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Fuzzy label methods for constructing imprecise limit state functions

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Abstract

In reliability analysis of engineering systems it is conventional to represent the limit state function as a precise surface. Uncertainty in the limit state function may be represented by introducing one or more additional random variables. However, the meaning of the additional random variable(s) is unclear and seldom does justice to the uncertainties in the subtle combination of expert judgement and sometimes scarce data from which the limit state function is constructed. Two new methods based on linguistic covering of the state space with fuzzy labels are introduced and used to generate an imprecise limit state function from very scarce experimental data. An example from flood defence engineering is used to demonstrate how plausible relaxations of the strong assumptions in the conventional probabilistic approach can generate wide bounds on the probability of system failure.

Keywords: Reliability analysis; Imprecise failure probabilities; Random sets; Possibility analysis; Linguistic labels

1. Introduction

Engineering reliability analysis is conventionally based on precise probabilistic information about the system state variables and a precise limit state function. If the system is characterised by a vector $\mathbf{x} = (x_1, \ldots x_n)$ of basic variables on $X = R^n$ then the probability of failure P_f is

$$P_f = P(g(\mathbf{x}) \leqslant 0) \tag{1}$$

where g is the limit state function and the probability of failure is the probability of limit state violation. If $f_X(\mathbf{x})$ is the joint probability density function over the basic variables, then

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$$P_f = P(g(\mathbf{x}) \leqslant 0) = \int_{g(\mathbf{x}) \leqslant 0} f_X(\mathbf{x})\mathrm{d}\mathbf{x}. \tag{2}$$

If the n basic variables are independent then

$$f_x(\mathbf{x}) = \prod_{i=1}^{n} f_{X_i}(x_i). \tag{3}$$

Whilst reliability methods are well established in theoretical terms, form the basis for modern codes of practice and have found some application, they have been criticised on several grounds [1–3]. Amongst the most significant is the criticism that in any given application there is seldom sufficient information to define precisely the joint probability distribution describing the basic variables or indeed the form of the limit state function.

1.1. Reliability calculations with random set arguments

The problem of conducting reliability analysis when there is insufficient information to define probability distributions for the basic variables has been addressed quite widely by reformulating reliability calculations to accept information in a range of formats, including probability intervals [4], fuzzy sets [1,5,6], possibility distributions [7], convex sets [3,8], imprecise probabilities [9,10] and random sets [11–13]. This paper adopts random set notation in order to develop a general framework for combining probabilistic and possibilistic analysis [14]. Random set theory was originally conceived independently by Kendall [15] and Matheron [16] in connection with stochastic geometry. Random set theory has found application particularly in the field of pattern recognition where multiple sensors are observing an unknown multiple number of targets [17]. The situation is analogous to the reliability problem of detecting some unknown and possibly stochastic system behaviour on the basis of imprecise observations.

Imprecise measurements, convex bounds on parameter values and linguistic judgements expressed as fuzzy sets (where fuzzy sets are interpreted as consonant random sets) can all be handled through random set theory, by extending the conventional probability distribution on X to the power set $\mathcal{P}(X)$. Following Dubois and Prade [14,18], a finite support random set on a universal set X is a pair $(\Im, m)$ and a *mass assignment* is a mapping

$$m : \Im \rightarrow [0, 1] \tag{4}$$

such that $m(\emptyset) = 0$ and

$$\sum_{A \in \Im} m(A) = 1 \tag{5}$$

Each set $A \in \Im$ contains the possible values of a variable $x \in X$, and $m(A)$ can be viewed as the probability that xA but does not belong to any special subset of A. The reliability problem then becomes that of finding the bounds on

$$P_f = P(g(\mathbf{x}) \leqslant 0) \tag{6}$$

subject to the available knowledge restricting the allowed values of $\mathbf{x}$. The dependency between $(x_1 \ldots x_n)$ can be expressed as a random relation R, which is a random set $(\mathcal{R}, \rho)$ on the Cartesian product $X_1 \times \ldots \times X_n$, in which case the range of g is the random set $(\mathcal{F}, m)$ such that [14]:

$$\begin{aligned} &\mathcal{F} = \left\{ g(R_i) \mid R_i \in \mathcal{R} \right\}, \quad g(R_i) = \left\{ g(\mathbf{x}) \mid \mathbf{x} \in R_i \right\} \\ &m(A) = \sum_{R_i : A = g(R_i)} \rho(R_i) \end{aligned} \tag{7}$$

If the set of failed states is labelled $F \subseteq X$, the upper and lower bounds on the probability of failure are then the plausibility $Pl(F)$ and belief $Bel(F)$ respectively:

$$Bel(F) \leqslant P_f \leqslant Pl(F) \tag{8}$$

where

$$Bel(F) = \sum_{A_i : A_i \subseteq F} m(A_i) \tag{9}$$

$$Pl(F) = \sum_{A_i : A_i \cap F \neq \varnothing} m(A_i) \tag{10}$$

Eqs. (7)–(10) form the basis for evaluation of the bounds on system reliability with random set variables, an approach that has been demonstrated in engineering geology [11,13] and flood defence engineering [19].

1.2. Handling uncertainty in the limit state function

The problem of constructing a limit state function from scarce experimental data, combined usually with some (incomplete) expert knowledge of the causal relationships within the system of interest is ubiquitous and fundamental in reliability analysis. Conventionally, uncertainty in the limit state function has been addressed by adding one or more random variables to the state variable set to represent uncertainty, so the system is then described by variables $\mathbf{x}' = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ [20,21]. The parameters $x_{n+1}, \ldots, x_{n+m}$ are supposed to be estimated from analysis of the residual difference between observations of $\mathbf{x} = (x_1, \ldots, x_n)$ at failure and the surface $g(\mathbf{x}) = 0$. These joint measurements are often obtained from laboratory tests, less frequently from back analysis of failures. Joint measurements of of $\mathbf{x} = (x_1, \ldots, x_n)$ at failure will invariably show some scatter, because $(x_1, \ldots, x_n)$ is merely a judicious sub-set of all of the possible variables that influence system behaviour. In other words there are 'hidden variables' not included in $(x_1, \ldots, x_n)$. Moreover, g may not be a good model of system behaviour throughout X.

If the distribution of residuals around $g(\mathbf{x})$ is assumed to be constant over X then the uncertainty is characterised by a single parameter x_{n+1}, which is often assigned a Gaussian distribution. The Gaussian distribution is justified on the grounds that all of the main influences on system behaviour have been captured in $x_1, \ldots, x_n$, so what remains is the combination of small influences of many hidden variables. Provided no single or small number of non-Gaussian hidden variables dominates the uncertainty then it is argued from the Central Limit Theorem that the combined influence should be Gaussian distributed. There is of course no reason to believe a priori that the model uncertainty is due to the combination of a large number of hidden variables each with a relatively small influence. Indeed when more knowledge is acquired about system behaviour, more often than not it results in a systematic reformulation of the model rather than merely a marginal reduction in the scatter around the original model by introducing another parameter or two.

The problem is particularly severe when experimental data is very expensive to obtain and there are few and poorly documented instances of prototype scale system failure (as is commonly the case in structural engineering). For example the data on failure of flood defence revetments presented later in this paper were obtained from expensive large-scale destructive tests. Under these circumstances, expert interpretation of scarce data on the basis of causal knowledge and analogous cases is fundamental to the process of constructing a limit state function. There is simply not enough data to let it 'speak for itself', by merely 'mining' relationships in the dataset. Even with the help of expert interpretation is may be impossible to justify one particular form of the limit state function in preference to any other. A further criterion of parsimony may be introduced to identify a relatively simple function, but it is still necessary to uniquely fit at least one parameter on the basis of limited empirical justification. The methods of robust statistics [22,23] may help to ensure that the parameter estimate is not overly sensitive to assumptions of a Gaussian distribution.

The conventional probabilistic format does not provide a convenient mechanism for representing varying states of knowledge or uncertainty about the limit state function. This issue is addressed by Bayesian methods, where increasing knowledge is represented by reducing variance in parameter distributions [24], though this uncertainty is then integrated out through the Bayes predictor, providing no explicit indication of the uncertainty in the probability of failure, without further analysis. Yet decision-makers may need and/or ask for a display of uncertainties in failure estimates before making an informed judgement [25].

A more fundamental argument against the conventional approach to parameterising model uncertainty is provide by Blockley [1,26], who argues that the meaning of the model parameter x_{n+1} is far from clear. He demonstrates, based on an argument originally developed by Rescher [27], that using a probabilistic parameter to represent the degree of belief in a model leads to logically absurd conclusions about the truth of the model. He concludes that probability theory should not be used to measure plausibility, credibility or dependability. Moreover, Blockley [2] argues that reducing model uncertainty to a single parameter is inadequate because the level of sophistication of handling such a difficult and important part of the total uncertainty is very much less than for the relatively straightforward issue of uncertainty in the system state variables. Paté-Cornell [25] concurs, suggesting that these uncertainties are "often ignored or under-reported, especially in public policy studies of controversial or politically sensitive issues".

The approach proposed in this paper aims to overcome some of the problems of the conventional treatment outlined above.

- In situations of very scarce data, a family of plausible limit state functions is used in the reliability analysis in order to avoid having to uniquely identify a single limit state function. Consequently, plausible bounds on the probability of system failure are generated in stead of a point value.
- Model uncertainty is not represented by additional basic system variables. It is represented in the imprecise format of the limit state function.

- Uncertainty is not represented using conventional probability theory. Instead a more general random set approach is adopted. This has the advantage of admitting imprecise information, which is an essential aspect of expert reasoning about model uncertainty.

In formal terms there is no distinction between

(i) treating uncertainty in the limit state function as one or more additional parameters in the reliability problem and,
(ii) explicitly representing uncertainty in the form of the limit state function.

In conceptual terms, however, it is useful to distinguish between those basic variables of the system that are measurable in nature, irrespective of the way the system is abstracted for the purpose of analysis, and those parameters that are fitted so that the system abstraction is a good representation of the current state of knowledge about the real world.

In Sections 3 and 4 of this paper, two alternative methods for combining expert knowledge with very scarce experimental data in order to generate an imprecise limit state function are proposed and are illustrated with an example from flood defence engineering. The first method (described in Section 3) uses expert knowledge to develop, on a case-by-case basis, a family of plausible limit state functions, which are then used to classify the state variable space. The second method (described in Section 4) uses prior expert knowledge to restrict the area of interest in state space, and then conditions scarce measured data on this imprecise prior knowledge. Both methods generate an imprecise conditional probability distribution of system failure. In other words, at any given point in state space a probability mass of unity is distributed between three states: 'failed', 'not failed' and 'unknown'. First, in the following section, definitions and notation used in both methods are introduced. In Section 5 both methods are demonstrated with an example of a reliability analysis of a flood defence embankment. The merits and disadvantages of the proposed approach are reviewed in Section 6 before drawing conclusions in Section 7.

2. Label descriptions of data

A dataset D of r points on $X_1\times\ldots\times X_n$:

$$D=\{(x_1(i),\ \ldots,\ x_n(i))\mid i=1,\ \ldots,\ r\} \tag{11}$$

can be represented by a mass distribution on a fuzzy set space $LA_1\times\ldots\times LA_n$, where LA_i is a finite set of fuzzy sets or 'labels' that describe X_i. The distribution on $LA_1\times\ldots\times LA_n$ can be thought of as a linguistic or 'label' description of the relationships in the dataset. Both of the methods proposed in this paper make use of label descriptions of, often scarce, datasets. The label description has the effect of generalising the point-valued data is a way that is analogous to non-parametric density estimation methods [28]. The approach has the additional attraction of generating a description of the dataset that has a direct semantic interpretation, which is intelligible to experts who are reasoning about and reflecting upon relationships in the data.

Three definitions are introduced:

1. The *linguistic covering* establishes how X_i is partitioned into a set of labels.
2. The *joint density estimate on labels* establishes how the mass distribution on $LA_1\times\ldots\times LA_n$ is constructed based on the dataset D.
3. The *posterior density on labels* establishes how a continuous distribution on $X_1\times\ldots\times X_n$ can be estimated, given the mass distribution on $LA_1\times\ldots\times LA_n$.

Definition 1 (Linguistic covering) [29]. A set of fuzzy sets $F_1, \ldots, F_n$ forms a linguistic covering of X if and only if $\forall x\in X$ max $(\mu_{F_1}, ..., \mu_{F_n})=1$.

For any $x\in X$, a unit mass can be distributed over the a of labels $\{l_1, \ldots, l_k\}$ covering that point, according to the fuzzy memberships of the labels. Suppose that $\{l_1, \ldots, l_k\}=\{l_i\in LA\ |\mu_{l_i}(x)>0\}$ and that $\{l_1, \ldots, l_k\}$ is ordered such that

$$\mu_{l_i}(x) \geqslant \mu_{l_{i+1}}(x) \text{ for } i=1,\ldots,k-1. \tag{12}$$

If $L_i=\{l_1, ..., l_i\}|1\leqslant i\leqslant k$ then the mass distribution at x can be written as a random set $(\mathcal{F}_x, m_x)$, where

$$\mathcal{F}_x = \{L_i \mid i=1, \ \ldots, \ k\} \tag{13}$$

$$m_x(L_i) = \begin{cases} \mu_{l_i}(x) - \mu_{l_{i+1}}(x) & |i=1,...,k-1 \\ \mu_{l_k}(x) & |i=k \end{cases}$$

This is referred to as the label description of x. For example if the set of labels {small}, {medium} and {large} provides a linguistic covering of the space of $x\in[0,100]$ as shown in Fig. 1, then $\mathcal{F}_{30}=\{small, medium\}$: 0.5, $\{small\}$: 0.5.

Notice that in order for $\mathcal{F}_x$ to be a normalised random set, in the sense that zero mass is allocated to the empty set for every x, then the set of labels LA must form a linguistic covering as given in Definition 1. Normalised random sets are desirable in this context since otherwise mass is associated with the possibility that none of the sets in LA are appropriate as labels for some x and this makes prediction more problematic.

The idea of a label description of a point can be extended to obtain a label description of a database of measurements, so that each element in the database will be described by a random set signifying a mass value for each subset of LA.

Definition 2 (Joint density estimate on labels). Suppose that each of the r elements in the database has two attributes that can be classified against label sets LA_1 and LA_2 respectively, then $\forall(S_1, S_2) \subseteq LA_1 \times LA_2$ the label description of the database D

$$D = \left\{(x_1(i), \ x_2(i)) \mid i=1, \ \ldots, \ r\right\} \tag{14}$$

on $2^{LA_1} \times 2^{LA_2}$ is defined by

$$m_D(S_1, S_2) = \frac{1}{n}\sum_{i=1}^{n} m_{x(i)}(S_1).m_{y(i)}(S_2) \tag{15}$$

This approach can be applied to the classification of the variable space in a reliability problem based on incomplete information. Suppose knowledge about the system behaviour comprises a

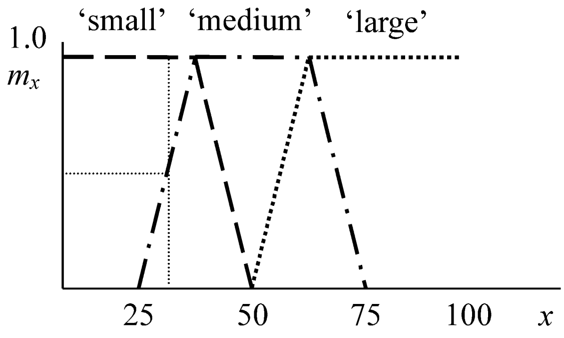

Fig. 1. Linguistic covering of $x \in [0,100]$.

database of r tests of system response and in each test the response has been categorised as belonging to one of three classes, $C_1 = \{failed\}$, $C_2 = \{not\ failed\}$, $C_3 = \{failed,\ not\ failed\}$ *i.e.* 'unknown'. Now consider the sub-database of instances with class C_j,

$$D_j = \{(x_1(i),\ \ x_2(i)) \mid C(i) = C_j\} \tag{16}$$

m_{Dj} can be evaluated according to Eq. (15) to find the mass assignment on $LA_1 \times LA_2$ describing class C_j.

Definition 3 (Posterior density from labels) The posterior density from labels $p(x_1, x_2|m_{D[C_j]})$ is defined as

$$p(x_1, x_2|m_{D[C_j]}) = u(x_1, x_2)\sum_{S \times R}\frac{m_{D[C_j]}(S_1, S_2)}{m_1(S_1).m_2(S_2)}m_{D[x_1]}(S_1).m_{D[x_2]}(S_2) \tag{17}$$

where

$u(x_1,x_2)$ is the prior density,
$m_{D[C_j]}(S_1, S_2)$ is the mass on the (joint) label (S_1,S_2),
$m_{D[x_1]}(S_1)$ and $m_{D[x_2]}(S_2)$ are derived from membership levels at x_1 and x_2 [see Eq. (13)], and
$m_1(S_1)$ and $m_2(S_2)$ normalise.

The posterior density can be interpreted as a density on X conditional upon the information on LA space. Therefore, if $p(x_1,\ x_2 \mid m_{D_j})$ is used as an estimate of $p(x_1,\ x_2 \mid C_j)$ the distribution at point $(x_1,\ x_2)$ over the states C_j is given by

$$m_{x,y}(C_j) = \frac{p(x, y|m_{D_j})p(C_j)}{\sum_{j=1}^{3}p(x, y|m_{D_j})p(C_j)}, \quad \sum_{j=1}^{3}m_{x,y}(C_j) = 1 \tag{18}$$

Each point $(x_1,\ x_2)$ then has an imprecise conditional probability of failure

$$P_f\ (x_1,\ \ x_2) \in [m_{x,y}(C_1),\ \ m_{x,y}(C_1) + m_{x,y}(C_3)] \tag{19}$$

The conventional reliability problem in Eq. (2) becomes

$$P_f = P(g(\mathbf{x}) \leqslant 0) = \int_{g(\mathbf{x}) \leqslant 0} f_X(\mathbf{x}).P_f\ (\mathbf{x})\mathrm{d}\mathbf{x}, \tag{20}$$

which will yield bounds on the probability of failure even if the basic variables are precise probability distributions.

3. Method 1: imprecise classification of state variable space based on a family of permissible limit state functions

Consider the typical situation where there are some scarce experimental results, measured at failure, which relate basic variables $x_1, \ldots, x_n$. In this case, for simplicity of representation, $n=2$ (Fig. 2), but the method is extendable to higher dimensions, albeit with some practical limitations, which are discussed in Section 6. An expert will typically use causal understanding of the system (besides the knowledge they have already used to select the basic variables) to select a parameterised family of functions to fit to the data. Suppose in the case illustrated that there is no reason to believe that the limit state function is linear. All that is known is that it must be continuous and monotonic and must pass through the origin. Monotonicity is a common feature of reliability problems for the following reason. Suppose that x_1 represents the strength and x_2 represents the load. Then an increase in x_2 for constant or reducing value of x_1 will make the structure more prone to failure. Similarly an increase in x_1 for constant or reducing x_2 will make the structure less prone to failure.

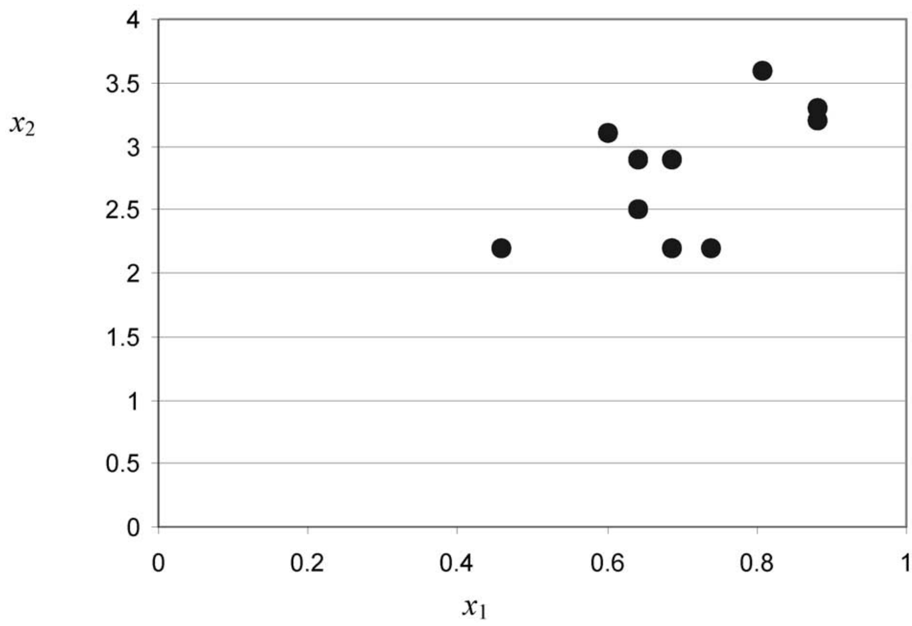

Fig. 2. Typical experimental measurements on of (x_1, x_2).

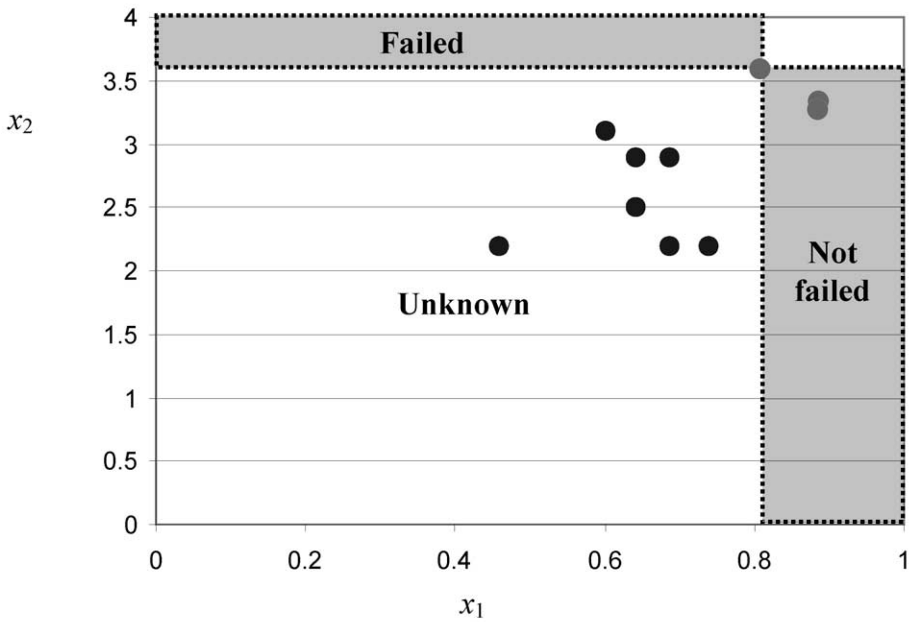

Fig. 3. Classification of $X_1 \times X_2$ on the basis of one experimental point.

Now consider the information that each experimental point provides. Provided measurement errors are negligible, each point can be considered as a deterministic realisation of an underlying random process. Any given point (one such point has been identified arbitrarily in Fig. 3) can be used to classify the parameter space on the basis of that deterministic realisation of the experiment and the assumed form of the limit state function. If the limit state function were assumed to be linear with fixed gradient then each experimental point could be thought of as precisely partitioning the space into a 'failed' region and a 'not failed' region. If all that can be assumed is that the limit state function is monotonic increasing, then each experimental point partitions the space into a 'failed' quadrant, a 'not failed' quadrant and two 'unknown' quadrants (Fig. 3). An intermediate classification would, for example, be a family of quadratic curves. The classification can be repeated on the basis of each point to build up a database of classifications. Even with the weak assumption of monotonicity there will usually be some conflict in the classification of some regions of the space.

The database of classifications of the space can be used to construct a label description on $LA_1 \times LA_2$ of the data. A convenient way of doing this is to discretize $X_1 \times X_2$ into a grid of t regularly spaced points $\{x_1(i), x_2(i) \mid i = 1, \ldots, t\}$. Each point in the grid can be classified on the basis of each of r experimental point to generate a database of $r \times t$ points. These can then be mapped onto label description for each of the three classes, C_1, C_2, C_3, on $LA_1 \times LA_2$ [Eqs. (14) and (15)]. This has the effect of constructing a linguistic description of the relationship in the data.

Finally the posterior distribution on $X_1 \times X_2$ is obtained from Eq. (17). Recall that this is the posterior distribution on $X_1 \times X_2$ given the information that is represented by the label description on $LA_1 \times LA_2$. The problem arises as to what prior distribution $u(x)$ should be adopted. One approach would be to identify a set of distributions that must contain $u(x)$ [30] and then obtain interval probabilities by taking the upper and lower bounds. If there is no information to restrict this set of distributions, taking upper and lower bounds across all distributions gives $p(I|L_i) \in [0,1]$ for any I, a measurable subset of X, for which

$$\int_I \mu_{L_i}(x)\mathrm{d}x > 0. \tag{21}$$

In Method 2 (introduced in Section 4 below) expert knowledge is used to construct a set of priors. For the time being in Method 1 the simple and more conventional assumption that $u(x)$ is the uniform distribution has been adopted. This has the advantage of being the maximum entropy distribution and hence, of all precise probability distributions, introduces minimum prior information into the model. Such a property in itself would seem to lend some justification to this choice as it gives maximum possible weighting to the information contained in the data.

The posterior density assigns an imprecise conditional probability of failure to each point on $X_1 \times X_2$, [Eqs. (18) and (19)], which is the imprecise limit state function.

In summary, Method 1 involves the following steps

1. Discretize $X_1 \times X_2$ into a regular grid of t points.
2. On the basis of each of r experimental points, classify each of t points in the grid as being in class C_1, C_2 or C_3, thereby obtaining a database of $r \times t$ classified points.
3. Construct linguistic covering on LA_1 and LA_2 of the axes X_1 and X_2, respectively (Definition 1).
4. Calculate the membership in $LA_1 \times LA_2$ of each point t [Eq. (13)].
5. Use the database of $r \times t$ classified points to generate a mass distribution on $LA_1 \times LA_2$ for each of the three classification states: C_1, C_2 or C_3 [Eqs. (14) and (15)].
6. Assuming a uniform prior distribution on X, estimate the posterior distribution on X, conditional on the label description on $LA_1 \times LA_2$ for each of the three classified states [Eqs. (17) and (18)]. This is the distribution of the imprecise conditional probability of failure.

4. Method 2: conditioning with scarce data on prior linguistic information about the form of the limit state function

In the method proposed in the previous section a uniform prior was assumed in Eq. (17) in order to construct the posterior density on X. An alternative approach is to exploit the Bayesian form of Eq. (17) to capture, in the prior distribution or family of priors, expert knowledge about the form of the limit state function. Data relating to the limit state function can then be used to condition on this prior information. The prior information will typically be derived from physical understanding of the processes at work and analogous cases. It therefore seems natural to represent this prior information as an imprecise classification of the parameter space. The approach is consistent with engineering practice. Fig. 4, which has been extracted from engineering guidelines

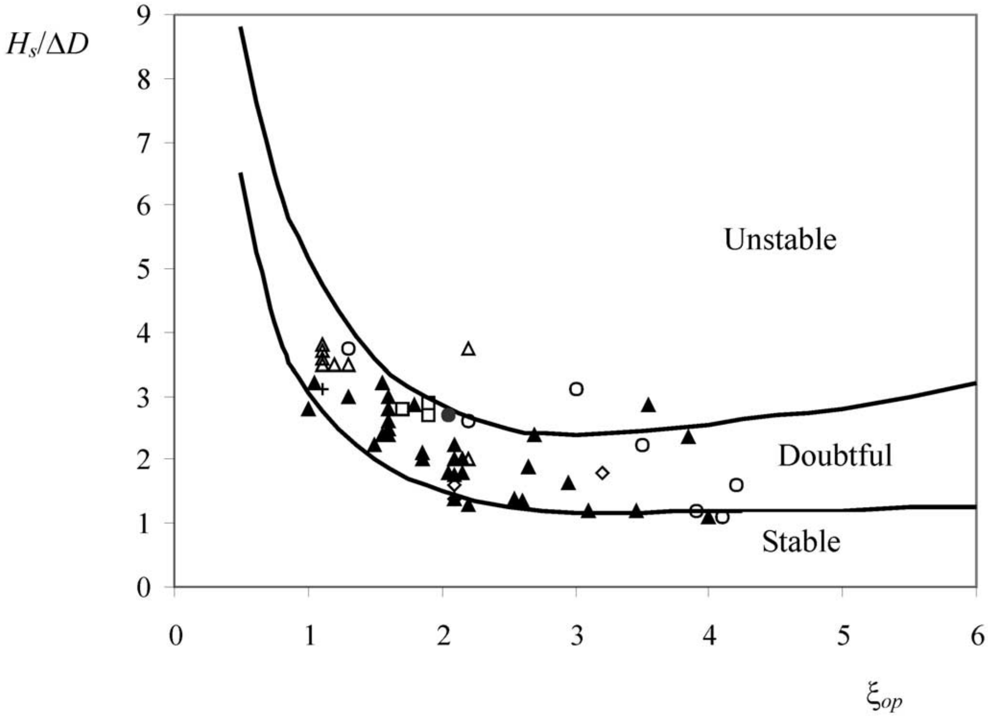

Fig. 4. Classification of parameter space in condition assessment guidance for flood defence revetments [31].

on the assessment of revetment structures on flood defence dikes [31], shows, in effect, an imprecise classification of the parameter space. The limit state function can be thought of as lying somewhere in the 'doubtful' region.

The expert classification of the space shown in Fig. 4 provides precise boundaries to the classification regions, whilst the classifications are themselves linguistic, suggesting a degree of fuzziness in the definition of the regions. Examination of the data that the experts used to inform the classification suggests that the boundaries are by no means crisp. Thus it is desirable to 'fuzzify' the boundaries to the classification, which can be achieved using a similar approach to Method 1, described in the previous section. That is, the space is discretised into a rectangular grid and each point in the grid is classified according to the region within which it lies. The database of points is then mapped to a label classification on $LA_1 \times LA_2$ and then mapped back to generate a prior density estimate on X. The prior density estimate will be smoother if there are fewer labels in LA. In other words, a greater number of labels represents more confidence in the precise classification of X. Rather than having to make an inevitably somewhat arbitrary choice of the number of labels in LA, it is possible to proceed with a family of estimates in which case a family of density estimates will be generated, resulting in an imprecise classification of the space.

It may be more convenient to elicit the label description of the limit state function directly in linguistic terms rather than mapping it from a classification of X. The label description on $LA_1 \times LA_2$ can be thought of as a linguistic description of the relationship between X_1 and X_2. Thus experts could be asked to describe, according to certain protocols, the relationship between x_1 and x_2, from which the mass distribution on $LA_1 \times LA_2$ could be constructed [32].

Next, data specific to the system in question are used to construct a label description of the limit state function on $LA_1 \times LA_2$. Unlike the Method 1, the data are not used to classify the space. In stead, each experimental point measured at failure is mapped onto $LA_1 \times LA_2$ according to Eq. (15). This has the effect of generalising the scarce experimental data. The label description of the data is then used to condition the prior description of the space by means of the Bayesian inversion in Eq. (17). The posterior density estimate is modified, and in particular regions where it is physically impossible for the limit state function to lie is excluded, by conditioning on the prior expert knowledge, which captures physical reasoning about plausible location of the limit state function. The result is, as in Method 1, an imprecise description of the limit state function, which can be used in to calculate an imprecise probability of system failure [Eq. (20)].

The degree of smoothness of the posterior density estimate reflects the number of labels used in the prior estimate and the label description of the data. The sum across X of the local directional derivatives of the density estimate is used as a measure of smoothness. The likelihood of the sample data is used as a measure of goodness of fit. A family of distributions that represent a compromise between these two criteria are used to generate the family of plausible limit state functions.

1. In summary, Method 2 involves the following steps:

(a) Construct the imprecise prior conditional probability distribution of failure
(b) Discretize $X_1 \times X_2$ into a regular grid of t points.
(c) Elicit a prior expert classification of the parameter space.
(d) On the basis of each of the regions identified by the experts, classify the each of t points in the grid as being possibly on the limit state function (i.e. in the 'doubtful' region) or not possibly on the limit state function (i.e. outside the 'doubtful' region).
(d) Construct a linguistic covering on LA_1 and LA_2 of the axes X_1 and X_2 (Definition 1).
(e) Calculate the membership in $LA_1 \times LA_2$ of each point on the grid on $X_1 \times X_2$ [Eq. (13)] within the 'doubtful' region and generate a mass distribution on $LA_1 \times LA_2$ [Eq. (15)], which is a label description of the limit state function.
(f) Assuming an initial uniform prior distribution on X, estimate a new prior distribution on X [Eq. (17)] conditional on the label description on $LA_1 \times LA_2$. This is the prior limit state function.

2. Construct the imprecise posterior conditional probability of failure

(a) Construct a linguistic covering on LA_1 and LA_2 of the axes X_1 and X_2 (Definition 1). This is not necessarily the same linguistic covering as in 1(a).
(b) Calculate the membership in $LA_1 \times LA_2$ of each experimental point on $X_1 \times X_2$ [Eq. (12)].
(c) Use the database of r classified points to generate a mass distribution on $LA_1 \times LA_2$ [Eq. (15)], which is a label description of the data.
(d) Using the prior distribution on $X_1 \times X_2$ developed in Step 1, estimate the posterior distribution on $X_1 \times X_2$ [Eq. (17)], conditional on the label description on $LA_1 \times LA_2$. This is the distribution of the imprecise conditional probability of failure.

3. Test the sensitivity of the posterior distribution to the label definition and identify a plausible family of limit state functions.

5. Example from flood defence engineering

The example is based on a previous conventional reliability analysis of a dike on the Frisian coast in the Netherlands, along the Wadden Sea [33]. The behaviour of the concrete block revetment on the seaward slope of the dike is described by basic variables $\mathbf{x} = (\Delta, D, H_s, \alpha, M, s_{op})$ where

Δ	is the density of the revetment blocks,
D	is the diameter of the revetment blocks,
H_s	is the significant wave height,
α	is the slope of the revetment,
M	is a model parameter and
s_{op}	is the offshore peak wave steepness.

The limit state function $g(\mathbf{x})$ is given by

$$g(\mathbf{x}) = \Delta D - H_s \frac{\xi_{op}}{M\cos\alpha} \tag{22}$$

where $\xi_{op} = s_{op}^{-0.5}\tan\alpha$. The wave height H_s in shallow water is related to the water depth. Given a particular water level h, the wave height is reported to be Gauss distributed with

$$\mu_{H_s|h} = 0.224h + 0.117, \quad \sigma_{H_s|h} = 0.04h - 0.05 \tag{23}$$

where h is Gumbel distributed with parameters $\alpha = 0.36$, $\xi = 2.91$ and there are 3 storm events each year [33]. The wave height therefore also conforms reasonably closely to a Gumbel distribution with $\alpha = 0.12$ and $\xi = 0.94$. The other parameters in the original analysis were assumed to be Gauss distributed and independent (Table 1). The probability of failure of 9×10^{-4} per year was calculated according to Eq. (3), which is conveniently solved using first order second moment (FOSM) or Monte Carlo methods [34].

Sensitivity factors (directional cosines $\cos\alpha_i$) [34] from the FOSM analysis are also listed in Table 1. They illustrate that the probability of failure is most strongly influenced by the variance in M, yet only ten experimental points were available to establish the distribution of M (plotted in Fig. 2 with $x_1 = \cos\alpha/\xi_{op}$ and $x_2 = H_s/\Delta D$). In the following analysis the uncertainty parameter M is removed and imprecise limit state functions are constructed using the two methods introduced in Sections 3 and 4.

From Eq. (23), when $g(\mathbf{x}) = 0$

$$\frac{H_s}{\Delta D} = \frac{M\cos\alpha}{\xi_{op}} \tag{24}$$

Now $H_s \geqslant 0$ so, if the limit state function is described by Eq. (24), in a plot of $x_1 = H_s/\Delta D$ against $x_2 = \cos\alpha/\xi_{op}$ the limit state function will be a line passing through the origin with a slope

Table 1
Means and standard deviations of basic variables [33]

Variable	Distribution	Parameter 1	Parameter 2	α_i
H_s	Gumbel	$\alpha=0.12$	$\xi=0.94$	0.515
s_{op}	Gaussian	$\mu=0.036$	$\sigma=0.004$	−0.154
$\tan\alpha$	Gaussian	$\mu=0.33$	$\sigma=0.01$	0.087
Δ	Gaussian	$\mu=1.62$	$\sigma=0.02$	−0.032
D	Gaussian	$\mu=0.70$	$\sigma=0.02$	−0.076
M	Gaussian	$\mu=4.06$	$\sigma=0.698$	−0.834

M. There are physical arguments constraining the limit state function to pass through the origin. There are no physical reasons for it to be linear, though the assumption of linearity is explicit in Eq. (22). The experimental data do not provide strong support for the assumption of linearity.

5.1. Method 1

Each of the ten experimental point was used to classify the parameter space into C_1: 'failed', C_2: 'not failed' and C_3: 'unknown' regions, as illustrated in Fig. 3. The variable space was covered with regular grid of 10×10 points (x_1, x_2) on [0,1]×[0,4]. Each point on the grid was then classified as belonging to C_1, C_2, or C_3 on the basis of each experimental point in turn. The label sets LA_1 and LA_2 for X_1 and X_2 were five uniform trapezoidal fuzzy sets on the respective universes. These could be thought of as corresponding to the labels 'very small', 'small', 'medium', 'large', 'very large', which formed a linguistic covering, according to Definition 1, on each of the axes. Fig. 5 illustrates the label set on X_1. The mass assigned to the label set was calculated according to Eq. (15) for each point on the grid of 100 points. The database of 10×100 points was then used to calculate the joint mass assignment on the label set for the three classes, C_1, C_2, and C_3, according to Eqs. (15) and (16). The mass assignment for the 'failed' state (class C_1) is illustrated

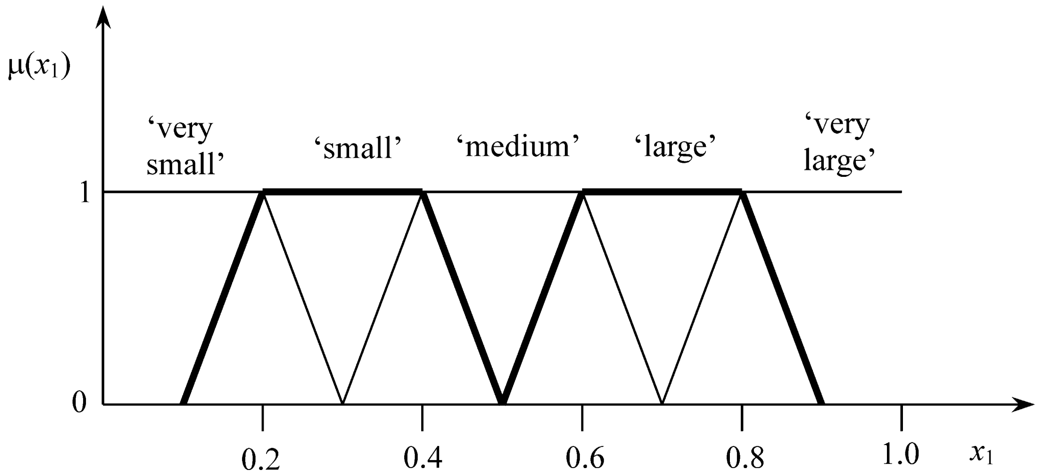

Fig. 5. Marginal label set on X_1.

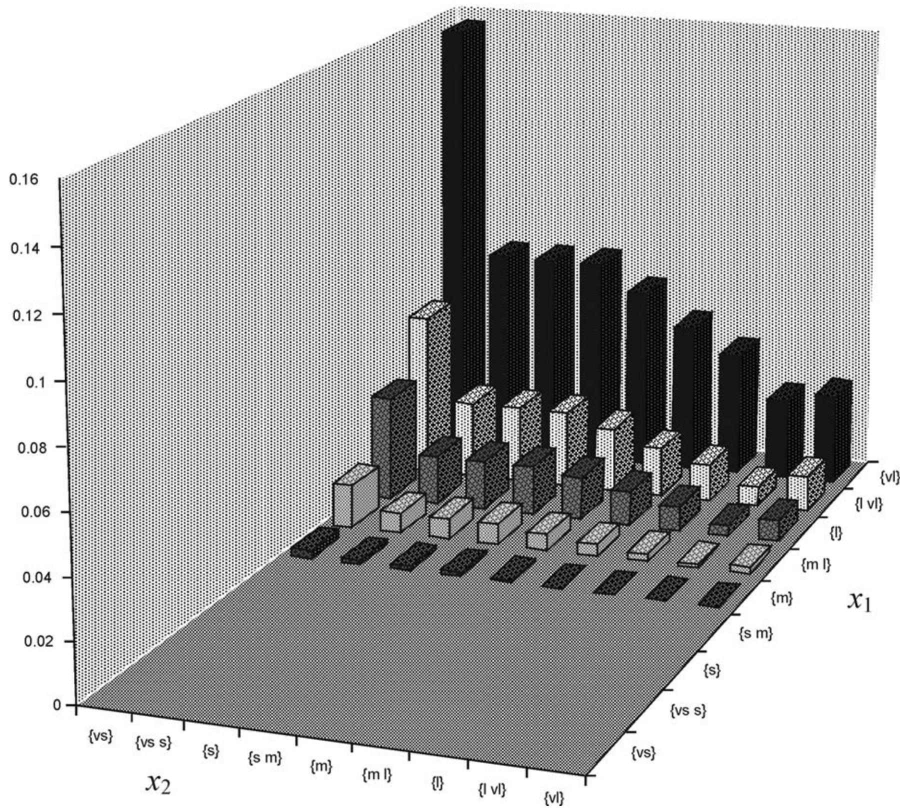

Fig. 6. Label description of the 'failed' region.

in Fig. 6. Using Eqs. (17) and (18) the posterior density $m_{x,y}(C_j)$ was calculated for $j = 1$ to 3. In Fig. 7 each point in the grid has been given classification C_c where

$$m_{x,y}(C_c) = \max_j \left(m_{x,y}(C_j) \right). \tag{25}$$

Note how the extent of the 'unknown' region reflects the distribution of data points. The distribution of $m_{x,y}(C_1)$ is illustrated in Fig. 8.

Integrating the joint distribution of the basic variables **x** (Table 1) over the imprecise limit state function [Eq. (20)] yields a probability of system failure of [2×10^{-5}, 0.87]. These wide bounds reflect the relative weakness of the monotonicity assumption about the limit state function when compared with the linear assumption implicit in the conventional probabilistic method (which yielded a point probability of failure of 9×10^{-4}). Nonetheless, as was argued previously, there are scant grounds to substantiate the assumptions of linearity and Gaussian distribution. The implications of plausibly weakening these assumptions are dramatic.

The sensitivity of the analysis to the definition of the label sets LA_1 and LA_2 was investigated by repeating the methodology with 4 and 6 labels on each axis rather than the 5 used in the original analysis (Table 2). The method is reasonably robust to changes in definition of the linguistic covering.

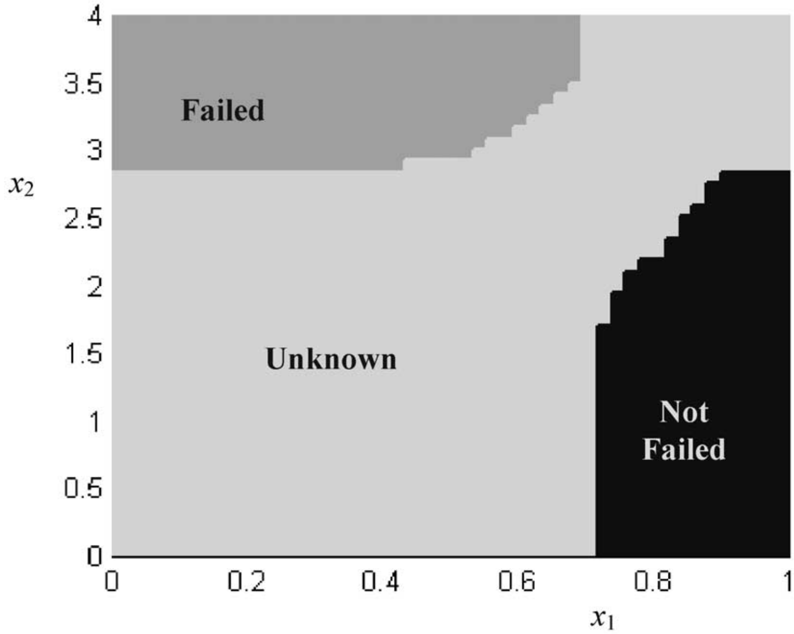

Fig. 7. Imprecise limit state function.

Table 2
Probability of system failure from Method 1 for different numbers of labels

Number of labels on each axis	$P(C_1)$	$P(C_2)$	$P(C_3)$
7	0.228	7×10^{-6}	0.772
9	0.236	4×10^{-6}	0.764
11	0.231	2×10^{-6}	0.769

Further insight into the information carried on the label set can be obtained by considering alternative strategies for reallocating the points classified as 'unknown'. Two strategies were tested:

(i) The points classified as 'unknown' were removed from the database and the mass assignments $m_{x,y}(C_1)$ and $m_{x,y}(C_2)$ normalised to $m_{x,y}*(C_1)$ and $m_{x,y}*(C_2)$, respectively, by setting $m_{x,y}*(C_1)=m_{x,y}(C_1)/(m_{x,y}(C_1)+m_{x,y}(C_2))$ and $m_{x,y}*(C_2)=m_{x,y}(C_2)/(m_{x,y}(C_1)+m_{x,y}(C_2))$. At points where $m_{x,y}(C_1)=m_{x,y}(C_2)=0$, the normalistion is undefined, so $m_{x,y}*(C_1)$ and $m_{x,y}*(C_2)$ were set to 0.5.

(ii) $m_{x,y}(C_3)$ was reallocated equally between $m_{x,y}(C_1)$ and $m_{x,y}(C_2)$, so $m_{x,y}*(C_1)=m_{x,y}(C_1)+m$-$*(C_1)=m_{x,y}(C_1)+m_{x,y}(C_3)/2$ and $m_{x,y}*(C_2)=m_{x,y}(C_2)+m_{x,y}(C_3)/2$. This corresponds to Smets' pignistic probability distribution [35] and Baldwin's least prejudiced distribution [36].

The classification between C_1 and C_2 is the same for both of the reallocation strategies and is illustrated in Fig. 9. The zone marked as 'unknown' corresponds to the area where $m_{x,y}(C_1)=m_{x,y}(C_2)=0$, so reallocation strategy (i) is undefined. Compare this with the conventional

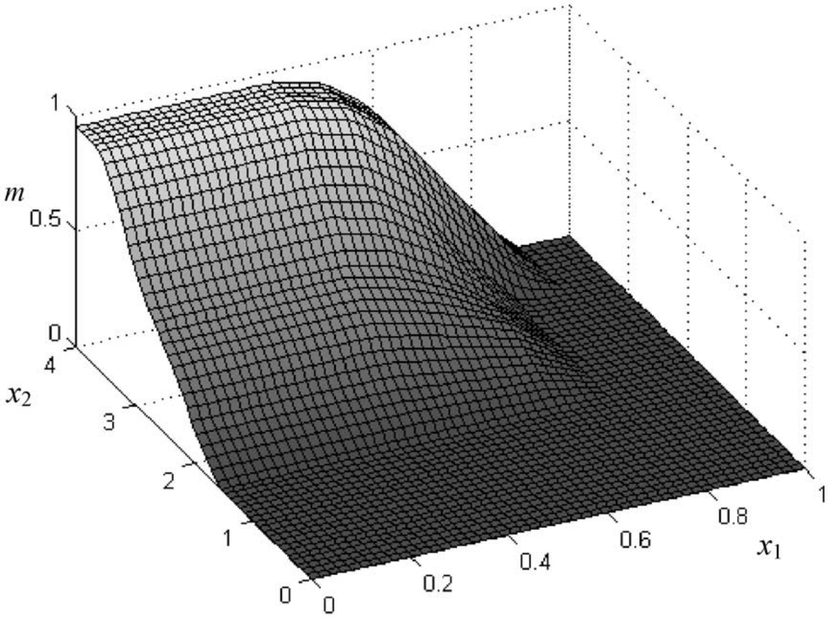

Fig. 8. Distribution of $m_{x,y}(C_1)$.

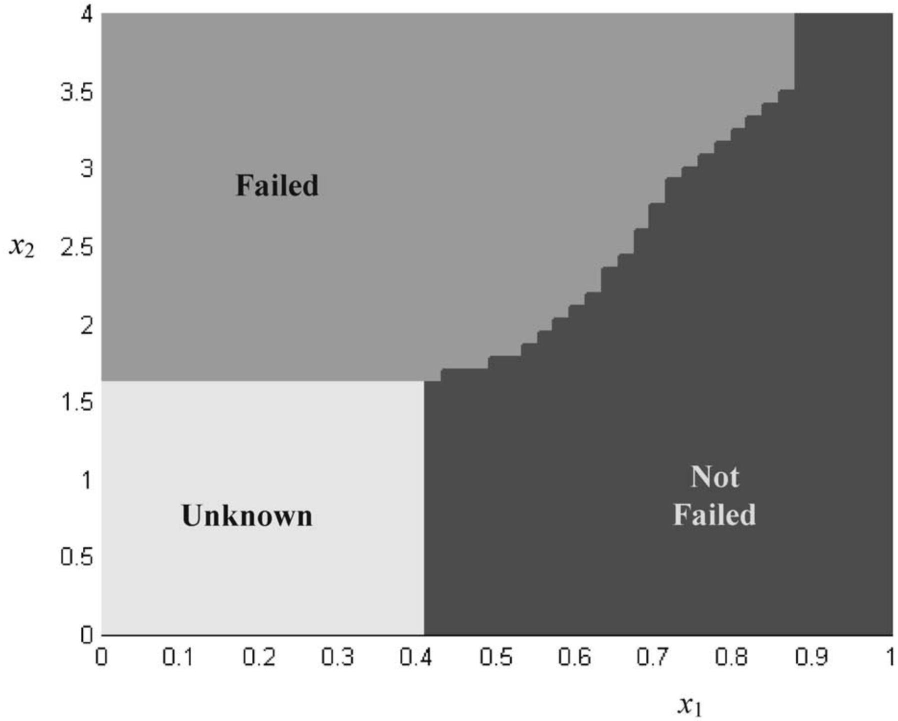

Fig. 9. Precise limit state function after reallocation of uncertain probability.

probabilistic approach to parameterising the uncertainty with M, which corresponds to a straight classification boundary passing through the origin with gradient 4.06. The fuzzy classification mechanism has learnt a more subtle classification. Whilst the classification of the space with the two reallocation methods is the same, the probability distribution is significantly different. Fig. 10 shows the normalised distribution and Fig. 11 shows the least prejudiced distribution.

Integrating over the precise limit state functions derived from normalisation and from the least prejudiced distribution yields system probabilities of failure of 2×10^{-4} and 0.43, respectively. The normalisation assumption is analogous to the conventional probabilistic approach of insisting upon a precise limit state function, and yields a similar probability of failure. The least prejudiced assumption yields a probability of failure mid-way between the bounds obtained from the imprecise limit state function.

5.2. Method 2

The prior probabilities of failure were obtained from the imprecise classification shown in Fig. 4. In order to construct the imprecise priors probability distribution, the space shown in Fig. 4 was covered in a regular grid on [0.5,4]×[0,4]. The points classified as being 'doubtful' were then mapped onto $LA_1 \times LA_2$ to construct a label description of this region. A typical prior density estimate obtained from this label description (using 4 labels on each dimension) is illustrated in Fig. 12. Each of the 10 experimental points was then mapped to $LA_1 \times LA_2$ to generate a label description of the data. A posterior density for the limit state function is illustrated in Fig. 13. Note how the method generalizes from the scarce data, but far from the data the predicted density is zero. Note also how the form of the prior has been modified by the data.

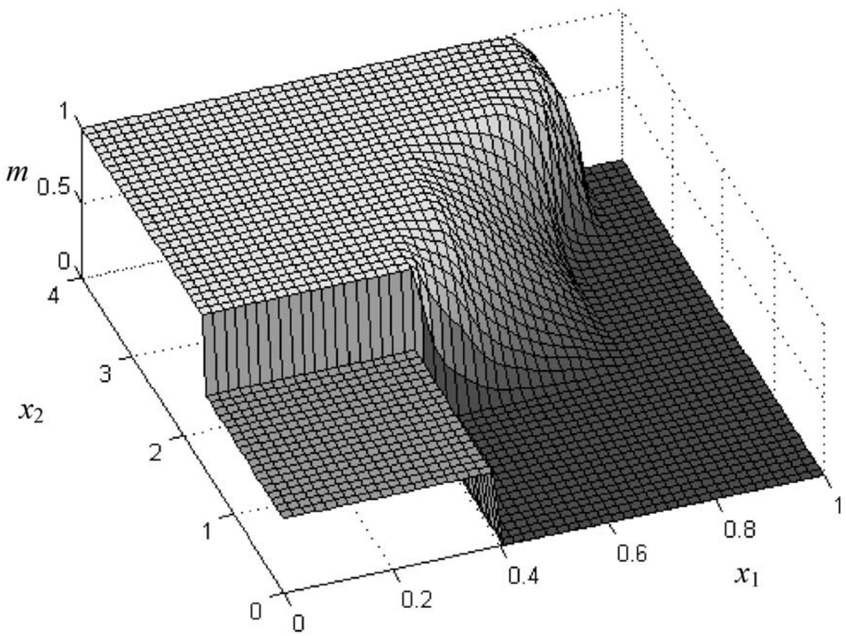

Fig. 10. Normalised distribution of $m_{x,y}*(C_1)$.

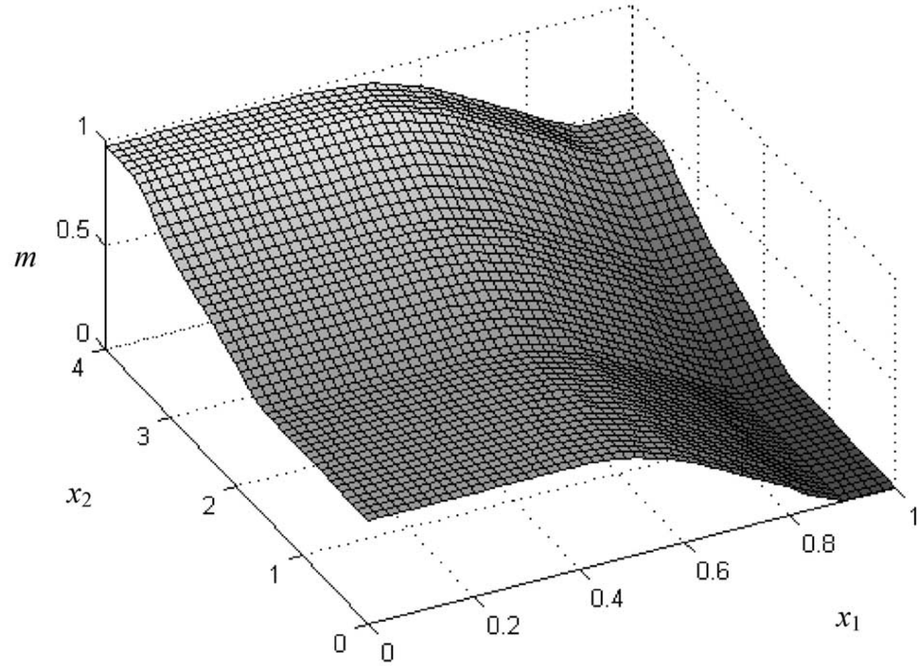

Fig. 11. Least prejudiced distribution of $m_{x,y}{}^{*}(C_1)$.

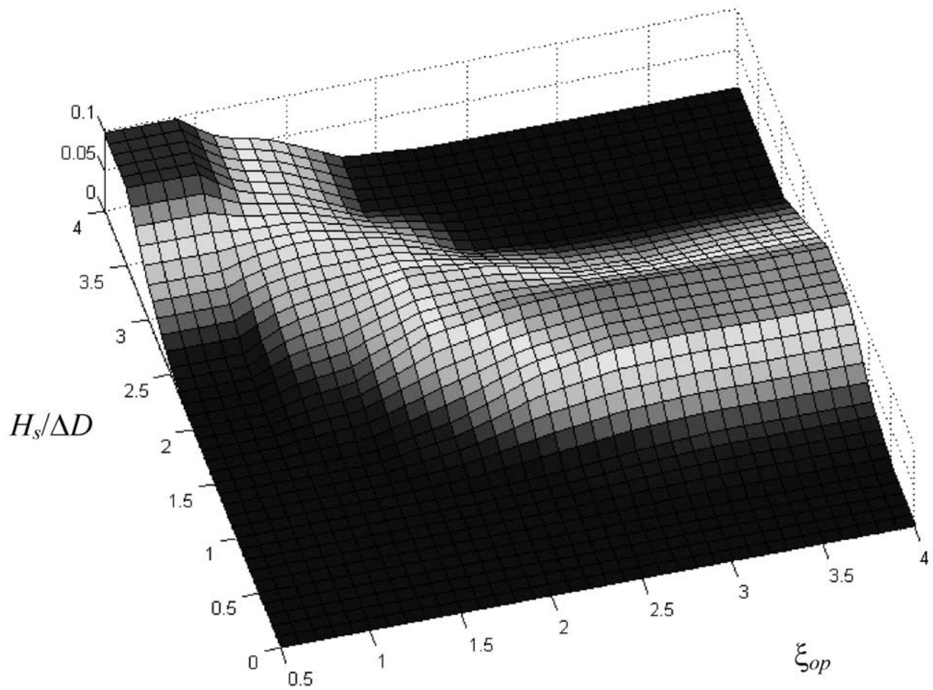

Fig. 12. Typical prior density estimate for the limit state function (based on 4 labels on X_1 and X_2).

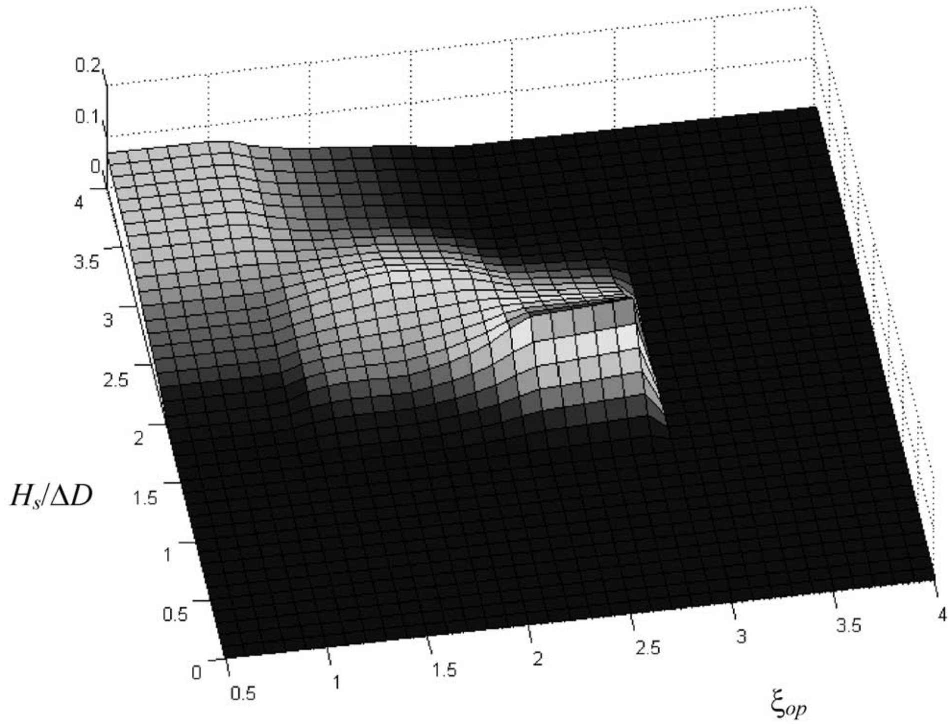

Fig. 13. Posterior density estimate for the limit state function (3 prior labels and 3 posterior labels).

The effect on:

(i) estimated probability of failure;
(ii) smoothness of the density estimate for the limit state function and;
(iii) sample likelihood;

of varying the number of labels in the prior an posterior label description is illustrated in Tables 3–5 respectively. Note that in all cases the probability of failure is less than the 9×10^{-4}

Table 3
Probability of failure from Method 2

Number of prior labels	Number of posterior labels			
	2	3	4	5
2	3.5×10^{-5}	4.6×10^{-6}	1.5×10^{-6}	1.0×10^{-6}
3	2.4×10^{-5}	3.4×10^{-6}	1.3×10^{-6}	8.8×10^{-7}
4	2.0×10^{-5}	3.1×10^{-6}	1.3×10^{-6}	9.0×10^{-7}
5	1.2×10^{-5}	3.0×10^{-6}	1.3×10^{-6}	8.8×10^{-7}
6	9.1×10^{-6}	2.7×10^{-6}	1.2×10^{-6}	8.8×10^{-7}
7	6.8×10^{-6}	2.4×10^{-6}	1.2×10^{-6}	8.9×10^{-7}

Table 4
Smoothness (sum of local directional derivates: smaller numbers indicate greater smoothness) of limit state function for different numbers of labels

Number of prior labels	Number of posterior labels			
	2	3	4	5
2	0.002	0.005	0.029	0.074
3	0.004	0.009	0.026	0.080
4	0.007	0.010	0.034	0.077
5	0.013	0.015	0.030	0.106
6	0.015	0.017	0.041	0.087
7	0.028	0.025	0.042	0.111

Table 5
Sample log likelihood for different numbers of labels

Number of prior labels	Number of posterior labels			
	2	3	4	5
2	−13.7	−11.9	−7.6	−5.2
3	−12.2	−11.1	−6.9	−4.6
4	−12.1	−10.9	−7.3	−4.9
5	−11.3	−9.8	−6.7	−4.2
6	−11.1	−9.3	−6.5	−4.1
7	−11.2	−9.1	−6.5	−4.0

predicted by the conventional reliability method. This is because the data is somewhat skewed away from the joint distribution of basic variables (see Fig. 3). In the original probabilistic analysis the model uncertainty parameter M was obtained from the first two moments only of the population of 10 data points, so did not reflect this skewness in the data.

As expected, increasing the number of labels increased the sample likelihood but reduces the smoothness. The number of labels used in the classification of the experimental data has more influence than then the number of labels used in the prior density estimate. From the values in Table 3 a range of plausible probabilities of failure need to be identified. For more than 7 prior labels the definition of the 'doubtful' zone in the limit state function was too precise when compared with the data upon which the definition was based. Examination of the limit state function revealed that for more than 3 posterior labels the limit state function was implausibly unsmooth. On the basis of this reasoning, the bounds on the probability of system failure were narrowed to $[7\times10^{-6}, 3.5\times10^{-5}]$.

The effect of obtaining more data can be demonstrated by simulating a large population of data (996 points), assuming that the limit state function is indeed linear when $H_s/\Delta D$ is plotted against $\cos\alpha/\xi_{\mathrm{op}}$ with gradient M equal to 4.06. The probability of system failure is recorded in Table 6. In a data-rich situation the method is less influenced by the priors and by the number of fuzzy

Table 6
Probability of system failure from Method 2, based on 996 experimental data points simulated from a linear model with Gaussian distributed residuals

Number of prior labels	Number of posterior labels						
	3	5	7	11	16	21	31
3	0.0043	0.0027	0.0024	0.0017	0.0017	0.0015	0.0014
5	0.0010	0.0011	0.0013	0.0013	0.0015	0.0014	0.0013
7	0.0003	0.0004	0.0007	0.0008	0.0012	0.0013	0.0012
9	0.0002	0.0003	0.0004	0.0006	0.0008	0.0009	0.0009

labels, and generates a probability of failure that coincides closely with the result from FORM or Monte Carlo analysis.

6. Discussion

Conventional probabilistic reliability theorists may be concerned by the degree of judgement involved in the methods proposed in this paper. Method 1 is influenced by the choice of the family of limit state functions for the classification procedure. Method 2 is influenced by the prior classification of X space and by the number of labels used to construct the label description of the priors classification and experimental data. However, under conditions of very scarce data it is inevitable that a great deal of judgement will have to be exercised in the design of a reliability analysis. In Method 1, when only weak assumptions were made, nearly vacuous bounds on the probability of system failure were generated. The proposed methods bring the role of expert judgements to the fore, and provides mechanisms for exploring the influence of expert judgements on the probability of system failure. It avoids the strong assumptions that have to be adopted in order to generate a precise probability of system failure.

The difficulty of identifying a limit state function will increase rapidly as the number of basic variables $x_1, \ldots x_n$ increases. The quantity of data required to identify a limit state function to a given level of precision increases rapidly with n, as can the computational expense. Moreover, there is a practical limit to the dimensionality of the space on which expert judgements can be elicited. These problems are by no means unique to the methods introduced in this paper. Nor indeed are some of the potential solutions, which involve critical scrutiny of candidate basic variables to establish whether they are sufficiently influential or identifiable to justify their inclusion in the analysis. Judicious independence assumptions can be used to sub-divide problems. The definition of the linguistic covering need not be at the same granularity on all dimensions, so can be modified to reflect the identifiably of each parameter. The use of fuzzy sets is attractive in high dimensional problems because it provides a mechanism for generalising from scarce data points.

The proposed approach has the merit of generating a linguistic description of the relationship in the data, which may be used to inform subsequent human or machine reasoning. This is achieved at the expense of a methodology that is more elaborate than conventional FORM or Monte Carlo analysis, though is no more demanding than other more advanced methods. A linguistic covering is defined by a relatively large number of parameters (compared, for example, to a simple linear regression model), though it is arguably no less parsimonious than a multivariate kernel density estimation method, to which it is in some respects analogous. However, the number of parameters required by a methodology is a rather crude indicator of its parsimony. It has been demonstrated that Method 1 is not particularly sensitive to the definition of the linguistic covering. It has been demonstrated how Method 2 is sensitive to the definition of the linguistic covering. In this case further judgements of smoothness and goodness of fit are then used to identify the plausible range of results.

Elicitation of expert knowledge and encoding it in mathematical terms is by no means a straightforward endeavour, as the extensive literature on subjective probability attests [37–42]. This literature does now provide some well-tested mechanisms for encoding judgements in individual and group contexts. The literature on elicitation of fuzzy sets and imprecise probabilities is for the time being less well developed [43,44], but operational definitions of fuzzy sets, including the voting model [36] and the label semantics of Lawry [32] upon which this paper is based, do provide the basis from practical elicitation.

Whilst, as has been noted, the fuzzy label method is to some extent analogous to multi-variate kernel density estimation, the Bayesian form of Definition 3 means that it provides a more convenient mechanisms for including prior knowledge. Conventional empirical density estimation is targeted at data analysis, whilst the methods proposed here aim to supplement scarce data with expert knowledge. In common with non-parametric density estimation, the methods have the attractive property that they do not depend on any assumption about the form of the underlying limit state function, beyond the weak assumptions in Method 1 and the prior expert classification in Method 2.

7. Conclusions

In reliability analysis of engineering systems it is conventional to represent the limit state function as a precise surface. Uncertainty in this surface may be represented as one or more additional random variables in the reliability problem. However, fitting these random variables can require strong assumptions about the form of the underlying function. Without further analysis, the resulting precise estimate of the probability of failure provides no indication of the implications of uncertainty in the limit state function. It has been argued that probability theory is not the most appropriate approach to representing uncertain knowledge about model uncertainty.

This paper has introduced two new methods for constructing limit state functions from very scarce data combined with expert knowledge about the possible form of the limit state function. The concept of a label description of data has been introduced, which is a mass distribution on a space of fuzzy sets. The label description has been used to generalise scarce data. It also provides a linguistic description of the relationships in the data. In this context, fuzzy sets have been given a fairly probabilistic interpretation, and it has been shown how an estimate of a probability density function can be constructed, conditional on a label description. The approach contributes to uniting the sometimes divergent courses of probabilistic and possibilistic analysis in reliability theory.

The first of the two new methods combines causal reasoning about the form of the limit state function with scarce experimental data to classify the state space. It has been used to demonstrate how plausible relaxations of some of the assumptions implicit in conventional reliability calculations result in wide bounds on the probability of system failure. The second method captures expert knowledge in the form of a prior distribution of the location of the limit state function. Expert judgement about the smoothness of the limit state function was used to constrain the range of plausible functions. As more data was acquired the bounds on the probability of failure converged towards the probabilistic solution.

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