A comparative study of different parameter estimation methods for statistical distribution functions in civil engineering applications

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ABSTRACT: In designing civil engineering structures use is made of probabilistic calculation methods. Stress and load parameters are described by statistical distribution functions. The parameters of these distribution functions can be estimated by various methods. An extensive comparison of these different estimation methods is given in this article. Main point of interest is the behaviour of each method for predicting p-quantiles, where p«1. We analyze the performance of the parameter estimation method with respect to its small sample behaviour. Monte Carlo simulations are used in this comparative study added with mathematical proofs. The popularity of the least squares method among engineers is analysed.

1 INTRODUCTION

In civil engineering practice many parameter estimation methods for probability distribution functions are in circulation. Well known are for example:
- the method of moments (B&C, 1970),
- the method of probability weighted moments, (Hosking, 1986),
- the method of L-moments (Hosking, 1990),
- the method of least squares (on the original or linearized data), (B&C, 1970),
- the method of weighted least squares, (Castillo),
- the method of maximum likelihood, (B&C, 1970),
- the method of minimum cross entropy, (Lind et.al.),
- the method of Bayesian estimation (Berger, 1980).

Many attempts (for instance, Burcharth et.al., 1994, Yamaguchi, 1996, and Goda et.al., 1990) have been made to find out which estimation method is preferable for the parameter estimation of a particular probability distribution in order to obtain a reliable estimate of the p-quantiles (the value which is exceeded by the random variable with probability p). In this study, we will in particularly investigate the performance of the parameter estimation method with respect to its small sample behaviour and the behaviour of under- and overestimation of p-quantiles.

2 MONTE CARLO SIMULATION STUDY

With a Monte Carlo simulation study, datasets can be generated from a “on beforehand” known probability distribution function (and known p-quantile). Different parameter estimation methods can be applied on these datasets and compared with respect to their estimates of the p-quantiles. The estimation method with the smallest bias and/or variance is then considered to be the best method for that particular distribution function. Under- or overestimation of the p-quantiles have an important meaning in civil engineering practice as well. Underestimation may give rise to unsafe structures whereas overestimation may lead to conservatism or too expensive structures. Therefore it is very useful to study the probabilities of under- and overdesign of a certain estimation method.

In the following we start with a simulation analysis of six estimation methods for the scale parameter of a one-parameter exponential distribution $F(x)=1-e^{-\lambda x}$. In figure 1 we see the performances of the six methods for one simulation of 20 values from EXP(λ=1.5). The different methods cause a very wide range of frequency lines. The once per 100 year return period of the theoretical distribution is 3.07m.

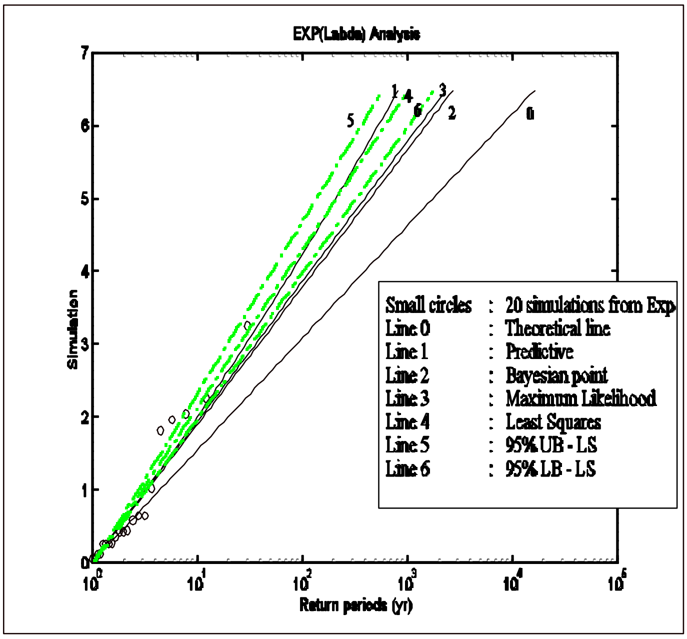

Figure 1, Six different estimation methods

The above simulation analysis is repeated 200 times and from each simulation we store per estimation method the results of the 1/100 year prediction. This excersise is also repeated for other sample sizes. Apart from n=20 values we look at n=3, 6, 9,..., 60 values. In the next figure we have plotted the mean and standard deviations of each estimation method.

3 ML VERSUS LS

In this section, we will in particularly concentrate on the under- and overestimation of the p-quantile of an Exponential and Gumbel distribution with a ML- and LS-parameter estimation method. Different sample sizes are considered (n=10, 30 and 100) for the same quantile of interest x_{100} such that $P(x>x_{100} \mid \text{Data}) = 1/100$. The following results were obtained:

Table 1, Probabilities of underdesign p_u

Data from Gumbel	n	ML	LS
Fitted by Gumbel	10	0.59	0.34
	30	0.56	0.36
	100	0.53	0.38
Fitted by Exponential	10	0.18	0.19
	30	0.01	0.12
	100	0.00	0.05

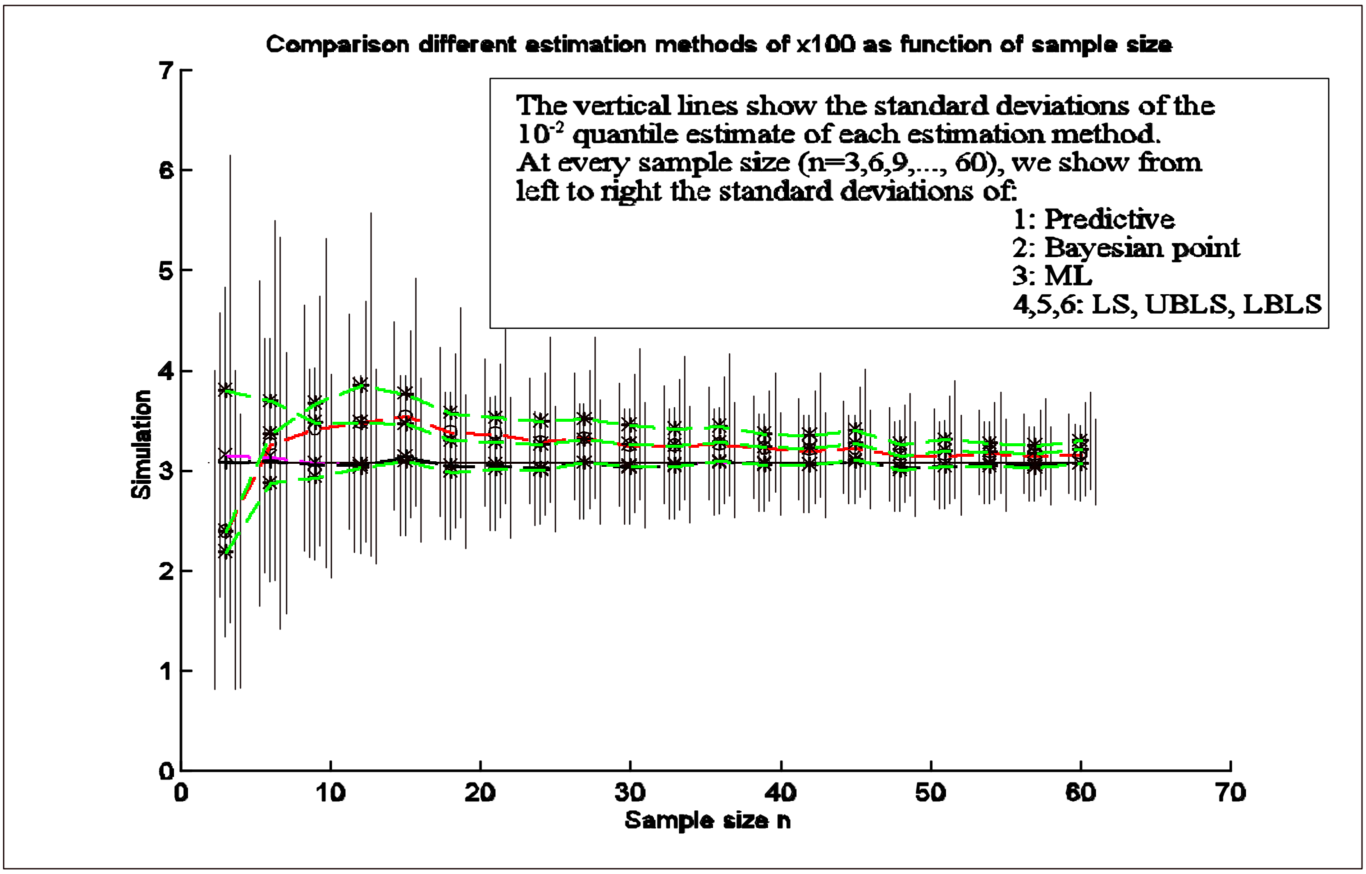

Figure 2, Simulation from Exponential Distribution.

Table 2, Probabilities of underdesign p_u

Data from Exponential	n	ML	LS
Fitted by Gumbel	10	0.85	0.50
	30	0.96	0.55
	100	0.99	0.67
Fitted by Exponential	10	0.59	0.37
	30	0.54	0.37
	100	0.52	0.37

The probabilities of overdesign follow from the relation p_o=1-p_u. From the tables 1 and 2, it follows that the least squares method usually gives lower probabilities of underdesign than the maximum likelihood method. That's why a least squares method is so popular under engineers. If we define assymetric loss-functions, in which we can model the risk aversion of a designer towards underdesign, we can determine optimal choices for distribution type and estimation method. For example if we penalize underdesign with a factor 4 more than overdesign we get the following table.

Table 3, Optimal choices for distribution type f and estimation method EM**

n	Gumbel		Exponential	
	f*	EM*	f*	EM*
10	Exp	ML	Exp	LS
30	G	LS	Exp	LS
100	G	LS	Exp	ML

From this table, we indeed notice a preference for the least squares method, except for large sample sizes from an exponential distribution where a ML-method is prefered and for small sample sizes from a Gumbel distribution which are better modeled by an exponential distribution with a ML-method for risk-averse engineers.

Another well known estimation method applied in civil engineering practice is the Method of Moments (MoM). The authors have developed a proof to show that the p-quantile estimates by a MoM are always smaller (and therefore less risk averse) than the p-quantile estimates by a LS-method. See Appendix.

4 CONCLUSIONS

In this paper, an overview and references are given of parameter estimation techniques that are well known in civil engineering practice. With a Monte Carlo simulation study, these estimation techniques can be compared. Special interest is given in the performance of p-quantile estimates. Under- and overdesign are important measures for the engineer. Based on simulations from an Exponential distribution (and a mathematical proof in the appendix) an ordening in risk aversion of the different estimation techniques can be made. The Maximum Likelihood, Bayesian point (mean of posterior distributions) and Method of Moments parameter estimation techniques give a relatively higher proportion of underdesign than the Bayesian predictive (integration over the posterior distribution) and Least Squares techniques. Assymetric loss criteria can be used to model the risk aversion of the engineer in mathematical terms. The optimal choice of the probability distribution function and the parameter estimation method can then be determined by minimizing the assymetric loss. In Van Gelder, 1996, this idea has been worked out for more types of loss functions and includes parameter and model uncertainty.

APPENDIX (THEORETICAL COMPARISON BETWEEN MOM- AND LS-ESTIMATION)

We first start by proving that:
The LS-estimate of the scale parameter of the Exponential and Gumbel probability distributions is always larger than its MoM-estimate.

Assume that we have n observations in sorted order given by $x_1, x_2, ..., x_n$.
We can linearize the probability distribution F(x) under consideration (with scale parameter B and location parameter A). So we can find a function g such that:

$g(F(x)) = (x-A)/B$.

For the Exponential distribution g is given by $g_E(\zeta) = -\ln(1-\zeta)$ and for the Gumbel distribution, we have $g_G(\zeta) = -\ln(-\ln(\zeta))$.

Define the vector y by $y = i/(N+1)$, the plot position of the i^{th} observation (i=1,...,N) and the vector $y^* = g(y)$.

From linear regression theory we have $B_{LS} = \sigma_x/\rho\sigma_{y^*}$ in which $\rho = cov(x,y^*)/\sigma_x\sigma_{y^*}$.

For the Exponential distribution we have $B_{MoM} = \sigma_x$.

We will proof that $B_{MoM}<B_{LS}$, or equivalently:

$\rho\sigma_{y^*}<1$, or equivalently (because $-1<\rho<1$):

$\sigma_{y^*}<1$.

Note that $y^* = g(y) = -\ln(1-i/(N+1)) = -\ln(z)$, in which z can be considered as a uniform distribution between δ and $1-\delta$ with $\delta = 1/(N+1)$.

So $E(y^*) = \int_\delta^{1-\delta} -\ln(z)dz = -\ln(1-\delta)+1+\ln(1-\delta)\delta-2\delta+\ln(\delta)\delta \uparrow 1 \ (\delta\to 0)$,

and $Var(y^*) = \int_\delta^{1-\delta} \ln^2(z)dz - E^2(y^*) \uparrow 1 \ (\delta\to 0)$.

So $\sigma_{y^*}<1$ (for all N).

For the Gumbel distribution we have $B_{MoM} = \sigma_x \sqrt{6}/\pi$

and $E(y^*) = \int_\delta^{1-\delta} -\ln(-\ln(z))dz = (\delta-1)\ln(-\ln(1-\delta)) + \delta\ln(-\ln(\delta)) + Ei(1,-\ln(1-\delta)) + Ei(1-\ln(\delta)) \uparrow \gamma \ (\delta\to 0)$,

and $Var(y^*) \uparrow \pi^2/6 \ (\delta\to 0)$,

so $\sigma_{y^*}<\pi/\sqrt{6}$ (for all N).

Since the estimation of the p-quantile is linearly related with the estimation of the scale parameter B of the exponential distribution:

$x_p = A-B\ln(1-p) = A+4.61B$ (p=0.99),

and of the Gumbel distribution:

$x_p=A-B\ln(-\ln(p)) = A+4.60B$ (p=0.99),

we can conclude that the LS-estimate of a p-quantile of an Exponential or Gumbel model is always larger than the MoM-estimate.

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