3.4 Uncertainties in the treatment of statistical data and influences on the structural reliability assessment*

3.4.1 Introduction

Structural reliability methods allow us to account for the uncertain part of data in order to quantify the structure integrity. They lead to results in accordance with the definition of the probabilistic representation of the data concerning the structure or its environment (geometry, material, loading, ...) and a mechanical model resulting from physical, mathematical and numerical considerations describing the structural behavior. A limitation of the reliability approach is that the probabilistic data formulation is generally based on a limited number of observations. As a consequence the reliability results depend on the actual inherent variability of the structure but also on imperfections in the probabilistic data representation.
This paper aims at showing the influence of these imperfections on the reliability results and at proposing methods to account for them in order to evaluate reliability function of the strict statistical information available. After the review of the uncertainties arising from the probabilistic data representation and the proposed methods to account for them, the main concepts are illustrated for the case of a cracked pipe subjected to bending.

3.4.2 The different sources of uncertainties

Statistical methods are available to fit the observed data to mathematical models in order to represent them in a probabilistic way. However when fitting data to these models, uncertainties arise due to the limited number of observations (Fig. 3.15):

- First, in the choice of a particular model which is generally chosen among the parametric models that are not rejected by an hypothesis test at a certain confidence level (Saporta [16]), (Ditlevsen [5]).
- Second, when using the fitted model in structural reliability assessment, the most probable failure point may have its coordinates in tails where very little information is available and where the failure probability is very sensitive (Caers & Maes [3]), (Maes and Huyse [10]).
- Finally, errors in the estimation of the model parameters occur due to the limited observations in the samples (Der Kiureghian [4]), (Ditlevsen & Madsen [6]).

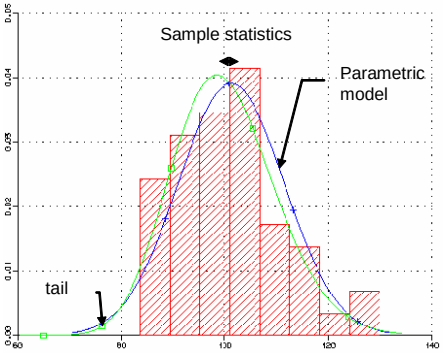

Figure 3.15. The different sources of statistical uncertainties

In the following the developments are focused on the first and the third kind of uncertainty, considering that the tail behavior is a consequence of the definition of a particular parametric density function and that the problem has been treated in the previous section (3.3 Extremes laws and tail distribution).

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3.4.2.1 Uncertainties in the use of a particular parametric model

The choice of a particular density function to describe the physical uncertainties may influence a lot the reliability results obtained in output (Ardillon et al. [1]). Uncertainty which arises from this choice is mainly due to the use of simplified mathematical models to describe the density functions. These models are generally chosen for the limited number of parameters that have to be identified. Moreover they often result from an expert judgment as hypothesis test are not always able to discriminate two types of density functions from central located data. The classical parametric model selection is then guided by a balance between ability to represent reality and the pragmatic need for having simple mathematical properties for the identification of the model.

For the reduction of this kind of uncertainty, A. Der Kiureghian [4] suggests to use mixed density functions. The identification of mixed models may however require a great number of observations that are generally not available.

As it has been shown by Tukey and Pearson [13,14], there is always a great similitude between distribution where the first four moments are identical. This suggests that the use of the first four sample moments may reduce the uncertainty and that the choice of a particular Johnson or Pearson density function defined by the use of these moments will provide a part of the answer to the problem. The use of Johnson and Pearson distributions is however not studied in this paper as it is preferred here to try to link directly the probabilistic representation with all the available information contained in the samples. It is suggested here to employ a description based on the successive sample moments with their related estimation uncertainty (see next paragraph). Edgeworth and Gram-Charlier expansions present such an advantage (Winterstein et al. [17]) but they may lead to non density functions as negative values in the representation of the density function may occur. It is then recommended here to use the Hermite transformation (Ditlevsen and Madsen [6]). Such a transformation present several advantages:

- it accounts for sample moments over the second order as far as desired,
- it may be directly used in the FORM/SORM process without the need to perform an additive isoprobabilistic transformation,
- it can cover a wide range of density shapes without specifying a simplified mathematical representation.

Applications of this transformation and its justification in a context of statistical uncertainties will be further examined in a follow-up paper.

3.4.2.2 Uncertainties by estimating the sample statistics

Let us denote θ_k the unknown vector of the parameters involved in the known joint probability function $f_{X_1,\ldots,X_n}(x_1 \ldots x_n, \theta_k)$ of the random vector $(X_1,\ldots,X_n)$. In structural safety problems statistical methods aim at estimating θ_k from independent and identically distributed samples $(x_i^{(1)},\ldots,x_i^{(m)})$ of m observations of the random variables X_i which may concern geometry, material properties or loading. Once an estimation of θ_k is found, the failure probability or the reliability index of the structure can be evaluated using the estimation $\tilde{\theta}_k$ of θ_k [6]:

$$P_f = \text{PROB}(G(X_1,...,X_n) \le 0)$$
$$= \int_{G(X_1,...,X_n)\le 0} f_{X_1,...,X_n}(x_1...x_n, \tilde{\theta}_k) dx_1 dx_2 ... dx_n \quad (1)$$
$$= P_f(\tilde{\theta}_k)$$

where G represents the limit state function which separates the physical space into a safety domain:
$G(X_1,\ldots,X_n) > 0$
and a failure domain:
$G(X_1,...,X_n) \le 0$
Since the sample is a sequence of m random realizations of X_i, its relative statistics such as the empirical mean or variance which are used to estimate θ_k become random. When m tends to infinity $\tilde{\theta}_k = \theta_k$, otherwise $\tilde{\theta}_k$ may be characterized by its probability density function (PDF) $f_{\Theta_k}(\theta_k)$.
The problem is then to find the PDF of Θ_k whose components are called hyper-parameters in the following, and to take it into account in the reliability assessment of the considered structure.

Characterization of $f_{\Theta_k}(\theta_k)$:

Let us consider a one-dimensional random vector of hyper-parameters Θ and a unique random variable X with a sample of n realizations $(x_1,\ldots,x_n)$ in order to be concise in notation. The main problem in determining the PDF of the hyper-parameter Θ is that generally one sample is available and thus a unique realization θ of Θ. To obtain more information about this vector two ways are today identified:

- the first is to resample in the available sample of data in order to create new samples with their relative statistics. Two resampling techniques are proposed:
  - the bootstrap, in which a given number of new samples, each of the same size as the observed data, are drawn with replacement from the observed data. This method proposed by B. Efron [7] shows that accurate results concerning estimates of bias, mean, and standard error for the replicate statistics may be obtained;
  - the Jackknife, in which a statistic is calculated for all the possible samples of size *n-1*, each with one observation left out from the observed data sample. Variants with more than one observation removed are also available. This method leads to estimate of bias, mean and standard error calculated differently than the equivalent bootstrap statistics (Reeds [15]).
    Disposing then of statistics on statistics, a classical probabilistic modelization of the hyper-parameters can be performed.
- the second, that is focused on, is the bayesian approach (Berger [2]), (Ditlevsen & Madsen [6]) which can incorporate subjective information in order to increase the poor information available in the data:
  $$f_{\Theta|x_1,x_2,...,x_n}(\theta) = k.L(\theta; x_1, x_2, ..., x_n) f'_{\Theta}(\theta) \quad (2)$$
  which means that the posterior density of Θ is proportional to the product of the sample likelihood $L(\theta; x_1, x_2, ..., x_n)$ by the prior density $f'_{\Theta}(\theta)$ of Θ which can result from an expert judgment or from a previous posterior density that is actualized with new observations. The likelihood is defined from the known PDF $f_X(x,\theta)$ through:
  $$L(\theta; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_X(x_i, \theta) \quad (3)$$

Analytical expression of the posterior distribution is available when we restrict ourselves to the exponential family because of the existence of conjugate prior densities. Otherwise numerical computations are required.

Structural safety assessment under uncertainties in the estimation of the hyper-parameters.

For a detailed description of this aspect, the reader can refer to Pendola [18].
Once the PDF of the hyper-parameters is determined, three ways are identified in order to take into account the uncertainties concerning the hyper-parameters in the evaluation of the structural integrity:
the first (Der Kiureghian [4]) is to consider the reliability index β as a function of the hyper-parameters. The hyper-parameters becoming random, the reliability index becomes random too. The reliability index classically evaluated with deterministic hyper-parameters is then a particular realization of the random variable $\beta(\Theta)=B$. The cumulative distribution function (CDF) of B can then be evaluated by writing:

$$F_B(\beta_0) = \mathrm{PROB}(B \le \beta_0) = \int_{\beta(\theta)-\beta_0 \le 0} f_\Theta(\theta) d\theta \quad (4)$$

where β_0 is a value fixed by the analyst. It is noted that the integral 4 has the same form as the integral 1 which suggests that the same reliability techniques can be used to compute this integral. The CDF can then be approximated as accurately as desired by making β_0 varying. One should note, however, that computation of the integral 4 requires a nested application of the reliability methods because of the dependence of $\beta(\theta)$ on θ. Thus this computation may be very cost intensive in particular when a reliability computation requires several finite element computations for the mechanical model (see Pendola et al. [19]).

In order to decrease this potentially cost intensive computation, A. Der Kiureghian suggests in [4] to use approximations of the first and second moments of B instead of its CDF because it generally suffices in order to have a representation of B in term of a central measure and a measure of dispersion. He shows that the mean and the variance of B can be written as:

$$\begin{aligned} \mu_B &\approx \beta(\mu_\Theta) \\ \sigma_B^2 &\approx \nabla_\theta \beta(\mu_\theta).\Sigma_{\theta\theta}.\nabla_\theta \beta(\mu_\theta)^T \end{aligned} \quad (5)$$

in which μ_θ and $\Sigma_{\theta\theta}$ are the mean vector and covariance matrix of the vector θ_k and $\nabla_\theta \beta(\mu_\theta)$ is the row vector of the partial derivatives of β with respect of the components θ_i (the sensitivities of the reliability index, see (Madsen et al. 1986)) evaluated at the mean point. Thus the approximations 5 requires a unique evaluation of the reliability index with its sensitivities.
the second [4,6] is based on the Bayes' theorem combined with the totality probability rule and aims at determining the predictive distribution of the variable X:

$$f_{X|x_1,\ldots,x_n} = \int_{D_\Theta} f_X(x,\theta) f_{\Theta|x_1,\ldots,x_n}(\theta) d\theta \quad (6)$$

in which D_Θ is the domain where the posterior density of Θ is defined. The computation of the integral 1 is performed by combining the inherent variability classically represented by the PDF of X and the uncertainty on Θ in the predictive distribution of X. This predictive distribution is then used instead of $f_X(x,\theta)$ in order to compute a reliability index denoted β_p in the following.
the third is to add in the reliability problem formulation the random variables concerning the hyper-parameters. The reliability index is then evaluated by using compounded random variables, that is to say random variables function of other random variables. The problem of compound variables is not fundamentally different from the classical formulation as the distribution can also be seen as a conditional distribution: in the classical representation the density

$$f_{X_1,\ldots,X_n}(x_1 \ldots x_n, \theta_k)$$

is conditional to deterministic parameters θ_k and in the case of compound variables the same density is conditional to random variables θ_k. The reliability calculus is then performed by using the classical Rosenblatt's transformation by taking care of the transformation order: we must begin with uncompound variables, then follow the order of compounding levels (Mohamed [11]).
This reliability index is denoted β_a in the following.
These three different methods will be compared in the next section on an industrial application.

How to deal with uncertainties

If we are involved with calculating the expected probability distribution for a random variable X, then the inferences we make on X should reflect the uncertainty in the parameters θ. In the Bayesian terminology we are interested in the so-called predictive function:

$$F(x) = \int_{\Omega} F_X(x,\theta) f_\Theta(\theta) d\theta \tag{7}$$

where $F(x|\theta)$ is the probabilistic model of X, conditional on the parameters θ and F(x) is the predictive distribution of the random variable x, now parameter free. In popular words: "the uncertainty in the θ parameters has to be integrated out".

The predictive distribution can be interpreted as being the distribution $F(x|\theta)$ weighted by $f(\theta)$. *In making inferences on a random variable it is important to use the predictive function for x,* as opposed to the probabilistic model for x with some estimator for the parameter set θ, i.e. $f(x|\theta^*)$. This is because using point estimators for uncertain parameters underestimates the variance in the random variable X.

The techniques will be illustrated with an exponential distribution:

$$F(x) = 1 - e^{-\frac{x-\xi}{\alpha}} \qquad x \geq \xi \tag{8}$$

The influence of statistical uncertainty in the shift parameter ξ will be considered by writing ξ as $\xi + \epsilon$ in which $\epsilon \sim N(0,\sigma_\xi)$. The PDF of ϵ is given by:

$$f(\varepsilon) = \frac{1}{\sigma_\xi \sqrt{2\pi}} e^{-\frac{\varepsilon^2}{2\sigma_\xi^2}} \tag{9}$$

According to Eqn. (7), we can write:

$$F(x) = \int F(x|\varepsilon) f(\varepsilon) d\varepsilon = \int (1 - e^{-\frac{x-\xi-\varepsilon}{\alpha}}) \frac{1}{\sigma_\xi \sqrt{2\pi}} e^{-\frac{\varepsilon^2}{2\sigma_\xi^2}} d\varepsilon =$$

$$= 1 - \frac{1}{\sigma_\xi \sqrt{2\pi}} e^{-\frac{x-\xi}{\alpha}} \int e^{\frac{\varepsilon}{\alpha} - \frac{\varepsilon^2}{2\sigma_\xi^2}} d\varepsilon = 1 - \frac{1}{\sigma_\xi \sqrt{2\pi}} e^{-\frac{x-\xi}{\alpha}} \int e^{-\frac{1}{2\sigma_\xi^2}\left(\varepsilon - \frac{\sigma_\xi^2}{\alpha}\right)^2} d\varepsilon \, e^{\frac{\sigma_\xi^2}{2\alpha^2}} = \tag{10}$$

$$= 1 - e^{-\frac{x-\xi}{\alpha}} e^{\frac{\sigma_\xi^2}{2\alpha^2}} = 1 - e^{-\frac{x-\xi+\frac{\sigma_\xi^2}{2\alpha}}{\alpha}}$$

Notice that the probability of exceedance curve is translated with $\sigma_\xi^2/2\alpha$.

Now we will consider the influence of statistical uncertainty in the scale parameter. For that purpose we rewrite the CDF as $F(x) = 1 - e^{-(ax - b)}$. Note that $a = 1/\alpha$ and $b = \xi/\alpha$. Assume a statistical uncertainty in the a parameter: $a = a + \epsilon$ in which $\epsilon \sim N(0,\sigma_a)$. Then:

$$F(x \mid \epsilon) = 1 - e^{-((a+\epsilon)x - b)} \tag{11}$$

and

$$f(\varepsilon) = \frac{1}{\sigma_a \sqrt{2\pi}} e^{-\frac{\varepsilon^2}{2\sigma_a^2}} \tag{12}$$

So:

$$F(x) = \int F(x \mid \epsilon)\, f(\epsilon)\, d\epsilon = \int (1 - e^{-((a+\epsilon)x - b)}) \tfrac{1}{2}\sigma_a^{-1}\pi^{-1/2} \exp(-\tfrac{1}{2}(\epsilon/\sigma_a)^2)\, d\epsilon =$$
$$= 1 - e^{-(ax - b)} \exp(\tfrac{1}{2}\sigma_a^2 x^2) \tag{13}$$

This can be written again in terms of ξ and α like:

$$F(x) = 1 - \exp\{-(x - \xi - \tfrac{1}{2}\sigma_a^2 x^2 \alpha)/\alpha\} \tag{14}$$

Note that $\sigma_a = \sigma(1/\alpha) = f(\sigma_\alpha)$ is difficult to express as a function of σ_α. The approximation $\sigma(1/\alpha) \approx 1/\sigma(\alpha)$ may not be used. However the relation $CV(1/\alpha) \approx CV(\alpha)$ is quite good. We therefore use as a first approximation $\sigma_a = \sigma_\alpha/\alpha^2$. Substitution in Eqn. (14) leads to:

$$F(x) = 1 - \exp\{-(x - \xi - \tfrac{1}{2}\sigma_a^2 x^2 \alpha)/\alpha\} = 1 - \exp\{-(x - \xi - \tfrac{1}{2}\sigma_\alpha^2 x^2 \alpha^{-3})/\alpha\} =$$
$$= 1 - \exp\{-(x-\xi)/\alpha\} \exp\{\sigma_\alpha^2 x^2 / 2\alpha^4\} \tag{15}$$

Notice that the probability of exceedance is translated as a function of σ_a and x. So apart from a shift also the slope of the survival function 1-F increases. Summarizing these results leads to Table 20

Table 20
Multiplication factors

Exponential Distribution	Shift Parameter	Scale Parameter
Multiplication Factor	$e^{\frac{\sigma_\xi^2}{2\alpha^2}}$	$e^{\frac{\sigma_\alpha^2 x^2}{2\alpha^4}}$

From Table 20 we notice the influence of the x-value in the multiplication factor for the scale parameter. The influence of the x-value in the multiplication factor for the shift parameter has disappeared.

Summarized; if $F(x) = 1 - e^{-(x-\xi)/\alpha}$ has an uncertainty in the scale parameter (given by σ_α which should be not too large), then in making inferences on X the original exponential distribution should be "replaced".

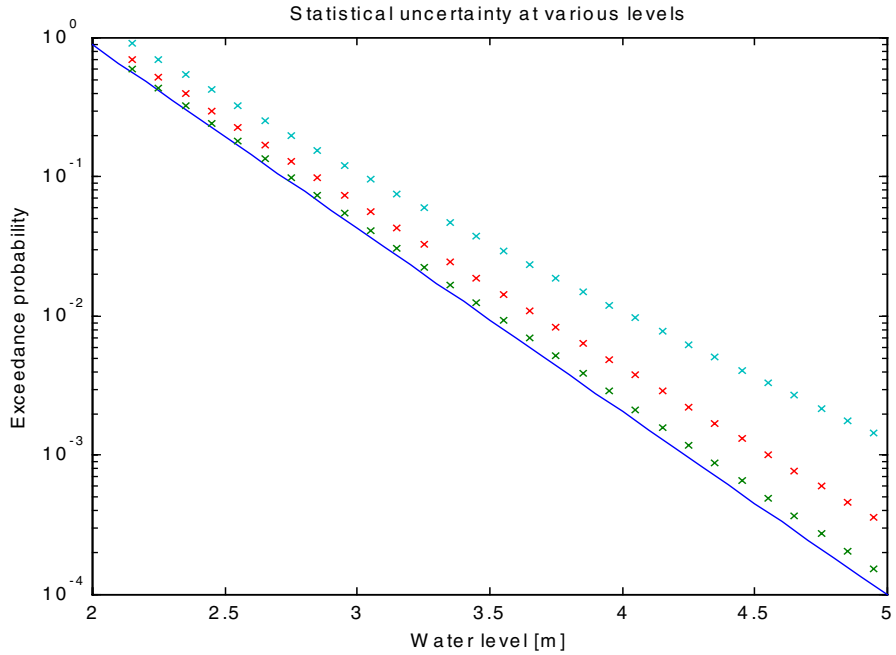

Figure 3.16. Translation and rotation of the frequency curves as Φ_B increases from 5%, 11% to 17% in the case study of extreme water levels at the location Hook Of Holland (The Netherlands)

The equation is applied to the data set of extreme water levels at Hook of Holland ([11]). This set can be modelled with an exponential distribution with parameters A=1.96 and B=0.33. Different levels of uncertainty in B will be discerned: $\Phi_B = 0.17$, $\Phi_B = 0.11$, and $\Phi_B = 0.05$. The influence of the uncertainty is depicted in Figure 3.16 and appears to be quite large in this particular case study. Notice the combination of translation and rotation of the frequency curves.

Uncertainty and sensitivity analyses are similar in that both strive to evaluate the variation in results arising from the variations in the assumptions, models, and data. However, they differ in scope, and the information they provide.

Uncertainty analysis attempts to describe the likelihood for different size variations and tends to be more formalized than sensitivity analysis. An uncertainty analysis explicitly quantifies the uncertainties and their relative magnitudes, but requires probability distributions for each of the random variables. The assignment of these distributions often involves as much uncertainty as that to be quantified.

Sensitivity analysis is generally more straightforward than uncertainty analysis, requiring only the separate (simpler) or simultaneous (more complex) changing of one or more of the inputs. Expert judgement is involved to the extent that the analyst decides which inputs to change, and how much to change them. This process can be streamlined if the analyst knows which variables have the greatest effect upon the results. Variation of inputs one at a time is preferred, unless multiple parameters are affected when one is changed. In this latter case, simultaneous variation is required.

3.4.3 APPLICATION ON A CRACKED PIPE IN PURE BENDING

3.4.3.1 Description of the problem

Let us consider the pipe of the Figure 3.17 subjected to a bending moment M and with a semi-elliptic crack of height a and angle 2γ.

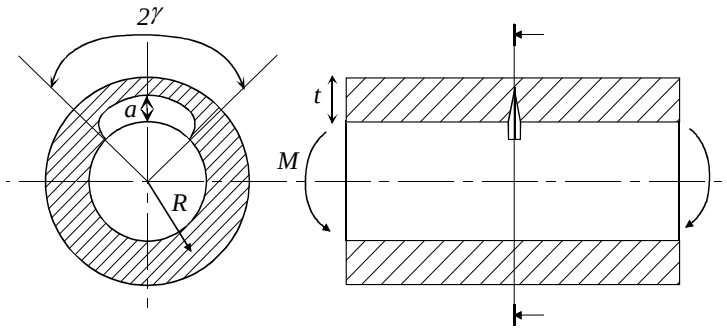

Figure 3.17. Representation of the studied pipe

We consider in this study the crack initiation in ductile tearing. Then, we can consider having failure of the component as soon as:

$$J \geq J_{0.2}$$

Where J is the Rice's integral and $J_{0.2}$ the material fracture toughness.
In the problem studied here, some parameters are supposed to be deterministic. They are given in Table 21.

Table 21
Description of the deterministic parameters

Parameter	Value
R	386 mm
t	40 mm
M	3 MN.m
2γ	90°

The random variables are given in Table 22. They describe in fact the crack depth and the material properties in term of stress strain curve and fracture toughness. The given hyper-parameters and P.D.F. constitute the theoretical probabilistic model, that is to say the model that would be used if the statistical information were complete. Moreover the random variables are assumed to be uncorrelated.

Table 22
Description of the random variables

Variable	Density law	Mean	Standard deviation
$J_{0.2}$	normal	200 N/mm	20 N/mm
E	normal	180,000 N/mm²	30,000 N/mm²
σ_0	normal	150 N/mm²	5 N/mm²
a	normal	20 mm	1 mm

J can be determined with different models but we consider here the mechanical model described below.

3.4.3.2 Deterministic model

The considered deterministic model is based on results from EPRI (Kumar et al [8]) relative to a crack with constant depth assumed to be equivalent to a semi-elliptic crack in the following.
It supposes that the material stress-train curve is described by a Ramberg-Osgood law which can be written:

$$\varepsilon = \frac{\sigma}{E} + \alpha \frac{\sigma_o}{E} \left(\frac{\sigma}{\sigma_o} \right)^n \quad (11)$$

where ε is the strain, σ is the stress, E is the Young's modulus, σ_0 is the yield strength, n is the strain hardening exponent and α is the coefficient of the Ramberg-Osgood law. n and α are dependent of the value of E and σ_0. They are computed from a given stress-strain curve assuming that the plastic strain remains constant for a given value of σ/σ_0.
The J-integral is then estimated by superposing the elastic and the fully plastic contributions (Kumar & German [8]):

$$J = J^{el} + J^{p}$$

in which each term is a function of the load and the geometry:

$$J^{el} = \frac{1-\nu^2}{E} K_I^2$$

where ν is the Poisson coefficient, K_I is the stress intensity factor in the opening mode (mode I) and:

$$J^{p} = \alpha \frac{\sigma_0^2}{E} (t-a) \frac{a}{t} h_1 \left(\frac{A}{A_{nc}} \right)^{n+1} \left(\frac{M}{M_0} \right)^{n+1}$$

where h_1 is a constant function of the geometry and the material that can be found in [8], A is the area of the crack, A_{nc} is the section area without crack and M_0 is the moment corresponding of the full plastification of the section:

$$M_0 = 2\sigma_0 t \left[(R + t/2)^2 + (R - t/2)^2 \right]$$

3.4.3.3 Influence of the uncertainties in the estimation of the hyper-parameters

When disposing of a sample of size N for a normal random variable X, the Bayesian approach allows us to determine the joint distribution of the mean μ and the variance σ^2 [4]: for a non-informative prior, the joint distribution is such that μ is conditionally normal with mean $\overline{X}$ where:

$$\overline{X} = \frac{1}{N} \sum_{k=1}^{N} x_k$$

is the sample mean, and variance σ^2/N. Moreover $1/\sigma^2$ is gamma distributed with parameters $(N-1)/2$ and $S^2(N-1)/2$ where:

$$S^2 = \frac{1}{N-1} \sum_{k=1}^{N} (x_k - \overline{X})^2$$

is the sample variance.
For the predictive distribution we use the fact that the predictor distribution of

$$\sqrt{\frac{N}{N+1}} \frac{X - \overline{X}}{S_X}$$

is a Student distribution with $N-1$ degrees of freedom.

In order to show the influence of this kind of uncertainty on the failure probability of the structure, thousand samples of size N are simulated for each random variable (it is assumed that all the samples have the same size). For each set of samples the three following reliability indexes are computed using the software *PROBAN* and FORM approximations (Olesen [12]):

- b which is a realization of the random variable B,
- β_p resulting from the characterization of the predictive distribution as suggested by Der Kiureghian [4] and Ditlevsen et al. [6] ;
- β_a which is proposed in this paper.

Figure 3.18 shows the 90% confidence interval for the probability of failure as a function of the sample size N. This confidence interval is evaluated thanks to the thousand realizations of each reliability index by writing that

$$P_f = \Phi(-\beta)$$

where Φ is the standard normal CDF.

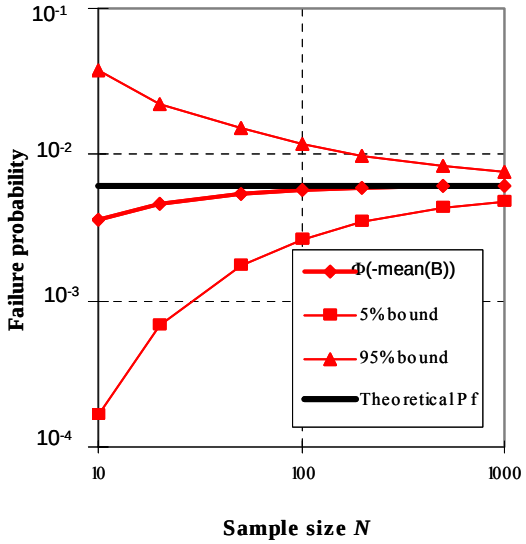

Figure 3.18. Confidence interval for the probability of failure

Only the results relative to b are represented but the results relative to β_a and β_p are very close. It is interesting to observe in Figure 3.18 that:

- the confidence interval gradually narrows as the sample size increases,
- the means relative to B asymptotically approach the theoretical result as N approaches infinity, i.e., as the statistical information becomes complete. The results show that the mean relative to B is located below the theoretical result and the mean relative to β_a above. This shows, in our example, that the classically evaluated reliability index leads to non conservative results in expectancy while β_a leads to conservative ones. Moreover each realization of β_p and β_a are smaller to the corresponding realization b.

As a consequence of these results on the case of the pipe, knowing the mean and standard deviation of the reliability index, the decision maker may fix a probability level that the target reliability index must not overcome [4]. For example a conservative reliability index that has a probability less than 5% to be above

which corresponds approximately to $mean(\beta)$-$1{,}64\sigma(\beta)$ if a normal density for β is assumed. Another possibility is to evaluate β_p or β_a which incorporate in their value the uncertainty in the estimation of the hyper-parameters. These reliability indexes can then be taken as the reliability index of the structure under a certain amount of statistical information.

3.4.4 Conclusion

This paper shows that the uncertainties arising from the estimation of the hyper-parameters can lead to reliability results that are far away from the results that could be obtained if the statistical information were complete. A method that takes into account these uncertainties in the evaluation of the reliability of the structure is proposed. This method leads to a reliability index different but conservative in comparison with the reliability index classically evaluated: it remunerates improvements performed by considering new observations and converges to the reliability index corresponding to a perfect statistical knowledge when the number of observations increases. As a consequence of this proposed method, safety factors that are function of the available statistical information may be calibrated.

3.4.5 References

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