The Sensitivity of Quantile Estimates to the Distribution Shape

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Summary

The paper illustrates the sensitivity of quantile bias and mean-square error to the type of distribution obtained by matching 3 sample L-moments. The simulation results substantiate the accuracy and efficiency of the method of L-moments.

Keywords: Extreme values, quantiles, risk analysis, L-Moments, Monte Carlo Simulation.

1. Introduction

The estimation of extreme quantiles corresponding to small probabilities of exceedance is commonly required in the risk analysis of flood protection structures and hydrologic problems. It is desirable that the quantile estimate be unbiased, i.e., its expected value should be equal to the true value. It is also desirable that an unbiased estimate be efficient, i.e., its variance should be as small as possible. The problem of unbiased and efficient estimation of extreme quantiles from small samples is commonly encountered in the civil engineering practice.

The first step in quantile estimation involves fitting an analytical probability distribution to represent adequately the sample observations. To achieve this, the distribution type should be judged from data and then parameters of the selected distribution should be estimated. Since the bias and efficiency of quantile estimates are sensitive to the distribution type, the development of simple and robust criteria for fitting a representing distribution to small samples of observations has been an on-going area of research of which this paper shows some results.

The paper is organised as follows. First a brief introduction is given on the methods of L-Moments. Then Sec. 3 presents a simulation-based study to examine the accuracy of extreme quantile values estimated from the method of L-moments. In Sec. 4 conclusions are given.

2. L-Moments

L-moments are certain linear combinations of probability weighted moments that are analogous to ordinary moments in a sense that they also provide measures of location, dispersion, skewness, kurtosis, and other aspects of the shape of probability distributions or data samples. An r^{th} order L-moment is mathematically defined as

$$l_r = \sum_{k=1}^{r} p^*_{r-1,k-1} b_k \tag{1}$$

where $p_{r,k}$ represents the coefficients of shifted Legendre polynomials (Hosking 1990). The following normalized form of higher order L-moments is convenient to work with.

$$t_r = \frac{l_r}{l_2} \quad , \quad r = 3,4,\ldots \quad \text{and } |\tau_r| < 1 \tag{2}$$

The normalized fourth order L-moment, τ_4, is referred to as L-kurtosis, of a distribution. Hosking and Wallis (1997) showed that L-moments are very efficient in estimating parameters of a wide range of distributions from small samples. The required computation is fairly limited as compared with other traditional techniques, such as maximum likelihood and least square. Furthermore, L-skewness (τ_3) and L-kurtosis (τ_4) are bounded so that $|\tau_r| <1$ for $r = 3,4$. To illustrate this point, annual maximum discharge data of 194 European rivers (Van Gelder et al. 2000) were used to plot the variation of τ_3 and τ_4 in Figures 1a and 1b, which shows the bounded nature of L-moments.

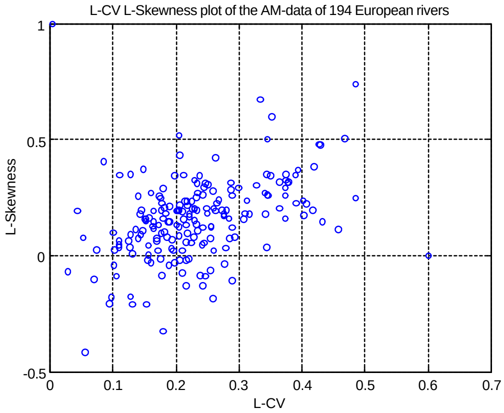

Fig. 1a L-CV / L-Skewness plot

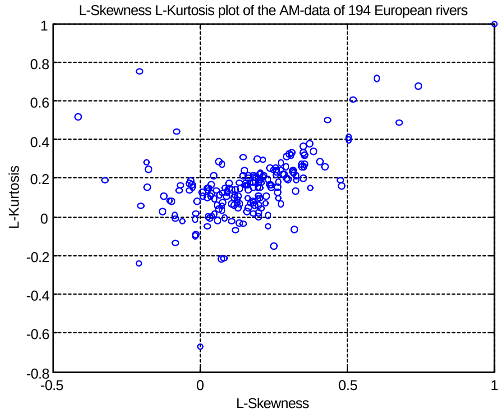

Fig. 1b L-Skewness / L-Kurtosis plot

Hosking and Wallis (1997) proposed a simple approach for quantile estimation using the L-moments of an available sample. The approach involves the computation of 4 L-moments from a given sample. By matching the first three L-moments, a set of 3-parameter distributions (see Table 1) can be fitted to the sample data. The distribution with its L-kurtosis value closest to the sample value of L-kurtosis is suggested to be the most representative distribution, which should be used for quantile estimation. In essence, L-kurtosis can be interpreted as a measure of resemblance between two distributions having common values of the first three L-moments.

No	**Probability Distribution**	**Notation**
1	Generalized Logistic	GLO
2	Generalized Extreme Value	GEV
3	Generalized Lognormal	GNO
4	Generalized Pareto	GPA
5	Generalized Gamma (Pearson Type III)	PE3

Table 1 Probability distributions considered in simulations

3. Simulation Experiments

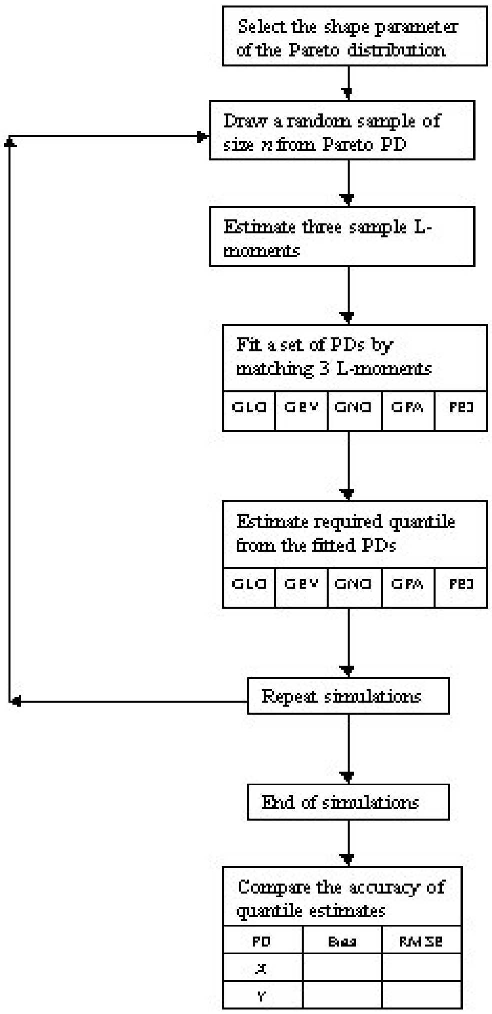

Figure 2: Steps in the simulation experiment

This section presents a simulation-based study to examine the accuracy of extreme quantile values estimated from the method of L-moments. In the simulation experiment, samples were generated from the generalized Pareto distribution (parent) and five 3-parameter distributions (Table 1) were fitted by matching the first three L-moments. Figure 2 shows the steps of the simulation experiment. In particular, the paper illustrates the sensitivity of quantile bias and mean-square error to the type of distribution obtained by matching 3 sample L-moments. For that purpose, various simulation experiments were designed to illustrate the effect of distribution shape and tail weight on the accuracy of quantile estimation.

Consider the first case in which a sample size 30 was used to estimate the quantile for a POE (probability of exceedance) $p=10^{-3}$. The shape parameter, *k*, of the parent Pareto distribution was varied from –0.4 to +0.1. Note that the tail weight of the Pareto distribution as defined by Maes (1995) is equal to –*k*, such that higher the tail weight, the longer and heavier the distribution tail (Smith and Weissman, 1987). In Figure 3, the normalized bias of the quantile estimated from the five candidate PDFs is plotted against the shape parameter.

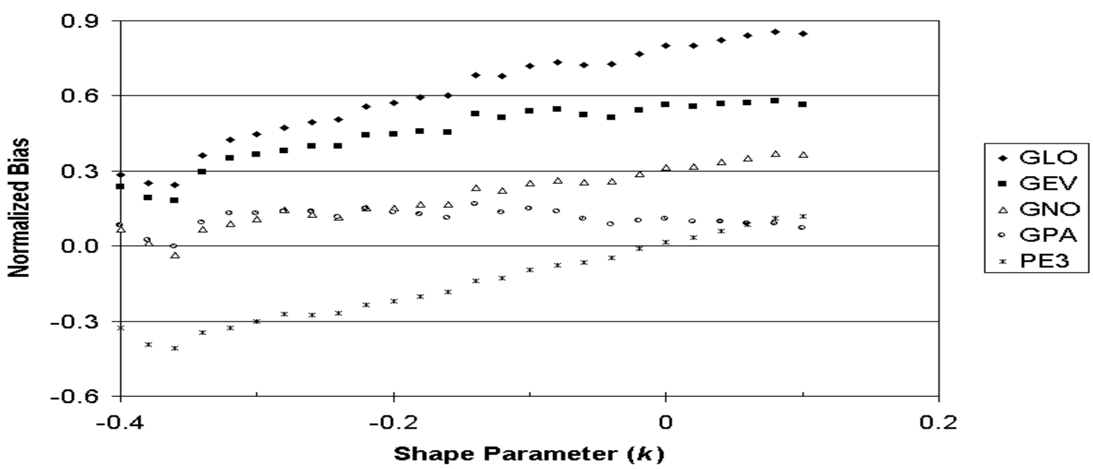

Figure 3: Normalized bias against the shape parameter of the GPA (POE=10^{-3}, n=30)

The normalized bias is minimum (< 10%) for the fitted GPA distribution, whereas it varies significantly (±80%) for other PDFs. It is interesting that the bias of GNO quantile estimates is fairly close to that of GPA for $k < 0$. The efficiency of the quantile estimate can be seen from Figure 4 where the normalized RMSE is plotted against the shape parameter. The RMSE of PE3 estimates is minimum, though the associated bias is quite large as seen from the previous Figure. Among the remaining four PDFs, GPA and GNO estimates have smaller RMSE.

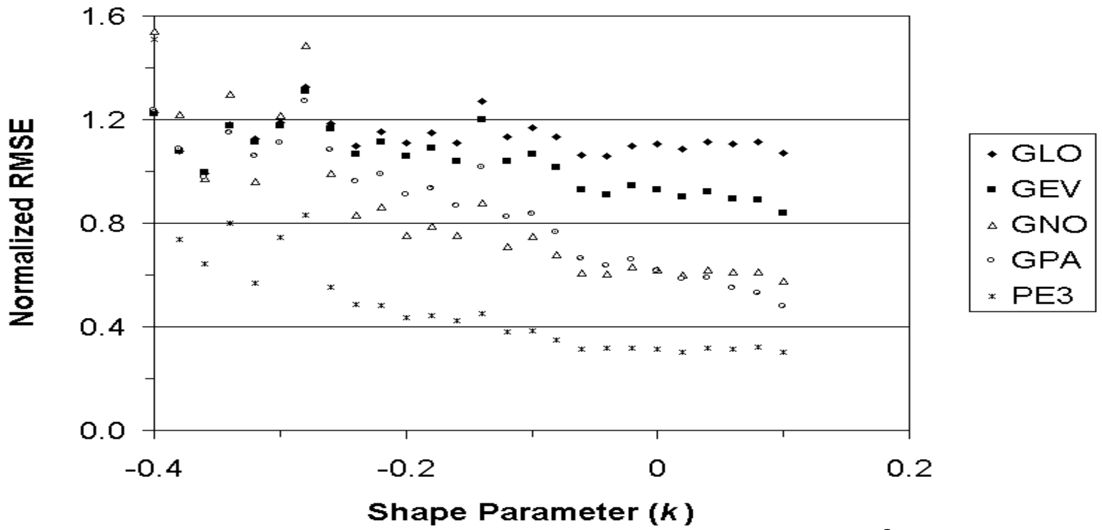

Figure 4: Normalized RMSE against the shape parameter (POE=10^{-3}, n=30)

However, in case of estimates of a lower quantile (probability of exceedence of 10^{-2}) the bias decreases significantly for all PDFs, as shown in Figure 5. In fact, for $k < -0.2$, all PDFs result in fairly small bias (<10%). As far the efficiency is considered, Figure 6 suggests that the RMSE is fairly insensitive to the distribution type. With few exceptions, the RMSE for all quantile estimates lies in a narrow band.

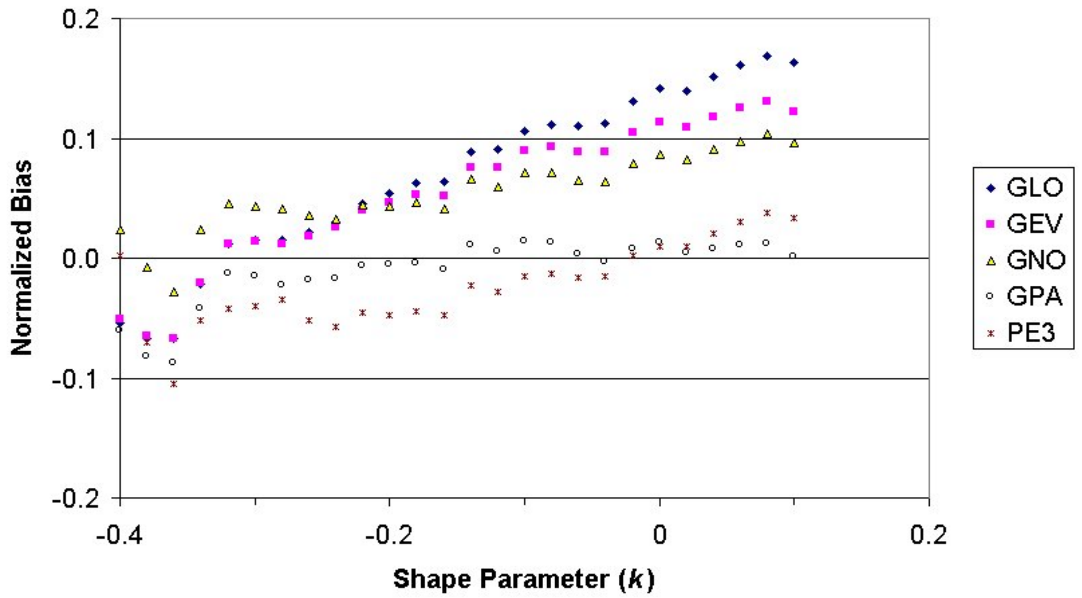

Figure 5: Bias of Pareto quantile estimates (POE=10^{-2}, n = 30)

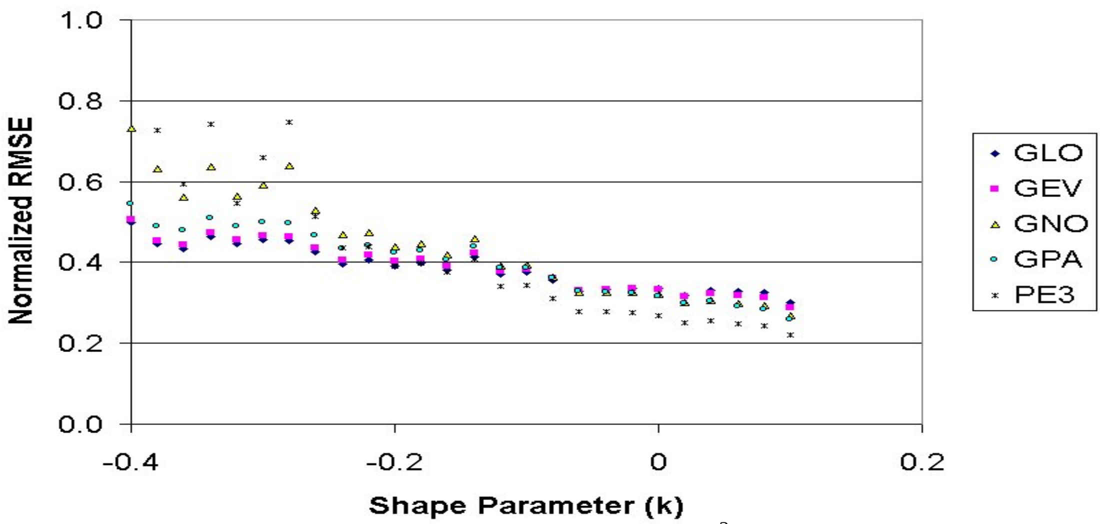

Figure 6: RMSE of Pareto quantile estimates (POE=10^{-2}, n = 30)

To analyze the effect of sample size on bias and efficiency, random samples of various sizes ranging from 50 to 500 were generated from GPA distribution with $k = -0.4$, and quantile values for 10^{-3} were estimated as before. The GPA quantile appears to be almost unbiased, followed by the GNO quantile estimates. It is interesting that the bias is almost insensitive to the sample size (for large sample sizes). The efficiency, however, improves with an increase in the sample size, as shown in Figure 7. In large samples, $n > 100$, the normalized RMSE appears to be insensitive to the distribution type. The RMSE still remains around 30% for the large sample sizes in accordance with Hosking and Wallis (1987). The quantile estimates for 10^{-2} exhibit fairly small bias as well as reduced RMSE . In fact, all five PDFs are able to provide fairly accurate estimates of low quantiles (results not shown here).

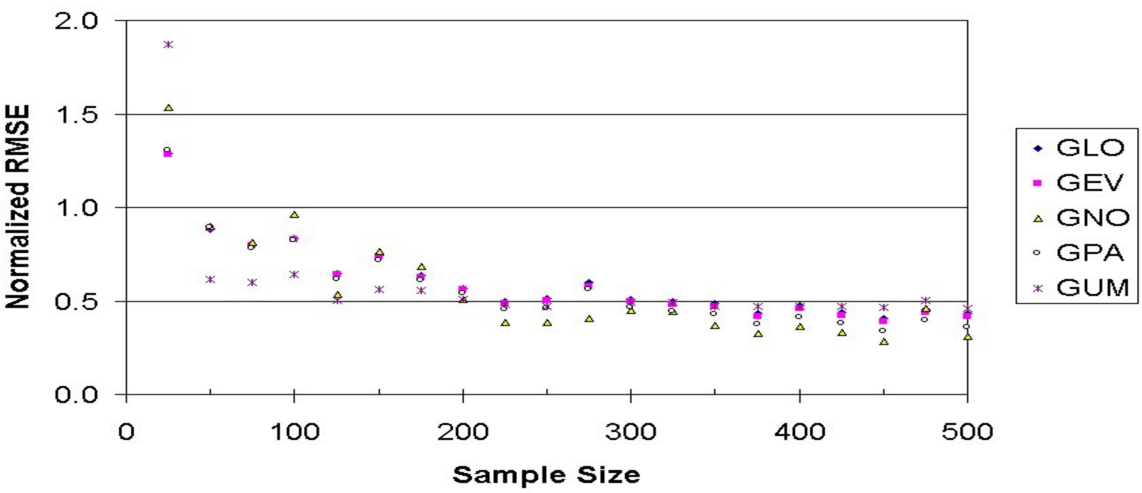

Figure 7: RMSE of Pareto quantile estimates for various sample sizes (prob. of exceedence 10^{-3}, k=-0.4)

The bias of an extreme quantile (with probability of exceedance < 10^{-3}) estimate is more sensitive (than the RMSE) to the distribution type fitted to the sample data. Therefore, the knowledge of the correct distribution type has relevance to the minimization of bias. Although the distribution type has major influence on the quantiles estimates, the dependence is not unique, i.e., more than one distribution can provide reliable estimates with low bias and high efficiency depending on the degree of (tail) equivalence between the population and the fitted distribution. In this respect, the importance of identifying exactly the parent distribution from small sample is limited, rather any distribution reasonably close to the parent can serve the purpose. It is proposed that L-kurtosis is such a measure that can quantify approximately the degree of closeness of the sample data to a candidate distribution (Hosking 1992).

4. Final Comments and Conclusions

Since the probability weighted moments of higher order (> 2) can be reliably estimated from small samples, their use in distribution fitting and quantile estimation has distinct advantages. PWMs are directly related to the expectations of order statistics, as discussed in Section 2. L-moments are the linear combinations of PWMs, and in a simple sense they are analogous to ordinary moments. The L-kurtosis, which is the fourth order L-moment normalized by that of the second order, is considered in the literature as an effective measure of distribution shape and tail behaviour.

The impact of distribution shape on the accuracy of extreme quantile estimates is discussed in Sec. 3. Numerical results indicate that the bias of quantile estimates is more sensitive than RMSE to the distribution shape. It also highlighted that there can be more than one distribution with similar tails that can provide accurate quantile estimates. In other words, the importance of identifying exactly the parent distribution from small sample is somewhat limited, rather any distribution reasonably close to the parent can serve the purpose. It is accepted that L-kurtosis is such a measure that can quantify approximately the degree of closeness of the sample data to a candidate distribution as proposed by Hosking (1992). L-kurtosis is a reliable indicator of distribution shape and its use in quantile estimation is very effective. However, this conclusion is subject to limitations of present investigation, and it would benefit from further validation. Remarkable simplicity of computation makes the L-Kurtosis criterion an attractive tool for distribution fitting.

5. Acknowledgements

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6. References

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