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Extreme value distributions for nonlinear transformations of vector Gaussian processes

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Abstract

Approximations are developed for the marginal and joint probability distributions for the extreme values, associated with a vector of non-Gaussian random processes. The component non-Gaussian processes are obtained as nonlinear transformations of a vector of stationary, mutually correlated, Gaussian random processes and are thus, mutually dependent. The multivariate counting process, associated with the number of level crossings by the component non-Gaussian processes, is modelled as a multivariate Poisson point process. An analytical formulation is developed for determining the parameters of the multivariate Poisson process. This, in turn, leads to the joint probability distribution of the extreme values of the non-Gaussian processes, over a given time duration. For problems not amenable for analytical solutions, an algorithm is developed to determine these parameters numerically. The proposed extreme value distributions have applications in time-variant reliability analysis of randomly vibrating structural systems. The method is illustrated through three numerical examples and their accuracy is examined with respect to estimates from full scale Monte Carlo simulations of vector non-Gaussian processes.

Keywords: Extreme value distributions; System reliability; Non-Gaussian random processes; Nonlinear transformations; Importance sampling; Multidimensional integration; Multiple design points

1. Introduction

Failures in randomly vibrating structures are defined to occur if the response exceeds permissible thresholds, within specified time durations. Estimating the failure probability involves characterizing the probability of exceedance of the structure response, which is a random process in time. An elegant approach for addressing this time-variant reliability problem lies in expressing the failure probability in terms of the probability distribution function (PDF) of the extreme values associated with the response.

In structural series systems, failure of any of the individual components signals system failure. The system reliability is thus expressible in terms of the joint probability of exceedance of the component response processes. Often, the loads acting on the various components of a system have common source, and hence, the component responses, and in turn, their extremes, are mutually dependent. This emphasizes the need for characterizing the joint PDF of these extreme values for estimating series system reliability.

In many real life situations, the safety of a structural component is measured in terms of response metrics which are non-Gaussian, such as the Von Mises stress. Estimating the reliability of the structure as a system requires taking into account the correlation that exist between the various component failures. The focus of this paper is to develop a methodology, based on the theory of random processes, by which approximations may be obtained for joint extreme value distributions associated with a vector of non-Gaussian processes. This, in turn, can be used to make predictions about the structural system reliability. The class of non-Gaussian processes considered in this study is limited to those which are expressible as nonlinear transformations of vector Gaussian processes.

2. Literature review

A common approach in characterizing the extreme value distributions for random processes, is to study the associated first

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passage failures, based on the assumption that level crossings can be modeled as Poisson counting processes. The parameter of the counting process is related to the mean outcrossing rate, which in turn, can be estimated if the joint probability density function (pdf) of the process and its instantaneous time derivative, is available [29]. Since for Gaussian random processes, the process and its instantaneous time derivative are mutually independent, their joint pdf is readily available. Consequently, closed form expressions for the extreme value distributions for Gaussian processes are easy to develop. The joint pdf for a non-Gaussian process and its instantaneous time derivative is, however, not so readily available, which makes characterizing its extreme value distribution difficult.

In some of the earlier studies on scalar non-Gaussian processes, linearization schemes have been applied to determine bounds on their exceedance probabilities [1,27,33]. Grigoriu [9] obtained approximations for mean outcrossing rates of non-Gaussian translation processes by studying the outcrossing characteristics of Gaussian processes, obtained from Nataf's transformation of the parent non-Gaussian process. The application of this method requires information only on the marginal pdf of the parent non-Gaussian process and its covariance structure.

Extreme value distributions for the class of non-Gaussian processes obtained as quadratic combination of Gaussian random processes have been extensively studied. Naess [25] considered the problem of computing the outcrossing rate of a second order stochastic Volterra series. The mean outcrossing rate was expressed in terms of the joint characteristic function of the process and its instantaneous time derivative. Explicit expressions for the joint characteristic function was derived from the knowledge of the joint moments of the process and its time derivative. The saddle point integration method was subsequently used to numerically calculate the outcrossing rate [25,26]. McWilliam [22] proposed alternative expressions for the joint characteristic function and proposed an exact method and a method based on maximum entropy for constructing the joint pdf of the process and its time derivative. A non-Gaussian process which is obtained as a quadratic combination of vector Gaussian processes and which has received significant attention in the literature is the Von Mises stress in linear structures subjected to Gaussian excitations. Madsen [20] adopted a geometric approach for developing analytical expressions for the mean outcrossing rate of Von Mises stress in Gaussian excited linear structures. Here, the problem had been transformed into the polar coordinate space and expressions have been developed for the pdf of the Von Mises stress and for its outcrossing rate. The underlying assumption in this study is that the uncorrelatedness of the process and its time derivative, at the same time instant, implies their independence. This, though true for stationary Gaussian processes, is not true for non-Gaussian processes. This assumption has been bypassed in [13] where the maximum entropy method has been used to construct the joint pdf for the Von Mises stress and its time derivative in a linear structure under Gaussian excitations. In constructing the joint pdf, the higher order joint moments between the process and its time derivative were considered. This ensured that though the process and its instantaneous time derivative were uncorrelated, they were not independent as their higher order joint moments were not zero. Computational algorithms, such as the response surface method, has also been used to study the extremes of Von Mises stress in nonlinear structures under Gaussian excitations [11].

Outcrossing rates of vector random processes have been studied in the context of problems in load combinations [24] and in structural reliability [6,7,14,15,28,32,34]. The focus of many of these problems has been in determining the probability of exceedance of the sum of the component processes, and the outcrossing event has been formulated as a scalar process outcrossing. Some of these results have been used in the geometrical approach considered by Leira [18,19] in the studies on development of multivariate extreme value distributions for vector Gaussian/non-Gaussian random processes. Multivariate extreme value distributions associated with a vector of Gaussian random processes have been developed recently [12], based on the principle that the multi-point processes can be used to model the level crossing statistics associated with the vector Gaussian process.

The methodology proposed in this paper is based on similar principles. Here, its is assumed that for high threshold levels, the outcrossing of a non-Gaussian process may be modelled as Poisson random variables. A discussion on the validity of this assumption is included later. The proposed method is built on the work of Naess [23] where analytical expressions have been derived for the mean outcrossing rate of scalar non-Gaussian processes, obtained as nonlinear transformations of stationary, Gaussian processes. This principle is extended to develop approximations for the multivariate extreme value distributions for a vector of random processes obtained as nonlinear transformations of stationary, Gaussian processes. The new contributions that are made in this study can be summarized as follows:

(1) Development of an analytical formulation for approximating the multivariate extreme value distribution, associated with a vector of mutually dependent, non-Gaussian processes, obtained as nonlinear functions (not limited to quadratic forms) of stationary Gaussian processes.
(2) A computational algorithm has been developed for problems not amenable for closed form analytical expressions for the extreme value distributions, both for the scalar and the vector case.
(3) The approximations for the extreme value distributions developed in this paper are useful in determining the system reliability of linear/nonlinear structures under stationary Gaussian excitations.

3. Problem statement

We consider a linear structure excited by a n-dimensional vector of mutually correlated, stationary, Gaussian loads $\{Y_i(t)\}_{i=1}^n$. The structure response at location j, is given by

$$Z_j(t) = \hat{g}_{[j]}[Y_1(t), \ldots, Y_n(t)], \tag{1}$$

where, $\hat{g}_{[j]}[\cdot]$ is a deterministic function which could be linear or nonlinear. A structure failure is defined to occur when $Z_j(t)$ exceeds specified threshold levels. Thus, the failure probability, P_f, is given by

$$P_f = 1 - P[Z_j(t) \leq \alpha_i; \forall t \in (0, T)]. \tag{2}$$

Here, α_j denotes the threshold level, t is time, T is the duration of interest and $P[\cdot]$ is probability measure. Introducing the random variable Z_{m_j}, given by

$$Z_{m_j} = \max_{0 \leq t \leq T} Z_j(t), \tag{3}$$

Eq. (2) is rewritten as:

$$P_f = 1 - P[Z_{m_j} \leq \alpha] = 1 - P_{Z_{m_j}}(\alpha_j), \tag{4}$$

where, $P_{Z_{m_j}}(\cdot)$ is the PDF of the random variable Z_{m_j}.

We assume that the specified threshold levels, α_j, to be high and the spectral bandwidth ratio of the process $Z_j(t)$ to be such that the out-crossings of $Z_j(t)$, across α_j, are rare and are statistically independent of each other [30,31]. This enables modelling of the level crossings as Poisson point process. For a stationary process, this leads to the expression for the PDF of Z_{m_j}, of the form:

$$P_{Z_{m_j}}(\alpha_j) = \exp[-\nu^+(\alpha_j)T]. \tag{5}$$

Here, $\nu^+(\alpha_j)$ is the mean outcrossing rate of the process $Z_j(t)$ across α_j. The mean outcrossing rate is determined by the Rice formula [29], given by:

$$\nu^+(\alpha_j) = \int_0^\infty \dot{z} p_{Z\dot{Z}}(\alpha_j, \dot{z}; t, t) \mathrm{d}\dot{z}, \tag{6}$$

where, $p_{Z\dot{Z}}(\cdot, \cdot)$ is the joint pdf of the process $Z_j(t)$ and its instantaneous time derivative $\dot{Z}_j(t)$, at time t. As is evident from Eq. (6), a crucial step in this formulation lies in determining the joint pdf $p_{Z\dot{Z}}(\cdot, \cdot)$.

For a structural series system comprising m components, the structural system is deemed to have failed if any of the constituent m components fail. Thus, the system failure, denoted by P_{f_s}, is expressed as

$$\begin{aligned} P_{f_s} &= 1 - P\left[\bigcap_{j=1}^{m} \{Z_{m_j} \leq \alpha_j\}\right] \\ &= 1 - P_{Z_{m_1} \ldots Z_{m_m}}(\alpha_1, \ldots, \alpha_m). \end{aligned} \tag{7}$$

Here, $P_{Z_{m_1} \ldots Z_{m_m}}(\cdot)$ is the m-dimensional joint PDF for the vector of extreme value random variables $\{Z_{m_j}\}_{j=1}^m$. Assuming that the respective thresholds α_j, corresponding to each component process $Z_j(t)$, are sufficiently high for the respective outcrossings to be rare, the level crossings, denoted by $\{N_j(\alpha_j)\}_{j=1}^m$, can be modelled as Poisson random variables. Since the different components have common source of excitations, $\{Z_j(t)\}_{j=1}^m$ are mutually correlated. Consequently, $\{Z_{m_j}\}_{j=1}^m$ are also expected to be mutually dependent. This implies the need for developing approximations for the joint multivariate PDF for the extreme values. In this paper, we follow the approach by Naess [23], to obtain expressions for the mean outcrossing rate. First, we discuss the method as applicable when $Z_j(t)$ is scalar. Subsequently, we extend the formulation to when the structure response constitutes a m-dimensional vector $\{Z_j(t)\}_{j=1}^m$.

4. Extreme value distribution for scalar random process

Using standard linear transformations, we first transform $\{Y_i(t)\}_{i=1}^n$ to a vector of mutually independent, zero-mean, stationary, Gaussian random processes, $\{X_i\}_{i=1}^n$. The structure response at location j, is now expressed as

$$Z_j(t) = g_{[j]}[X_1(t), \ldots, X_n(t)], \tag{8}$$

where, $\hat{g}_{[j]}[\mathbf{Y}(t)] = \hat{g}_{[j]}[\mathbf{L}\mathbf{X}(t)] = g_{[j]}[\mathbf{X}(t)]$ and $\mathbf{L}$ is a matrix of linear multipliers. For the sake of clarity, the subscripts in $Z_j(t)$ and $g_{[j]}[\cdot]$ are omitted further in this section. We rewrite $p_{Z\dot{Z}}(\cdot, \cdot)$ in the form

$$\begin{aligned} &p_{Z\dot{Z}}(z, \dot{z}; t, t) \\ &= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} p_{X_2 \ldots X_n Z\dot{Z}}(x_2, \ldots, x_n, z, \dot{z}; t, t) \mathrm{d}x_2 \ldots \mathrm{d}x_n, \end{aligned} \tag{9}$$

where, $p_{X_2 \ldots X_n Z\dot{Z}}(\cdot)$ is the joint pdf of random variables $X_2, \ldots, X_n$, Z and $\dot{Z}$, at time t. For the sake of conciseness, the parameter t is omitted from the above expressions. We now seek the transformation between the joint pdf $p_{X_2 \ldots X_n Z\dot{Z}}(\cdot)$ and $p_{X_1 \ldots X_n \dot{Z}}(\cdot)$. In order to achieve this, we assume that at time t, Z in Eq. (8), is a function of X_1 with all the other random variables being fixed. Assuming that for a given set of values $Z, X_2, \ldots, X_n$, there exists k solutions for X_1 from Eq. (8), we get:

$$\begin{aligned} &p_{X_2 \ldots X_n Z\dot{Z}}(x_2, \ldots, x_n, z, \dot{z}) \\ &\quad = \sum_{j=1}^{k} \left| \frac{\partial Z}{\partial X_1} \right|_j^{-1} p_{X_1 X_2 \ldots X_n \dot{Z}}(x_1, x_2, \ldots, x_n, \dot{z}). \end{aligned} \tag{10}$$

Implicit here is the assumption that $Z|_{X_2 \ldots X_n} \perp \dot{Z}|_{X_2 \ldots X_n}$. It must be noted that this is different from the assumption of $Z \perp \dot{Z}$, as has been made in earlier studies in the literature.

The joint pdf $p_{X_1 \ldots X_n \dot{Z}}(\cdot)$ is now written as

$$\begin{aligned} &p_{X_1 \ldots X_n \dot{Z}}(x_1, \ldots, x_n, \dot{z}) \\ &\quad = p_{\dot{Z}|X_1 \ldots X_n}(\dot{z}|X_1 = x_1, \ldots, X_n = x_n) \prod_{j=1}^{n} p_{X_j}(x_j). \end{aligned} \tag{11}$$

Since $\mathbf{X}$ is a vector of mutually independent Gaussian random variables, $p_{X_1 \ldots X_n}(x_1, \ldots, x_n) = \prod_{j=1}^n p_{X_j}(x_j)$.

$\dot{Z}(t)$ is obtained by differentiating Eq. (8) with respect to t, and when conditioned on $\{X_j = x_j\}_{j=1}^n$, is given by

$$\dot{Z}|_{\mathbf{X}} = \sum_{j=1}^{n} \left| \frac{\partial Z}{\partial X_j} \right|_{\mathbf{X}} \dot{X}_j = \sum_{j=1}^{n} g_j \dot{X}_j = \mathbf{G}\dot{\mathbf{X}}. \tag{12}$$

Here, the vector $\mathbf{G} = [g_1, \ldots, g_n]$, $\dot{\mathbf{X}} = [\dot{X}_1, \ldots, \dot{X}_n]'$, the superscript $(')$ denotes matrix transpose, $g_i = \partial Z / \partial X_i$

and when conditioned on $\mathbf{X}$, is a constant. $\dot{X}(t)$ are the time derivatives of $X(t)$ and are also zero-mean stationary, Gaussian random processes and at time t, are mutually independent. This implies that the covariance matrix of $\dot{\mathbf{X}}$, denoted by $\mathbf{C}_{\dot{\mathbf{X}}\dot{\mathbf{X}}}$ is a diagonal matrix. Thus, $\dot{Z}\,|_{\mathbf{X}}$ in Eq. (12), is a linear sum of Gaussian random variables and is a Gaussian random variable with parameters:

$$\mu_{\dot{Z}|\mathbf{X}} = \mathbf{G}\langle\dot{\mathbf{X}}|\mathbf{X}=\mathbf{x}\rangle = \mathbf{G}\langle\dot{\mathbf{X}}\mathbf{X}\rangle\mathbf{x},$$
$$\sigma^2_{\dot{Z}|\mathbf{x}} = \mathbf{G}\langle\dot{\mathbf{X}}\dot{\mathbf{X}}^*\rangle\mathbf{G}' = \mathbf{G}C_{\dot{\mathbf{X}}\dot{\mathbf{X}}}\mathbf{G}'. \tag{13}$$

Here, $\langle\cdot\rangle$ is the expectation operator and the superscript $*$ denotes complex conjugation.

Substituting Eqs. (9)–(13) in Eq. (6), the mean up-crossing rate is given by:

$$\nu^+(\alpha) = \sum_{j=1}^{k}\int_{\Omega_j}\cdots\int |g_1^{(j)}|^{-1}\left\{\int_0^\infty \dot{z}\,p_{\dot{Z}|\mathbf{X}}(\dot{z};\mathbf{x}^{(j)},t)\mathrm{d}\dot{z}\right\}$$
$$\times p_{X_1}(x_1^{(j)})\prod_{i=2}^{n} p_{X_i}(x_i)\mathrm{d}x_2\ldots\mathrm{d}x_n. \tag{14}$$

Here, $\mathbf{x}^{(j)} = [x_1^{(j)}, x_2, \ldots, x_n]$, $g_1^{(j)} = \left|\frac{\partial Z}{\partial X_1}\right|_{\mathbf{X}^{(j)}}$, and Ω_j denotes the domain of integration determined by the permissible set of values $x_2,\ldots,x_n$ for each solution of $\{x_1^{(j)}\}_{j=1}^k$. Since $p_{\dot{Z}|\mathbf{X}}(\cdot)$ is Gaussian, it can be shown that [23]

$$\int_0^\infty \dot{z}\,p_{\dot{Z}|\mathbf{X}}(\dot{z};\mathbf{x},t)\mathrm{d}\dot{z} = \sigma_{\dot{Z}|\mathbf{X}}\,\Psi\left(\frac{\mu_{\dot{Z}|\mathbf{X}}}{\sigma_{\dot{Z}|\mathbf{X}}}\right), \tag{15}$$

where, $\Psi(x) = \varphi(x) + x\,\Phi(x)$, with $\varphi(x)$ and $\Phi(x)$ denoting, respectively, the standard Gaussian pdf and PDF. Thus, Eq. (14) can be expressed as:

$$\nu^+(\alpha) = \sum_{j=1}^{k}\int_{\Omega_j}\cdots\int f(x_1^{(j)},x_2,\ldots,x_n)$$
$$\times p_{X_2}(x_2)\ldots p_{X_n}(x_n)\mathrm{d}x_2\ldots\mathrm{d}x_n, \tag{16}$$

where, $f(x_1^{(j)},x_2,\ldots,x_n) = |g_1^{(j)}|^{-1}\sigma_{\dot{Z}|\mathbf{X}}\,\Psi\left(\frac{\mu_{\dot{Z}|\mathbf{X}}}{\sigma_{\dot{Z}|\mathbf{X}}}\right)p_{X_1}(x_1^{(j)})$.

The difficulties involved in evaluating $\nu^+(\alpha)$ from Eq. (16) are:

(a) determining the domain of integration Ω_j, defined by the possible set of solutions for $X_1^{(j)}$,
(b) evaluation of the multidimensional integrals.

$$I_j = \int_{\Omega_j}\cdots\int f(x_1^{(j)},x_2,\ldots,x_n)$$
$$\times p_{X_2}(x_2)\ldots p_{X_n}(x_n)\mathrm{d}x_2\ldots\mathrm{d}x_n, \tag{17}$$

where the dimension of the integral, I_j is $(n-1)$.

In this study, we adopt a numerical strategy to overcome these difficulties. However, we first develop a methodology for constructing the extreme value distributions when the response $\{Z_j(t)\}_{j=1}^m$ constitutes a m-dimensional vector of random processes.

5. Extreme value distributions for vector random processes

The theory developed in the previous section is now extended to construct the multivariate extreme value distribution when the response processes, $\{Z_j(t)\}_{j=1}^m$, constitute a m-dimensional vector of mutually dependent random processes. Mathematically, each component of the vector non-Gaussian process is expressed as in Eq. (8), $(j = 1,\ldots,m)$. Here, $\{Z_j(t)\}_{j=1}^m$ represent similar response quantities at m different locations of the structure or m different response quantities at the same structure location or a combination of both. Extending the same argument as in the previous section, we assume that for sufficiently high thresholds, α_j, corresponding to each component process $Z_j(t)$, the number of level crossings can be modelled as Poisson random variables. Since $\{Z_j(t)\}_{j=1}^m$ are functions of $\{X_j\}_{j=1}^n$, it is expected that $\{Z_j(t)\}_{j=1}^m$ have mutual dependence. Consequently, the number of level crossings associated with each $Z_j(t)$ are expected to be mutually dependent. To construct the joint multivariate PDF for the level crossings, we adopt the procedure used in a recent study [12]. We first develop the joint PDF when $m = 2$ and subsequently show how the proposed methodology can be extended to higher dimensions ($m > 2$).

5.1. Bivariate vector of random processes

We consider the up-crossings of two non-Gaussian random processes $Z_1(t)$ and $Z_2(t)$. Let N_1 and N_2 be the number of level crossings of $Z_1(t)$ and $Z_2(t)$, across thresholds α_1 and α_2, in time of duration T. For high levels α_j, $(j = 1, 2)$, N_1 and N_2 are modeled as mutually dependent Poisson random variables. We introduce the transformations [16]

$$N_1 = U_1 + U_3,$$
$$N_2 = U_2 + U_3, \tag{18}$$

where, $\{U_j\}_{j=1}^3$ are mutually independent, Poisson random variables with parameters $\{\lambda_j\}_{j=1}^3$ which are, as of yet, unknowns. Taking expectations on both sides of Eq. (18), we get

$$\begin{bmatrix}1&0&1\\0&1&1\\0&0&1\end{bmatrix}\begin{Bmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{Bmatrix} = \begin{Bmatrix}\langle N_1\rangle\\\langle N_2\rangle\\\mathrm{Cov}[N_1,N_2]\end{Bmatrix}. \tag{19}$$

It has been shown [16] that the bivariate extreme value distribution is

$$P_{Zm_1Z_{m_2}}(\alpha_1,\alpha_2) = \exp\left[-\sum_{j=1}^{3}\lambda_j\right], \tag{20}$$

where $\{\lambda_j\}_{j=1}^3$ are obtained from Eq. (19). Thus, a crucial step in this formulation lies in evaluating $\langle N_1(\alpha_1)\rangle$, $\langle N_2(\alpha_2)\rangle$ and $\mathrm{Cov}[N_1, N_2]$.

Now, it is well known that $\langle N_j(\alpha_j)\rangle$, is related to the parent process $Z_j(t)$ through the mathematical relation [29]

$$\langle N_j\rangle = \int_0^T\int_0^\infty \dot{z}_j\,p_{Z_j\dot{Z}_j}(\alpha_j,\dot{z}_j;t)\mathrm{d}\dot{z}_j\mathrm{d}t. \tag{21}$$

Here, the integration of the inner integral leads to the mean outcrossing rate of $Z_j(t)$, across α_j, $(j = 1, 2)$ and is denoted by $\nu_j^+(\alpha_j)$. The covariance function is expressed as $\mathrm{Cov}[N_1, N_2] = \langle N_1 N_2\rangle - \langle N_1\rangle - \langle N_2\rangle$ and requires evaluation of the joint expectation $\langle N_1 N_2\rangle$, given by:

$$\langle N_1 N_2\rangle = \int_0^T \int_0^T \int_0^\infty \int_0^\infty \dot{z}_1 \dot{z}_2 p_{Z_1 Z_2 \dot{Z}_1 \dot{Z}_2} \times(\alpha_1, \alpha_2, \dot{z}_1, \dot{z}_2; t_1, t_2)\mathrm{d}\dot{z}_1 \mathrm{d}\dot{z}_2 \mathrm{d}t_1 \mathrm{d}t_2. \quad (22)$$

If $Z_1(t)$ and $Z_2(t)$ are stationary random processes, Eq. (22) can be reduced to the form:

$$\langle N_1 N_2\rangle = \int_{-T}^{T} (T - |\tau|) I(\tau)\mathrm{d}\tau, \quad (23)$$

where, $\tau = t_2 - t_1$ and $I(\tau)$ denotes the inner 2-d integral in Eq. (22). Details of the derivation of Eq. (23) is available in [12].

Techniques for evaluating Eq. (21) has been discussed in Section 4. For evaluating Eq. (22), we seek the joint pdf $p_{Z_1 Z_2 \dot{Z}_1 \dot{Z}_2}(z_1, z_2, \dot{z}_1, \dot{z}_2; t_1, t_2)$, which is usually not known. For the sake of clarity, we denote $Z_1(t)$ and $Z_2(t)$ by $U(t)$ and $V(t)$, respectively, such that:

$$U(t) = g[X_1(t), \ldots, X_n(t)]$$
$$V(t) = h[X_1(t), \ldots, X_n(t)], \quad (24)$$

where, g and h are respectively linear/nonlinear deterministic functions, obtained by the transformations $g(\cdot) = \hat{g}_{[1]}[\mathbf{L}\mathbf{X}(t)]$ and $h(\cdot) = \hat{g}_{[2]}[\mathbf{L}\mathbf{X}(t)]$. Following a similar argument as for the scalar case, we write:

$$p_{UV\dot{U}\dot{V}}(u, v, \dot{u}, \dot{v}; t_1, t_2) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{X_3 \ldots X_n U V \dot{U} \dot{V}} (x_3, \ldots, x_n, u, v, \dot{u}, \dot{v}; t_1, t_2)\mathrm{d}x_3 \ldots \mathrm{d}x_n, \quad (25)$$

where, the dimension of the integrals is $(n-2)$. Assuming that for fixed values of $X_3, \ldots, X_n$, U and V, the solution of the simultaneous Eq. (24) leads to k solutions for (X_1, X_2), we can write:

$$p_{X_3 \ldots X_n U V \dot{U} \dot{V}}(x_3, \ldots, x_n, u, v, \dot{u}, \dot{v}) = \sum_{i=1}^{k} |\mathbf{J}|_{(i)}^{-1} p_{X_1 \ldots X_n \dot{U} \dot{V}}(x_1^{(i)}, x_2^{(i)}, x_3, \ldots, x_n, \dot{u}, \dot{v}). \quad (26)$$

Here, $\mathbf{J}_{(i)}$ is the Jacobian:

$$\mathbf{J}_{(i)} = \begin{bmatrix} \partial U/\partial X_1 & \partial U/\partial X_2 \\ \partial V/\partial X_1 & \partial V/\partial X_2 \end{bmatrix}, \quad (27)$$

evaluated at $(x_1^{(i)}, x_2^{(i)})$. In writing the transformation in Eq. (26), it is assumed that: $p_{UV|X_3 \ldots X_n}(\cdot) \perp p_{\dot{U}\dot{V}|X_3 \ldots X_n}(\cdot)$.

Expressing $p_{X_1 \ldots X_n \dot{U} \dot{V}}(x_1^{(i)}, x_2^{(i)}, x_3, \ldots, x_n, \dot{u}, \dot{v}; t_1, t_2)$ in terms of the conditional pdf, we get:

$$p_{X_1 \ldots X_n \dot{U} \dot{V}}(x_1, \ldots, x_n, \dot{u}, \dot{v}; t_1, t_2) = p_{\dot{U}\dot{V}|X_1 \ldots X_n}(\dot{u}, \dot{v}|\mathbf{X} = \mathbf{x}; t_1, t_2) p_{X_1 \ldots X_n}(x_1, \ldots, x_n), \quad (28)$$

where, $p_{X_1 \ldots X_n}(x_1, \ldots, x_n) = \prod_{j=1}^{n} p_{X_j}(x_j)$, since $\{X_j\}_{j=1}^{n}$ constitute a vector of standard normal random variables.

The time derivatives, $\dot{U}(t_1)$ and $\dot{V}(t_2)$, conditioned on $\mathbf{X}$, is expressed as:

$$\dot{U}(t_1)|_{\mathbf{X}} = \sum_{i=1}^{n} \left|\frac{\partial g}{\partial X_i}\right|_{\mathbf{X}} \dot{X}_i(t_1) = \mathbf{G}\dot{\mathbf{X}}(t_1)$$
$$\dot{V}(t_2)|_{\mathbf{X}} = \sum_{i=1}^{n} \left|\frac{\partial h}{\partial X_i}\right|_{\mathbf{X}} \dot{X}_i(t_2) = \mathbf{H}\dot{\mathbf{X}}(t_2). \quad (29)$$

Here, $\mathbf{G} = [g_1, \ldots, g_n]$, $\mathbf{H} = [h_1, \ldots, h_n]$, where, $g_i = \partial g/\partial X_i$ and $h_i = \partial h/\partial X_i$, $(i = 1, \ldots, n)$, evaluated at $\mathbf{X} = \mathbf{x}$. Since $\mathbf{G}$ and $\mathbf{H}$ are constants and $\dot{\mathbf{X}}(t)$ constitutes a vector of zero-mean stationary, Gaussian random processes, $\dot{U}(t_1)|_{\mathbf{X}}$ and $\dot{V}(t_2)|_{\mathbf{X}}$ are zero-mean, stationary Gaussian processes. This implies that: $p_{\dot{U}\dot{V}|\mathbf{X}}(\dot{u}, \dot{v}|\mathbf{x}; t_1, t_2)$ is jointly Gaussian and is of the form

$$p_{\dot{U}\dot{V}|\mathbf{X}}(\dot{u}, \dot{v}; t_1, t_2) = 1/(4\pi^2|\mathbf{\Delta}|^{0.5}) \exp[-0.5 W \mathbf{\Delta}^{-1} W']. \quad (30)$$

Here, $W = [\dot{u}, \dot{v}]'$, $\mathbf{\Delta} \equiv \mathbf{\Delta}(t_1, t_2) = \mathbf{T}\mathbf{C}_{\dot{\mathbf{X}}\dot{\mathbf{X}}}(t_1, t_2)\mathbf{T}'$, the operator $|\cdot|$ denotes determinant of a matrix, $\mathbf{T} = [\mathbf{G}, \mathbf{H}]'$ and $\mathbf{C}_{\dot{\mathbf{X}}\dot{\mathbf{X}}}(t_1, t_2)$ is the covariance matrix $\langle \dot{\mathbf{X}}(t_1)\dot{\mathbf{X}}^*(t_2)\rangle$. Since $\dot{\mathbf{X}}(t)$ constitutes a vector of mutually independent, stationary, Gaussian random processes, $\mathbf{C}_{\dot{\mathbf{X}}\dot{\mathbf{X}}}(t_1, t_2) = \mathbf{C}_{\dot{\mathbf{X}}\dot{\mathbf{X}}}(\tau)$ is a diagonal matrix, where $\tau = t_2 - t_1$. Assuming that the covariance matrix $\mathbf{C}_{\dot{\mathbf{Y}}\dot{\mathbf{Y}}}(\tau)$, for the vector of correlated processes $\dot{\mathbf{Y}}(t)$ is known, we get:

$$\mathbf{C}_{\dot{\mathbf{X}}\dot{\mathbf{X}}}(\tau) = \mathbf{L}^{-1}\mathbf{C}_{\dot{\mathbf{Y}}\dot{\mathbf{Y}}}(\tau)(\mathbf{L}')^{-1}. \quad (31)$$

It must be noted that usually $\mathbf{C}_{\mathbf{YY}}(\tau)$ is available and if an element of $\mathbf{C}_{\mathbf{YY}}(\tau)$ is denoted by $r(\tau) = \langle Y(t)Y^*(t+\tau)\rangle$, the corresponding element in $\mathbf{C}_{\dot{\mathbf{Y}}\dot{\mathbf{Y}}}(\tau)$ is given by $\langle \dot{Y}(t)\dot{Y}^*(t+\tau)\rangle = -\partial r^2/\partial\tau^2$.

Substituting Eqs. (25)–(31) in Eq. (23), and rearranging the order of integrations, we get:

$$\langle N_1 N_2\rangle = \sum_{i=1}^{k} \int_{\Omega_i} \cdots \int |\mathbf{J}|_{(i)}^{-1} \left\{ \int_{-T}^{T} (T - |\tau|) F(\tau)\mathrm{d}\tau \right\} \times p_{X_1}(x_1^{(i)}) p_{X_2}(x_2^{(i)}) p_{X_3}(x_3) \ldots p_{X_n}(x_n)\mathrm{d}x_3 \ldots \mathrm{d}x_n, \quad (32)$$

where:

$$F(\tau) = \int_0^\infty \int_0^\infty \dot{u}\dot{v} p_{\dot{U}\dot{V}|\mathbf{X}}(\dot{u}, \dot{v}; \tau)\mathrm{d}\dot{u}\mathrm{d}\dot{v}. \quad (33)$$

Substituting Eq. (30) in Eq. (33) and using symbolic software MAPLE, closed form expressions for $F(\tau)$ are obtained. Thus, when $n = 3$, $\mathbf{\Delta}(\tau) = [c_{11}, c_{12}; c_{12}, c_{22}]$ is a 2×2 matrix and $F(\tau) = s_0(s_1 + s_2 + s_3 + s_4)$, where,

$$s_0 = 0.25(\pi c_{22})^{-3/2},$$
$$s_1 = \pi c_{12} c_{22}^{3/2},$$
$$s_2 = 2(c_{22}^2 c_{11} - c_{12}^2 c_{22})\sqrt{\pi c_{22}/(c_{11}c_{22} - c_{12}^2)},$$
$$s_3 = 2c_{12}^2(c_{22} - c_{12}^2/c_{11})\sqrt{\pi(c_{22}/(c_{11}c_{22} - c_{12}^2))},$$
$$s_4 = 2c_{22}c_{12}\sqrt{\pi}\left\{2c_{12}/\left[\sqrt{c_{22}(c_{11}c_{22} - c_{12}^2)}\right.\right.$$

$$\times (2 + 2c_{12}^2/(c_{11}c_{22} - c_{12}^2))\Bigg] + \tan^{-1}\left[c_{12}/\sqrt{c_{11}c_{22} - c_{12}^2}\right]\Bigg\}. \quad (34)$$

In Eq. (32), if the inner integral, with respect to τ is carried out numerically, the dimension of the remaining integrals is $(n-2)$.

5.2. Higher dimensional vectors of random processes

As has been shown earlier [12], the above construct for bivariate vector of Poisson random variables can be generalized easily for the case $m > 2$. The number of mutually independent Poisson random variables can be generalized to be given by $C_1^m + C_2^m$, where C_k^m denotes combination of m variables taken k at a time. Thus, for $m = 3$, we need six mutually independent Poisson random variables, $\{U_j\}_{j=1}^6$, with parameters $\{\lambda_j\}_{j=1}^6$. We define

$$\begin{aligned} N_1 &= U_1 + U_4 + U_5 \\ N_2 &= U_2 + U_4 + U_6 \\ N_3 &= U_3 + U_5 + U_6. \end{aligned} \quad (35)$$

The joint extreme value distribution is expressed as $P_{Z_{m_1}Z_{m_2}Z_{m_3}}(\alpha_1, \alpha_2, \alpha_3) = \exp[-\sum_{j=1}^6 \lambda_j]$. The equations relating $\{\lambda_j\}_{j=1}^6$ to the moments of $\{N_j\}_{j=1}^3$ can be shown to be given by:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{Bmatrix} = \begin{Bmatrix} \langle N_1 \rangle \\ \langle N_2 \rangle \\ \langle N_3 \rangle \\ \mathrm{Cov}[N_1, N_2] \\ \mathrm{Cov}[N_1, N_3] \\ \mathrm{Cov}[N_2, N_3] \end{Bmatrix}. \quad (36)$$

Thus, for $m > 2$, the formulation requires the evaluation of a set of integrals of the form in Eq. (23) and at no stage does the order of the integrals becomes greater than that of Eq. (23). In general, the number of such integrals that need to be evaluated is C_2^m.

5.3. Outcrossings of levels with positive and negative slopes

The formulation presented so far, considered only up-crossings, *i.e.*, level crossings with positive slopes. When level crossings with negative slopes are also included, $N_i = N_i^+ + N_i^-$. Here, N_i^- denotes the crossing of level $-\alpha_i$ with a negative slope. Thus, $\langle N_1 N_2 \rangle = \langle N_1^+ N_2^+ \rangle + \langle N_1^+ N_2^- \rangle + \langle N_1^- N_2^+ \rangle + \langle N_1^- N_2^- \rangle$. The evaluation of each of these expectation requires the solution of integrals of the form as in Eq. (23). Thus, four integrals of the form as in Eq. (23) needs to be evaluated for determining each $\langle N_i N_j \rangle$. The domain of integration with respect to $\dot{u}$ and $\dot{v}$, in evaluating $F(\tau)$ from equations of the form Eq. (33), depend on whether the level crossing being considered is with a positive slope or negative slope and is either $[0, \infty]$ or $[-\infty, 0]$.

6. Numerical algorithm for evaluation of multidimensional integrals

A crucial step in the above formulation lies in evaluating integrals of the type as in Eq. (17). Closed form solutions for I_j are possible only for a limited class of problems. Techniques such as numerical integration and Laplace's asymptotic approximation are of limited used, especially if Ω_j, is irregular and when the dimension of the integration is large [8]. A more robust technique for evaluating I_j is to use the Monte Carlo method.

Here, we first, recast Eq. (17) in the form:

$$I_j = \int_{-\infty}^{\infty} I[q(\mathbf{x}) \le 0] f(\mathbf{x}) p_{\mathbf{X}}(\mathbf{x}) \mathrm{d}\mathbf{x}, \quad (37)$$

where, $q(\cdot)$ defines the domain of integration and $I[\cdot]$ is an indicator function, which takes a value of unity if $q(\cdot)$ lies within Ω_j and zero, otherwise. The form of this equation is now similar to reliability integrals, which are given by:

$$I_j = \int_{-\infty}^{\infty} I[q(\mathbf{X}) \le 0] p_{\mathbf{X}}(\mathbf{x}) \mathrm{d}\mathbf{x}. \quad (38)$$

While direct application of the methods available in structural reliability literature (see [21] for a recent review) for evaluation of Eq. (37) is not possible due to the difference in the form of the integrand, we explore the possibility of using some of these techniques in conjunction with the Monte Carlo method.

6.1. Monte Carlo simulations

In the Monte Carlo method, the basic steps involved in approximating I_j, are:

(1) Digital generation of an ensemble of $\mathbf{X}$, of size N, with pdf $p_{\mathbf{X}}(\mathbf{x})$,
(2) Evaluation of the functions $f(\mathbf{X})$ and $q(\mathbf{X})$,
(3) An approximation for I_j is obtained as

$$I_j \approx \frac{1}{N} \sum_{j=1}^{N} I[q(\mathbf{X}_j) \le 0] f(\mathbf{X}_j). \quad (39)$$

Here, Monte Carlo simulations are carried out for a vector of random variables $\mathbf{X}$ and consequently, the computational effort is less than Monte Carlo simulation of time histories of random processes $\mathbf{Z}(t)$ and estimating their probability of exceedance. Moreover, since $p_{\mathbf{X}}(\mathbf{x})$ is standard normal, simulation of $\mathbf{X}$ is simple and can be carried out easily. As is well known, the accuracy of the estimate of I_j increases as $N \to \infty$. However, the associated difficulties in arbitrarily increasing the sample size are:

(i) Memory space: For large N, the memory space required for storing $\mathbf{X}$ can pose problems:
(ii) Computational time: The domain of integration is determined by computing the function $q(\mathbf{X})$ and the computational time required for this operation depends on the complexity involved in evaluating $q(\mathbf{X})$.

6.2. Importance sampling

The efficiency in approximating I_j using the Monte Carlo method can be increased using importance sampling. Eq. (37) is now recast as [8]:

$$I_j = \int_{-\infty}^{\infty} I[q(\mathbf{X}) \le 0] f(\mathbf{X}) \frac{p_{\mathbf{X}}(\mathbf{x})}{h_{\mathbf{Y}}(\mathbf{x})} h_{\mathbf{Y}}(\mathbf{x}) \mathrm{d}\mathbf{x}$$
$$= \frac{1}{N} \sum_{j=1}^{N} I[q(\mathbf{X}_j) \le 0] f(\mathbf{X}_j) \left[\frac{p_{\mathbf{X}}(\mathbf{x}_j)}{h_{\mathbf{Y}}(\mathbf{x}_j)} \right], \quad (40)$$

where, $h_{\mathbf{Y}}(\cdot)$ is the importance sampling pdf. Since the problem is formulated into the standard normal space $\mathbf{X}$, $h_{\mathbf{Y}}(\cdot)$ can be taken to be Gaussian with unit standard deviation and shifted mean. The difficulty lies in determining where $h_{\mathbf{Y}}(\cdot)$ should be centred. In structural reliability problems, for efficient importance sampling, $h_{\mathbf{Y}}(\cdot)$ is centred at the design point, *i.e.*, the point on the limit surface which lies closest to the origin in the standard normal space. If the limit surface $q(\mathbf{X}) = 0$ is available in explicit form, first order reliability methods can be used to determine the design point. If $q(\mathbf{X}) = 0$ is not available explicitly, an adaptive importance sampling strategy [3] can be adopted to determine the design point. For a given sample size, this procedure leads to more stable estimates of I_j. Alternatively, techniques like response surface method can be used if the function $q(\mathbf{X})$ is not available in explicit form. This, however, has not been explored in the present study.

6.3. Multiple importance sampling functions

In certain problems, the domain of integration, characterized by $q(\mathbf{X}) = 0$, may consist of multiple design points or multiple regions which contribute significantly to I_j. This is especially true when $q(\mathbf{X}) = 0$ is highly nonlinear, irregular or consists of disjointed regions. In these situations, it is necessary to construct a number of importance sampling functions, with each function centred at the various design points [17]. Techniques for determining multiple design points and regions of comparable importance have been discussed in the literature [2,5,10,17].

6.4. Determining the domain of integration

The domain of integration $\Omega \equiv q(\cdot) \le 0$ is defined to constitute the region

- in the $(X_2, \ldots, X_n)$-space which lead to admissible solutions for X_1, defined by the transformation in Eq. (10), for the scalar case,
- $(X_3, \ldots, X_n)$-space which lead to admissible solutions X_1 and X_2, defined by the transformation in Eq. (26), for the vector case.

Since this involves the solution of a set of simultaneous nonlinear equations (for the scalar case, a single nonlinear equation), not all the solutions are real. Admissible solutions for these equations constitute only solutions which are in the real plane since $\mathbf{X}$ is a vector of real random variables. In general, closed form solutions for these nonlinear equations are difficult to obtain and can only be determined numerically. Here, we propose the use of the following two numerical schemes:

(1) If possible, the nonlinear equation is first recast in a polynomial form of a single independent variable. The roots of this polynomial equation can be determined numerically. The corresponding solutions for the dependent variable is subsequently established. This method leads to real as well as imaginary roots.
(2) A more general approach is when the Newton–Raphson technique is used to determine the roots of nonlinear equations. This method converges to only real solutions and fails to converge in the absence of real solutions.

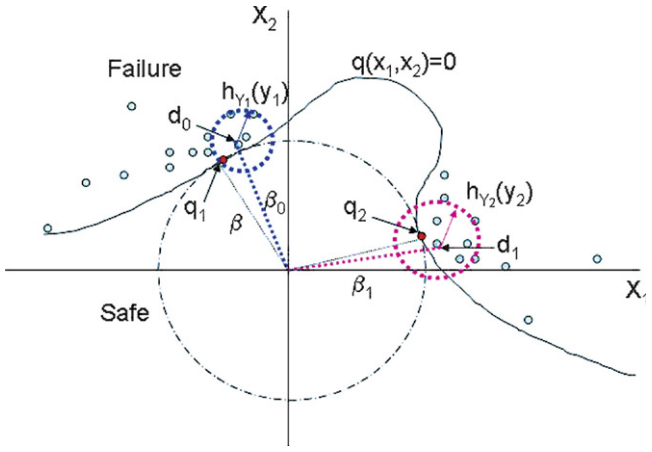

Fig. 1. Schematic diagram for numerical algorithm for evaluating multidimensional integrals; $q(x_1, x_2) = 0$ is the limit surface ion the $X_1 - X_2$ random variable space; $h_{Y_1}(y_1)$ and $h_{Y_2}(y_2)$ are the two importance sampling pdfs; two design points at distance β from the origin.

6.5. Numerical algorithm

The numerical algorithm for evaluating I_j is explained in the following steps, with reference to the schematic diagram Fig. 1:

(1) Carry out pilot Monte Carlo simulations in the $\mathbf{X}$-space. If too few or too many samples fall in Ω_j during the pilot simulations, the variance may need to be increased or decreased and the pilot simulations need to be repeated till a reasonable number of samples are obtained in Ω_j.
(2) The samples in Ω_j are sorted according to their distance from the origin and the realization closest to the origin is identified. Let this be d_0 and its distance from the origin be β_0. Here, we assume that there exists two design points, q_1 and q_2, located at a distance β, from the origin.
(3) A Gaussian importance sampling pdf, $h_{Y_1}(y_1)$, is constructed, centred at d_0.
(4) A check for samples in Ω_j, is carried out within a hyper-sphere of radius β_1, $\beta_1 - \beta_0 =$, where is a positive number.
(5) For samples lying within this hyper-sphere, we check for the sample d_1, which lie closest to the origin but is not located in the vicinity of d_0. This is checked by comparing the direction cosines of d_1 and d_0.

(6) By comparing the direction cosines of all samples lying within the hyper-sphere of radius β_1, we can identify the total number of design points that are to be considered. We construct importance sampling pdfs at each of these design points. If there exist no samples with direction cosines distinctly different from d_0, there is only one design point and a single importance sampling pdf is sufficient.
(7) During importance sampling procedure corresponding to a design point, for each sample realization, we check if X_1 and X_2 (X_1 only for the scalar case) expressed in terms of the random variables $(X_3, \ldots, X_n)$ are real. The indicator function is assigned a value of unity if real, and zero otherwise.
(8) An estimate of I_j is obtained from Eq. (40).

7. Numerical examples and discussion

We now illustrate the application of the proposed method through three numerical examples. We first consider a problem where the transformation function is linear and for which closed form solutions exist. In the second example, we consider the extreme value distribution for Von Mises stress. In the third example, we study the bivariate extreme value distribution for the Von Mises stress and the major principal stress, both of which are obtained as nonlinear combinations of vector Gaussian random processes. In the last two examples, the predictions are compared with those obtained from full scale Monte Carlo simulations, which serve as the benchmark in the absence of analytical solutions. In the full scale Monte Carlo method, an ensemble of vectors of time histories of the random processes are digitally generated using the well known spectral decomposition method. Form each of these samples, the extremes over a specified time duration, are detected. The resulting ensemble of vector of extreme values is processed subsequently to estimate the joint PDF.

7.1. Example 1: Linear combination of Gaussian processes

We consider the compound process

$$Z(t) = \sum_{j=1}^{6} X_j(t), \tag{41}$$

where, $\{X_j(t)\}_{j=1}^6$ are mutually independent, stationary Gaussian random processes, with auto-correlation function given by:

$$R_{jj}(\tau) = S_j^2 \exp[-\beta_j \tau^2], \quad j = 1, \ldots, 6. \tag{42}$$

Here, S_j and β_j, $(j = 1, \ldots, 6)$, are constants. $Z(t)$ being a linear combination of $\{X_j(t)\}_{j=1}^6$, is a stationary, Gaussian process, whose mean up-crossing rate is analytically given by [29]

$$\nu^+(\alpha) = \frac{1}{2\pi}\frac{\sigma_{\dot{Z}}}{\sigma_Z} \exp\left[-\frac{1}{2}\left(\frac{\alpha}{\sigma_Z}\right)^2\right]. \tag{43}$$

Here, σ_Z and $\sigma_{\dot{Z}}$ are the standard deviations of $Z(t)$ and $\dot{Z}(t)$, respectively. We employ the proposed method to determine $\nu^+(\alpha)$ and compare with the predictions from Eq. (43). The equation $z = g(X_1, x_2, \ldots, x_6)$ has just one solution, *i.e.*, $X_1 = z - \sum_{i=2}^6 x_i$. It can be shown that the expression for $\nu^+(z)$ reduces to the form [23]

$$\nu^+(z) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\left\{ \sum_{j=1}^{6} \sigma_{\dot{X}_j}^2 \right\}^{1/2} p_{X_1}(x_1) \right] \times \prod_{i=2}^{6} \{p_{X_i}(x_i) \mathrm{d}x_i\}. \tag{44}$$

Here, $\sigma_{\dot{X}_j}^2$ is the variance of the time derivatives of the processes $\{X_j(t)\}_{j=1}^6$. The integration in Eq. (44) is of 5-dimensions. Since X_1 is expressed as linear function of $(Z, X_2, \ldots, X_6)$, the domain of integration Ω is $[-\infty, \infty]$.

The extreme value distributions, for $T = 10$ s, are calculated using the following three methods:

- Method 1: Full scale Monte Carlo simulations where random time histories for $\{X_j(t)\}_{j=1}^6$ are digitally generated.
- Method 2: The mean outcrossing rate is computed from the closed form expression in Eq. (43).
- Method 3: $\nu^+(\alpha)$ is computed from Eq. (44) where the integrals are evaluated using Monte Carlo simulations.

Fig. 2 illustrates the estimates of extreme value distribution, $P_{Z_m}(\cdot)$ for $Z(t)$. In Method 1, we consider an ensemble of 2000 time histories. The sample size considered in Method 3, is taken to vary from $1 \times 10^4 - 1 \times 10^6$. It is observed that the accuracy of the integrals using the proposed method increases with the sample size. This emphasizes the efficiency of using importance sampling in increasing the accuracy in evaluation of the multidimensional integrals.

7.2. Example 2: Extreme value distribution for Von Mises stress

We consider the up-crossings of square of Von Mises stress process $Z(t)$. $Z(t)$ is obtained as a quadratic transformation of the constituent stress components, $\{Y_j(t)\}_{j=1}^6$, and is given by

$$Z(t) = Y_1(t)^2 + Y_2(t)^2 + Y_3(t)^2 + 3Y_4(t)^2 + 3Y_5(t)^2 + 3Y_6(t)^2 - Y_1(t)Y_2(t) - Y_1(t)Y_3(t) - Y_2(t)Y_3(t). \tag{45}$$

In linear structures under stationary Gaussian excitations, $\{Y_j(t)\}_{j=1}^6$ constitute a vector of stationary, Gaussian random processes. $Z(t)$ is however, non-Gaussian, whose pdf is not known. We assume $\{Y_j(t)\}_{j=1}^6$ to be stationary, Gaussian random processes, with auto-correlation function given by

$$R_{jj}(\tau) = S_j^2 \exp[-\beta_j \tau^2], \quad j = 1, \ldots, 6. \tag{46}$$

Here, S_j and β_j, $(j = 1, \ldots, 6)$, are constants. $\{Y_j(t)\}_{j=1}^6$ are functions of the same loading process and are correlated. However, in this example, we seek to illustrate the applicability

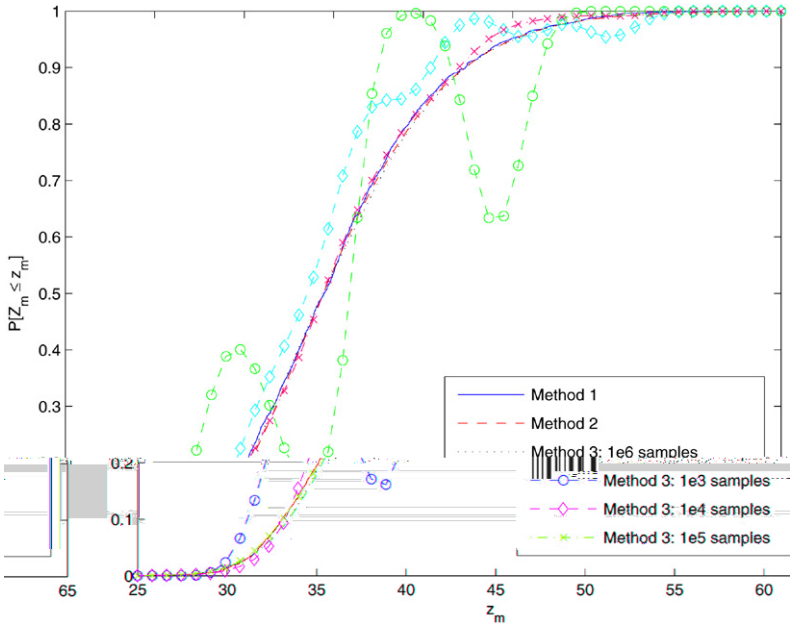

Fig. 2. Comparison of extreme value distribution and effect of sample size in Monte Carlo procedure in evaluating the multidimensional integrals; example 1.

of the proposed method to deal with nonlinear transformations of Gaussian processes. With this in view, we make the assumption that $\{Y_j(t)\}_{j=1}^{6}$ are mutually independent. The case of correlated stress components is considered in the next example.

Estimates of the exceedance probability are computed using three methods:

- Method 1: Full scale Monte Carlo simulations are carried out with sample size of 5000 time histories for $Z(t)$,
- Method 2: Proposed method where the 5-dimensional integrals are evaluated using the Monte Carlo procedure without importance sampling,
- Method 3: Proposed method where importance sampling based Monte Carlo is used to evaluate the 5-dimensional integrals.

The sample size considered in Methods 2 and 3 were taken to be 1×10^6. In applying Method 3, we observe that the domain of integration Ω, consists of two distinct regions which contribute significantly to the 5-d integral. Consequently, we construct two importance sampling pdfs at two different locations. A comparison of the estimates of exceedance probability by the above methods are illustrated in Fig. 3. From this figure, it can be concluded that the use of importance sampling based Monte Carlo method for evaluating multidimensional integrals lead to more accurate estimates of the mean-outcrossing rates, and hence, more accurate estimates of the exceedance probabilities.

7.3. Example 3: Bivariate extreme value distribution for Von Mises stress and major principal stress

We now consider a plane stress element with three non-zero stress components $[Y_1(t), Y_2(t), Y_3(t)]$, where $\{Y_j(t)\}_{j=1}^{3}$ constitute a vector of mutually correlated, stationary Gaussian random processes. The auto- and cross-PSDs of $Y_j(t)$ are, respectively, taken to be of the form:

$$S_{jj}(\tau) = \frac{s_j^2}{\sqrt{\pi\alpha_j}}\exp\left[-\frac{\omega^2}{4\alpha_j}\right], \quad j = 1, 2, 3, \tag{47}$$

and

$$S_{jk}(\omega) = c_{jk}(\omega)\sqrt{S_{jj}(\omega)S_{kk}(\omega)}\exp[-\mathrm{i}\gamma_{jk}(\omega)], \quad j \neq k, \tag{48}$$

where, $\gamma_{jk}(\omega)$ = phase spectrum, $c_{jk}(\omega)$ = coherence spectrum which takes vales in [0, 1], $\mathrm{i} = \sqrt{-1}$, s_j and α_j are constants and $(j, k = 1, 2, 3)$. For the sake of simplicity, we assume $c_{jk}(\omega)$ to have a constant value for all ω and is taken to be $c_{12} = 0.80$, $c_{13} = 0.50$ and $c_{23} = 0.30$. The phase spectrum is assumed to be of the form $\gamma_{jk}(\omega) = \omega/\psi_{jk}$, with $\psi_{12} = 4$, $\psi_{13} = 8$ and $\psi_{23} = 6$. Also, $S_1 = S_3 = 3\sqrt{2}$, $S_2 = 6\sqrt{2}$, $\alpha_1 = 100$, $\alpha_2 = 200$ and $\alpha_3 = 300$. The auto-PSD and the magnitude of the cross-PSDs are shown in Fig. 4.

The expressions for square of Von Mises stress, $V(t)$, and the major principal stress, $U(t)$, are respectively given by:

$$V(t) = Y_1^2(t) + Y_2^2(t) + 3Y_3^2(t) - Y_1(t)Y_2(t), \tag{49}$$

$$U(t) = \frac{1}{2}\left[Y_1(t) + Y_2(t) + \sqrt{Y_1^2(t) + Y_2^2(t) - 2Y_1(t)Y_2(t) + 4Y_3^2(t)}\right]. \tag{50}$$

The elements of the covariance matrix $\mathbf{C_{YY}}(\tau)$ is obtained from Fourier transforms of Eqs. (47) and (48). The matrix of linear multipliers is obtained by Cholesky decomposition of $\mathbf{C_{YY}}(\tau = 0)$ and is given by $\mathbf{C_{YY}}(\tau = 0) = \mathbf{L}\mathbf{L}'$. The covariance matrix $\mathbf{C_{\dot{Y}\dot{Y}}}(\tau)$ is derived from $\mathbf{C_{YY}}(\tau)$.

We first construct approximations for the marginal extreme value distributions for $V(t)$ and $U(t)$, using the proposed

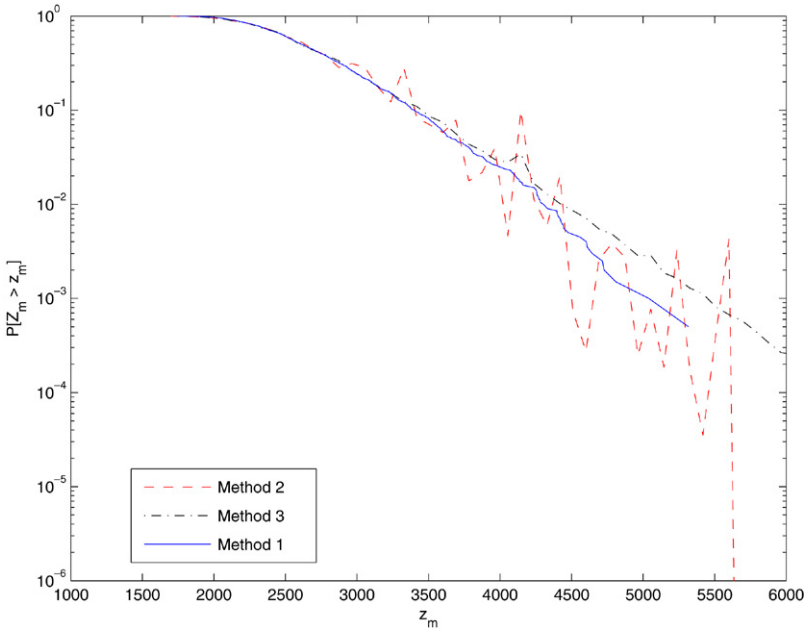

Fig. 3. Comparison of estimates of exceedance probability for Von Mises stress; example 2.

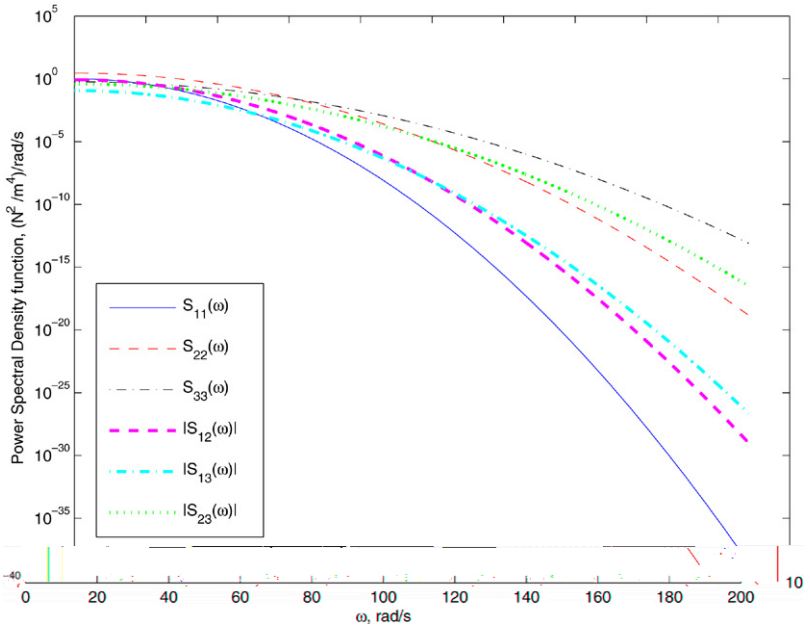

Fig. 4. Auto and cross power spectral density functions for the three stress components; example 3.

method. The time duration $T = 10$ s. The estimated exceedance probabilities for $V(t)$ and $U(t)$ are compared with those obtained from Monte Carlo simulations, on an ensemble of 5000 time histories of $V(t)$ and $U(t)$, and are illustrated in Figs. 5 and 6, respectively. Here, V_m and U_m respectively, denote the extreme values of $V(t)$ and $U(t)$. In both cases, while evaluating the multidimensional integrals of the form in Eq. (17) using the method proposed in this paper, we observe that the domain of integration Ω consists of primarily two distinct regions. This requires construction of two importance sampling pdfs.

We next construct approximations for the bivariate pdf $P_{V_m U_m}(v_m, u_m)$ using the theory developed in this paper; see Fig. 7. Contour plots of $P_{V_m U_m}(v_m, u_m)$, *i.e.*, plots of the relation between v_m and u_m, such that, $P_{V_m U_m}(v_m, u_m) = p$, for different values of p, are shown in Fig. 8, for the bivariate extreme value distribution of $V(t)$ and $U(t)$. The analytical predictions are compared with those obtained from Monte Carlo simulations. The analytical results are observed to be in fairly good agreement with those obtained from Monte Carlo simulations, especially in the higher end of the probability distribution functions. This is to be expected since the assumption of Poisson crossing is more realistic

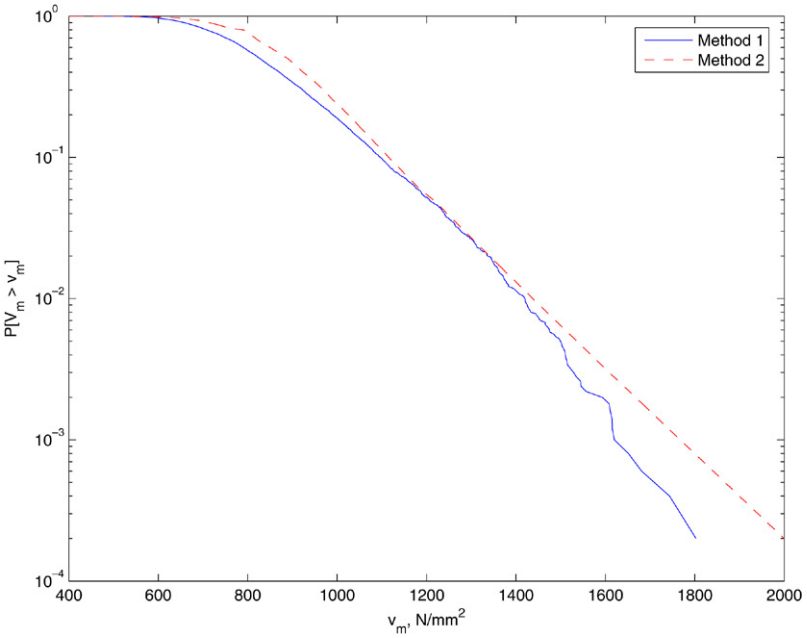

Fig. 5. Exceedance probability of Von Mises stress; example 3.

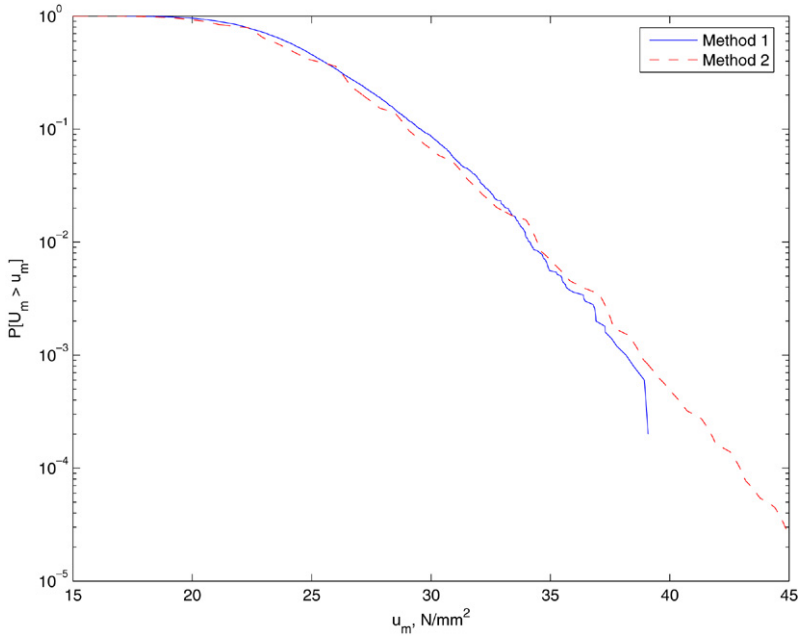

Fig. 6. Exceedance probability of major principal stress; example 3.

towards the tails of the probability distribution functions. The differences in the level curves along the diagonal of the plots can be attributed to inaccuracies due to the assumption that the vector process and its instantaneous time derivative, when conditioned on the process components, are independent. A comparison of the conditional distributions $P_{V_m|U_m}(v_m|U_m = u_m)$ with the univariate PDF, $P_{V_m}(v_m)$ for $V(t)$ is shown in Fig. 9. The corresponding comparison for $U(t)$ is shown in Fig. 10. These figures illustrate that the conditional PDFs differ significantly from the univariate PDFs and underline the importance of the effect of the dependence that exists between the two non-Gaussian processes. The accuracy in constructing the approximations for the joint PDF of the extremes depend on the accuracy in evaluating the multidimensional integrals, which in turn, depend on the sample size, N, used in the Monte Carlo procedure discussed in Section 5. It is to be noted that in the proposed formulation, the size of N depends on the dimension of the integrals I_j and is independent of the exceedance probability levels. On the other hand, in full scale Monte Carlo simulations, the sample size of non-Gaussian time histories that need to be generated is inversely proportional to the exceedance probability levels.

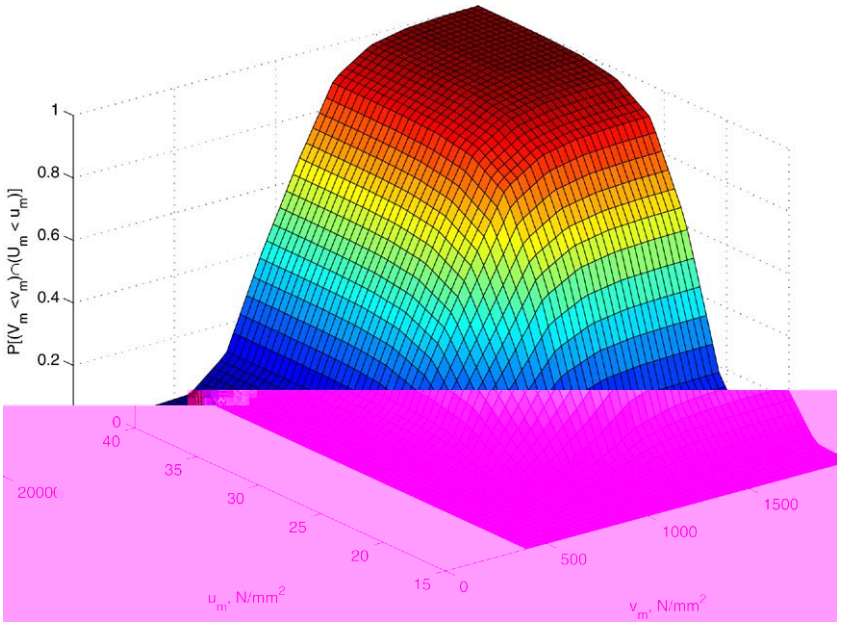

Fig. 7. Joint extreme value distribution for V_m and U_m; $T = 10$ s; example 3.

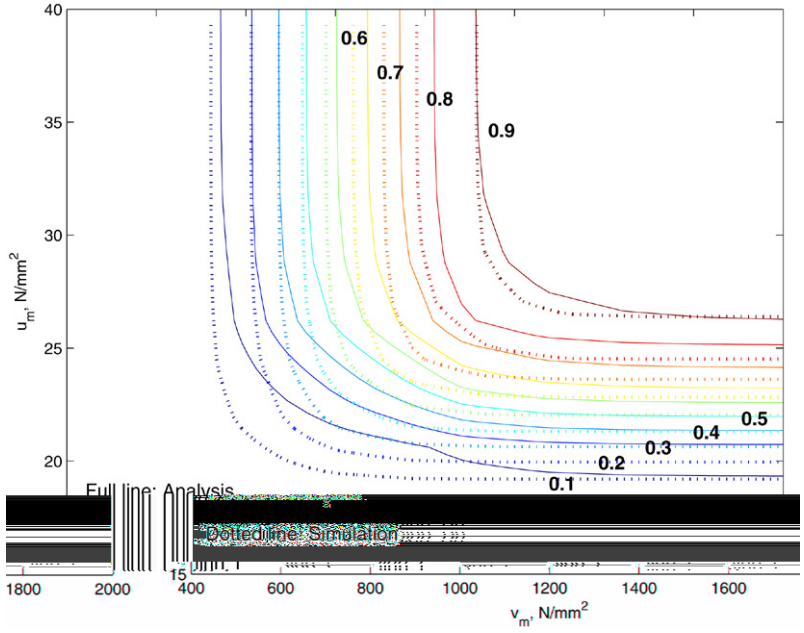

Fig. 8. Level curves of $P_{V_m U_m}(v_m, u_m)$ for the bivariate stationary non-Gaussian random processes, $V(t)$ and $U(t)$; the numbers on the figure indicate probability levels; example 3.

8. Concluding remarks

A methodology has been developed for approximating the joint probability distribution functions for the extremes associated with a vector of mutually dependent, non-Gaussian processes, obtained as nonlinear transformations of a vector of stationary Gaussian processes. A key feature in the development of the proposed method lies in the assumption, that for high thresholds, the number of level crossings of a non-Gaussian process can be modelled as a Poisson point process. The assumption of the outcrossings being Poisson distributed have been proved to be mathematically valid for Gaussian processes when the threshold approaches infinity [4]. However, it has been pointed out that for threshold levels of practical interest, this assumption results in errors whose size and effect depend on the bandwidth of the processes [30,31]. While it can be heuristically argued that for high thresholds, the outcrossings of non-Gaussian processes can be viewed to be statistically independent and hence can be modelled as a Poisson point process, to the best of the authors' knowledge, studies on the validity of this assumption for non-Gaussian processes, do not exist in structural engineering literature. The multivariate extreme value distributions obtained by the proposed method,

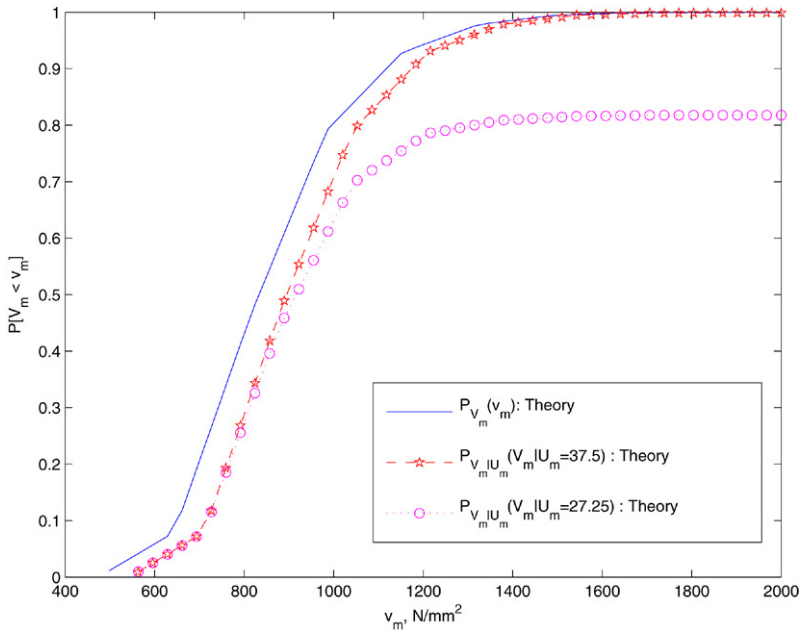

Fig. 9. Comparison of the conditional probability distribution function (PDF) $P_{V_m|U_m}(v_m|U_m = 32.8\ \mathrm{N/mm^2})$ with the univariate PDF $P_{V_m}(v_m)$; example 3.

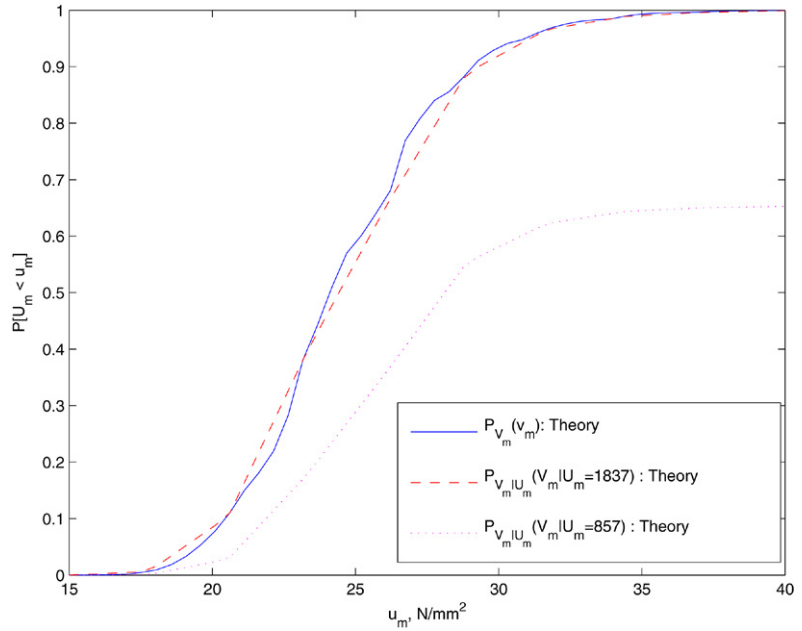

Fig. 10. Comparison of the conditional probability distribution function (PDF) $P_{U_m|V_m}(u_m|V_m = 1832.6\ \mathrm{N/mm^2})$ with the univariate PDF $P_{U_m}(u_m)$; example 3.

is thus expected to inherit the associated inaccuracies and limitations due to this assumption.

An additional limitation in the proposed formulation lies in the assumption that the non-Gaussian process and its instantaneous time derivative, when conditioned on $\mathbf{X}$, are independent. Though for stationary processes, a process and its time derivative are uncorrelated, it does not necessarily imply independence. Consequently, such assumptions lead to errors in approximating the extreme value distributions; this has been illustrated recently, for the scalar case [13]. The assumption made in the proposed formulation is, however, consistent with the assumptions made in some of the existing studies on scalar non-Gaussian processes [9,20]. The study presented in this paper can be viewed as an extension of these studies to the vector case, which to the best of the authors' knowledge, has not been attempted before. The proposed multivariate extreme value distribution functions have applications in reliability analyses of structural systems, with multiple limit states in series configurations.

References

[1] Breitung K, Rackwitz R. Nonlinear combination of load processes. J Struct Mech 1982;10(2):145–66.

[2] Breitung K, Faravelli L. Response surface methods and asymptotic approximations. In: Casciati F, Roberts JB, editors. Mathematical models for structural reliability analysis. CRC Press; 1996 [chapter 5].
[3] Bucher CG. Adaptive importance sampling-an iterative fast Monte Carlo procedure. Struct Saf 1988;5:119–26.
[4] Cramer H. On the intersections between the trajectories of a normal stationary stochastic process and a high level. Ark Mat 1966;6:337–49.
[5] Der Kiureghian A, Dakessian T. Multiple design points in first and second order reliability. Struct Saf 1998;20:37–49.
[6] Der Kiureghain A. The geometry of random vibrations and solutions by FORM and SORM. Probab Eng Mech 2000;15(1):81–90.
[7] Ditlevsen O. First outcrossing probability bounds. J Eng Mech 1984; 110(2):282–92.
[8] Evans M, Swartz T. Approximating integrals via Monte Carlo and deterministic methods. Oxford: Oxford University Press; 2000.
[9] Grigoriu M. Crossings of non-Gaussian translation processes. J Eng Mech 1985;110(4):610–20.
[10] Gupta S, Manohar CS. An improved response surface method for the determination of failure probability and importance measures. Struct Saf 2004;26:123–39.
[11] Gupta S, Manohar CS. Improved response surface method for time variant reliability analysis of nonlinear random structures under nonstationary excitations. Nonlinear Dynam 2004;36:267–80.
[12] Gupta S, Manohar CS. Development of multivariate extreme value distributions in random vibration applications. J Eng Mech 2005;131(7): 712–20.
[13] Gupta S, Manohar CS. Probability distribution of extremes of Von Mises stress in randomly vibrating structures. J Vib Acoust 2005;127:547–55.
[14] Hagen O, Tvedt L. Vector process out-crossing as parallel system sensitivity measure. J Eng Mech 1991;117(10):2201–20.
[15] Hagen O. Conditional and joint failure surface crossing of stochastic processes. J Eng Mech 1992;118(9):1814–39.
[16] Johnson NL, Kotz S. Discrete distributions. New York: John Wiley and Sons; 1969.
[17] Karamchandani A, Bjerager P, Cornell CA. Adaptive importance sampling. In: Ang AHS, Shinozuka M, Schueller GI, editors. Proc. of the 5th international conference on structural safety and reliability. New York: ASCE; 1989. p. 855–62.
[18] Leira BJ. Multivariate distributions of maxima and extremes for Gaussian vector processes. Struct Saf 1994;14:247–65.
[19] Leira BJ. Extremes of Gaussian and non-Gaussian vector processes: A geometric approach. Struct Saf 2003;25:401–22.
[20] Madsen HO. Extreme value statistics for nonlinear load combination. J Eng Mech 1985;111:1121–9.
[21] Manohar CS, Gupta S. Modeling and evaluation of structural reliability: Current status and future directions. In: Jagadish KS, Iyenagar RN, editors. Recent advances in structural engineering. University Press; 2005. p. 90–187.
[22] McWilliam S. Joint statistics of combined first and second order random processes. Probab Eng Mech 2004;19(1):145–54.
[23] Naess A. Prediction of extremes of stochastic processes in engineering applications with particular emphasis on analytical methods. Ph.D. thesis. Trondheim: Norwegian Institute of Technology; 1985.
[24] Naess A. A study of linear combination of load effects. J Sound Vib 1989; 129(2):83–98.
[25] Naess A. Crossing rate statistics of quadratic transformations of Gaussian processes. Probab Eng Mech 2001;16:209–17.
[26] Naess A, Karlsen HC. Numerical calculation of the level crossing rate of second order stochastic Volterra systems. Probab Eng Mech 2004;19(1): 155–60.
[27] Pearce HT, Wen YK. On linearization points for nonlinear combination of stochastic load processes. Struct Saf 1985;2:169–76.
[28] Rackwitz R. Computational techniques in stationary and non-stationary load combination. J Struct Eng 1998;25(1):1–20.
[29] Rice SO. A mathematical analysis of noise. In: Wax N, editor. Selected papers in random noise and stochastic processes. Dover Publications; 1956. p. 133–294.
[30] Vanmarcke E. Properties of spectral moments with applications to random vibrations. J Eng Mech 1972;98:425–46.
[31] Vanmarcke E. On the distribution of first passage time for normal stationary random processes. J Appl Mech 1975;42:215–20.
[32] Veneziano D, Grigoriu M, Cornell CA. Vector process models for system reliability. J Eng Mech 1977;103(EM3):441–60.
[33] Wen YK. Structural load modeling and combination for performance and safety evaluation. Amsterdam: Elsevier; 1990.
[34] Wen YK, Chen HC. System reliability under time varying loads: I. J Eng Mech 1989;115(4):808–23.