is θ , and the discount factor is α . For the purpose of sand nourishment, we cannot use the criterion of the expected average costs per unit time,

$$L(\theta, y) = \begin{cases} C(\theta, y) & \text{if } \theta > 0 \\ 0 & \text{otherwise} \end{cases} \quad (21)$$

because the contribution of the initial costs to the average costs is completely ignored: $\lim_{n \rightarrow \infty} c_0(y)/n = 0$.

In conclusion, we recommend to choose an optimal sand nourishment decision y^* satisfying $E(L_\alpha(\Theta, y^*)) = \min_{y \in [0, \infty)} E(L_\alpha(\Theta, y))$, where the probability density function of the lim-

iting average rate of permanent coastal erosion and accretion is given by the t distribution in (5).

CASE STUDY: ZWANENWATER, THE NETHERLANDS

In this section, the above decision model is applied to a sand nourishment project at Zwanenwater. This case study is selected because the ratio between nourishment volume and nourishment length is relatively small and the erosion rate is therefore not much affected by the nourishment. The parameters are given in Table 1. In Fig. 8, the expected average costs per unit time, $E(L(\Theta, y))$, and the expected equivalent average costs per unit time, $E((1 - \alpha)L_\alpha(\Theta, y))$, are shown as a function of the distance y between the nourished coastline and the basal coastline [see (21) and (20), respectively]. These expected values have been obtained by numerical integration using the trapezoidal rule. As α tends to 1, from below, the expected equivalent average costs approach the expected average costs by interchanging the order of the operations of expectation and passing to the limit through Lebesgue's Theorem of Dominated Convergence; see, e.g., Weir (1973, chapter 5).

Recall that for the purpose of sand nourishment the criterion of average costs cannot be used, because the costs of the first sand nourishment are neglected in this situation. Indeed, the cost figures in Fig. 8 show that the expected average costs per unit time are minimal for a sand nourishment that is unrealistically large. Actually, the average costs are minimal for $y = 98.5$ m and $\nu(y) = 4.75 \times 10^6$ m³, resulting in average costs of 0.81×10^6 Dutch guilders per unit time (Fig. 8, top), but equivalent average costs of 2.46×10^6 Dutch guilders per unit time (Fig. 8, bottom).

According to numerical integration of (20), the expected discounted costs over an unbounded time-horizon are minimal when the decision-maker chooses the distance between the nourished coastline and the basal coastline to be $y^* = 6.5$ m (Fig. 8, bottom, and Fig. 9). The corresponding optimal sand nourishment volume is $\nu(y^*) = 0.31 \times 10^6$ m³. For the optimal decision, the expected discounted costs over an unbounded horizon are 21.7×10^6 Dutch guilders, where the expected equivalent average costs per unit time are 1.03×10^6 Dutch guilders per unit time. If $y^* = 6.5$ m, the average number of sand nourishments per unit time is 0.17, which can be obtained by calculating the expected value of (14) using numerical integration.

One of the reasons why it is necessary to account for the

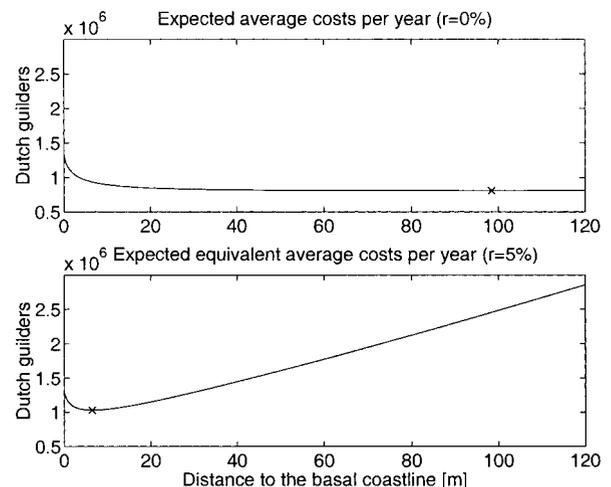


FIG. 8. Expected Average Costs of Sand Nourishment per Unit Time (Top) and Expected Equivalent Average Costs of Sand Nourishment per Unit Time (Bottom) at Zwanenwater, The Netherlands, Where r Is Annual Discount Rate

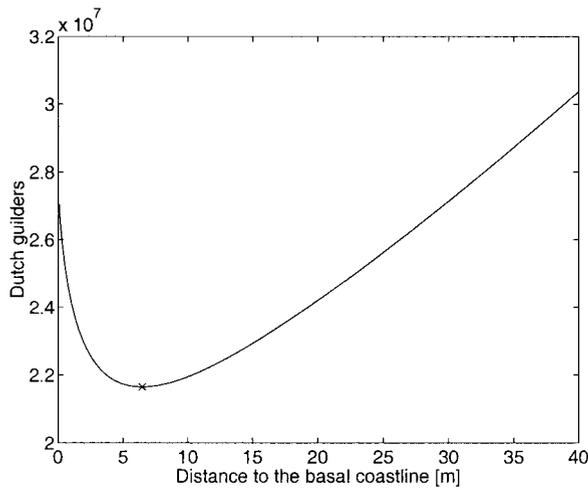


FIG. 9. Expected Discounted Costs of Sand Nourishment over Unbounded Time-Horizon at Zwanenwater, The Netherlands

uncertainty in the limiting average rate of permanent erosion is as follows. If the value of the limiting average rate of permanent erosion were equal to its 5th percentile (-4.5 m) or its 95th percentile (6.5 m), the optimal decision would be $y^* = 0$ m or $y^* = 15.4$ m, respectively. The corresponding respective discounted costs would be 0 or 66.3×10^6 Dutch guilders. Hence, since the error in using the above rates is quite large, the optimal decision can best be based on the probability distribution of the limiting average rate of permanent erosion. Although an optimal width of 6.5 m seems to be narrow in comparison to the 95th percentile of the limiting average rate, we should keep in mind that the probability of coastal accretion is rather large: 0.39 .

A great advantage of the explicit expression for the expected discounted costs over an unbounded horizon in (20) is that we can easily determine the cumulative probability distribution of the costs associated with the optimal decision $y^* = 6.5$ m. This distribution reflects the potential costs associated with the uncertainty in the limiting average rate of permanent erosion. For example, the 5th and 95th percentile of the expected discounted costs over an unbounded horizon are $L_\alpha(\theta_{0.05}, y^*) = 3.68 \times 10^6 = c_0(y^*)$ and $L_\alpha(\theta_{0.95}, y^*) = 67.7 \times 10^6$ Dutch guilders, respectively.

Our discussion thus far has considered the situation where both the fixed cost c_f and the variable cost c_v are known. However, randomness in these cost parameters may come about as a result of sand resources depletion, rising fuel costs, market forces, and technological improvements. The costs of sand nourishment can often only be predicted to approximately 10% accuracy. Since (11) is a linear function of the cost parameters c_f and c_v , it suffices to assume these cost parameters to be equal to their expected values. Hence, the variability in the cost parameters does not affect the optimal sand nourishment decision.

CONCLUSIONS

In this paper, we have presented a sand nourishment decision model that enables the decision-maker to optimize nourishment programs while taking account of the main uncertainties involved. As decision criterion, we recommend the calculation of expected discounted costs over an unbounded time-horizon for finding an optimum balance between initial costs and future costs. An important starting point is the probability distribution of the limiting average rate of permanent erosion, which is assumed to be unaffected by sand nourishment. The probability distribution can be given a priori and can later be refined on the basis of measurements using Bayes'

theorem. Because of the assumption that the rate of permanent erosion is not influenced by sand nourishment, the decision model can be applied so long as the ratio between nourishment length and nourishment width is on the order of $20-40$. In the event of ratios below this range, accelerated erosion can be expected immediately after placement of the nourishment. To account for this physical effect, the model should be adapted. On the basis of a case study, we endorse the conclusion of Bruun (1991) and Dette et al. (1994) that more frequent nourishments (once in about 5 years) with smaller quantities are to be preferred to less frequent nourishments (once in about 20 years) with larger quantities. The case study serves as an example how an optimal nourishment frequency could be obtained on the basis of an economic optimization.

APPENDIX I.

Theorem 1

Suppose the infinite sequence of random quantities $\{D_i; i \in \mathbb{N}\}$ is I_1 -isotropic and

$$X_n = \sum_{i=1}^n D_i$$

for all $n \in \mathbb{N}$, then

$$E([X_n]^m I_{[0,y]}(X_{n-1}) I_{(y,\infty)}(X_n) | \theta) = \left\{ \sum_{i=0}^m \frac{m!}{(m-i)!} y^{m-i} \theta^i \right\} \times \frac{1}{(n-1)!} \left[\frac{y}{\theta} \right]^{n-1} \exp \left\{ -\frac{y}{\theta} \right\} \quad (22)$$

for $n = 1, 2, \dots, m = 0, 1, 2, \dots, y \in [0, \infty)$, where $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$.

Proof

Since

$$X_n = \sum_{i=1}^n D_i$$

for all $n \in \mathbb{N}$, it follows that the integration bounds are determined by $0 \leq X_1 \leq \dots \leq X_{n-1} \leq X_n$. Moreover, $X_{n-1} \leq y$ and $X_n > y$, and the Jacobian equals one. Hence, we may write

$$E([X_n]^m I_{[0,y]}(X_{n-1}) I_{(y,\infty)}(X_n) | \theta) = \int_{x_1=0}^y \dots \int_{x_{n-1}=x_{n-2}}^y \int_{x_n=y}^{\infty} x_n^m \frac{1}{\theta^n} \exp \left\{ -\frac{x_n}{\theta} \right\} dx_n dx_{n-1} \dots dx_1$$

By applying the transformation $t = (x_n - y)/\theta$, and by using the binomial formula and the gamma function, we can integrate out the variable x_n :

$$\int_{x_n=y}^{\infty} x_n^m \exp \left\{ -\frac{x_n}{\theta} \right\} dx_n = \sum_{i=0}^m \frac{m!}{(m-i)!} y^{m-i} \theta^{i+1} \exp \left\{ -\frac{y}{\theta} \right\} \quad (23)$$

The remaining integral is the Dirichlet integral:

$$\int_{x_1=0}^y \dots \int_{x_{n-1}=x_{n-2}}^y 1 dx_{n-1} \dots dx_1 = \frac{y^{n-1}}{(n-1)!} \quad (24)$$

Combining (23) and (24) leads to the desired result.

ACKNOWLEDGMENTS

This work was partly supported by WL/Delft Hydraulics, The Netherlands, under project numbers Q1210, H1593 (the European Community), and H2068. The writers acknowledge helpful comments from Roger Cooke, Matthijs Kok, Tim Bedford, and three anonymous referees. The writers are grateful to the Dutch Ministry of Transport, Public Works and Water Management for providing the sand nourishment data.

APPENDIX II. REFERENCES

Barlow, R. E., and Mendel, M. B. (1992). "De Finetti-type representations for life distributions." *J. Am. Statistical Assn.*, 87(420), 1116-1122.

- Barlow, R. E., and Proschan, F. (1996). *Mathematical theory of reliability*. SIAM, Philadelphia.
- Bruun, P. (1991). "Optimum dredging procedures for artificial nourishment of beaches." *Water resour. plng. and mgmt. and urban water resour.; 18th Annu. Conf. and Symp.*, Jerry L. Anderson, ed., ASCE, New York, 303–307.
- Bruun, P., and Willekes, G. (1992). "Bypassing and backpassing at harbors, navigation channels, and tidal entrances: Use of shallow-water draft hopper dredgers with pump-out capabilities." *J. Coast. Res.*, 8(4), 972–977.
- DeGroot, M. H. (1970). *Optimal statistical decisions*. McGraw-Hill, New York.
- Dette, H.-H., Führböter, A., and Raudkivi, A. J. (1994). "Interdependence of beach fill volumes and repetition intervals." *J. Wtrwy., Port, Coast., and Oc. Engrg.*, ASCE, 120(6), 580–593.
- Feller, W. (1950). *An introduction to probability theory and its applications*, Vol. 1, John Wiley, New York.
- Flanagan, R., Norman, G., Meadows, J., and Robinson, G. (1989). *Life cycle costing; theory and practice*. BSP Professional Books, Oxford, U.K.
- Houston, J. R. (1996). "Simplified Dean's method for beach-fill design." *J. Wtrwy., Port, Coast., and Oc. Engrg.*, ASCE, 122(3), 143–146.
- Karlin, S., and Taylor, H. M. (1975). *A first course in stochastic processes*, 2nd Ed., Academic, San Diego.
- Misiewicz, J. K., and Cooke, R. M. (1999). "Isotropic sequences of random variables and stochastic rescaling." *Third Scandinavian-Ukrainian Conf. on Probability and Statistics*, Kyiv, Ukraine (in press).
- Rijkswaterstaat. (1990). *A new coastal defence policy for The Netherlands*. Ministry of Transport, Public Works and Water Management, The Hague, The Netherlands.
- Rijkswaterstaat. (1993a). *Coastline management; from coastal monitoring to sand nourishment*. Ministry of Transport, Public Works and Water Management, The Hague, The Netherlands.
- Rijkswaterstaat. (1993b). "Evaluatie van Zandsuppleties [Evaluation of sand nourishments]." *Tech. Rep. DGW-93054*, Ministry of Transport, Public Works and Water Management, The Hague, The Netherlands.
- Ross, S. M. (1970). *Applied probability models with optimization applications*. Dover, New York.
- Speijker, L. J. P., van Noortwijk, J. M., Kok, M., and Cooke, R. M. (2000). "Optimal maintenance decisions for dikes." *Probability in the Engineering and Informational Sciences*, 14, 101–121.
- van Noortwijk, J. M., Cooke, R. M., and Kok, M. (1995). "A Bayesian failure model based on isotropic deterioration." *Eur. J. Operational Res.*, 82(2), 270–282.
- van Noortwijk, J. M., Kok, M., and Cooke, R. M. (1997). "Optimal maintenance decisions for the sea-bed protection of the Eastern-Scheldt Barrier." *Engineering probabilistic design and maintenance for flood protection*, Roger Cooke, Max Mendel, and Han Vrijling, eds., Kluwer Academic, Dordrecht, The Netherlands, 25–56.
- van Noortwijk, J. M., and van Gelder, P. H. A. J. M. (1996). "Optimal maintenance decisions for berm breakwaters." *Struct. Safety*, 18(4), 293–309.
- Verhagen, H. J. (1993). "Method for artificial beach nourishment." *23rd Int. Conf. on Coast. Engrg.*, Billy L. Edge, ed., ASCE, New York, 2474–2485.
- Wagner, H. M. (1975). *Principles of operations research*, 2nd Ed., Prentice-Hall, Englewood Cliffs, N.J.
- Weir, A. J. (1973). *Lebesgue integration & measure*. Cambridge University Press, Cambridge, U.K.