Discretization and Symmetry

Rob F. Remis and Jörn T. Zimmerling

DCSE Fall School, Delft, November 4 – 8, 2019

Introduction

- Objective
  - Discretize Maxwell's equation to formulate model order reduction using linear algebra
  - Symmetry in the discrete domain
  - Application of this theory to imaging

Instantaneously Reacting Material: 1D

One-dimensional Maxwell equation

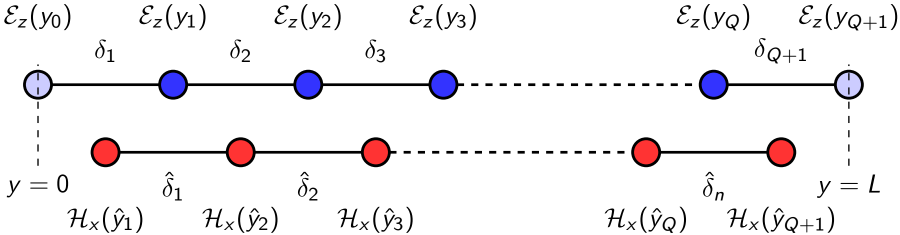
$$\left[\begin{pmatrix} 0 & \partial_y \\ \partial_y & 0 \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \partial_t\right] \begin{pmatrix} E_z \\ H_y \end{pmatrix} = - \begin{pmatrix} J_z^{\mathrm{ext}} \\ 0 \end{pmatrix}, \quad (1)$$

succinctly written as

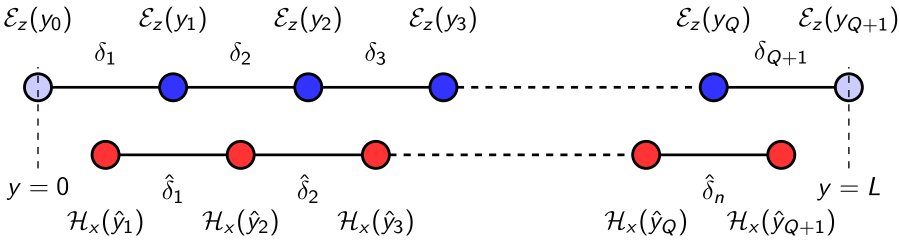
$$[\mathcal{D} \quad + \quad \mathcal{S} \quad + \quad \mathcal{M}\partial_t] \quad f \quad = - \quad q, \quad (2)$$

Discretization:
differential operators & functions $\mapsto$ matrices and vectors

Maxwell on a grid



Maxwell on a grid



We discretize the equation

$$\partial_y \mathcal{H}_x|_{y=y_q} + \sigma(y_q)\mathcal{E}_z(y_q,t)\varepsilon_{\mathrm{r}}(y_q)\partial_t\mathcal{E}_z(y_q,t) = -\mathcal{J}_z^{\mathrm{ext}}(y_q,t),$$

on primary grid (blue) for $q = 1, 2, \ldots, Q$ and

$$\partial_y \mathcal{E}_z|_{y=\hat{y}_q} + \mu(\hat{y}_q)\partial_t\mathcal{H}_x(\hat{y}_q,t) = 0$$

on the dual grid (red) for $q = 1, 2, \ldots, Q+1$.

Discrete Equations

We finally arrive at

$$\frac{h_x(\hat{y}_{q+1},t)-h_x(\hat{y}_q,t)}{\hat{\delta}_{y;q}}+\sigma(y_q)e_z(y_q,t)+\varepsilon(y_q)\partial_t e_z(y_q,t)=-j_z^{\mathrm{ext}}(y_q,t)$$

for $q = 1, 2, ..., Q$ and

$$\frac{e_z(y_q,t)-e_z(y_{q-1},t)}{\delta_{y;q}}+\mu(\hat{y}_q)\partial_t h_x(\hat{y}_q,t)=0,$$

for $q = 1, 2, ..., Q + 1$.
Where are the boundary conditions?

System Formulation

We collect the FD approximation in the vectors

$$\mathrm{e}_z = [e_z(y_1, t), e_z(y_2, t), ..., e_z(y_Q, t)]^T,$$
$$\mathrm{h}_x = [h_x(\hat{y}_1, t), h_x(\hat{y}_2, t), ..., h_x(\hat{y}_{Q+1}, t)]^T.$$

The source vector $\mathrm{j}_z^{\mathrm{ext}}$ is defined in a similar manner. In addition we introduce the differentiation matrices $\hat{\mathrm{Y}}$ and Y as

$$\hat{\mathrm{Y}} = \begin{pmatrix} -\hat{\delta}_{y;1}^{-1} & \hat{\delta}_{y;1}^{-1} & & & \\ & -\hat{\delta}_{y;2}^{-1} & \hat{\delta}_{y;2}^{-1} & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \\ & & & -\hat{\delta}_{y;Q}^{-1} & \hat{\delta}_{y;Q}^{-1} \end{pmatrix}, \quad \mathrm{Y} = \begin{pmatrix} \delta_{y;1}^{-1} & & & \\ -\delta_{y;2}^{-1} & \delta_{y;2}^{-1} & & \\ & -\delta_{y;3}^{-1} & \delta_{y;3}^{-1} & \\ & & \ddots & \ddots \\ & & & \ddots \\ & & -\delta_{y;Q}^{-1} & \delta_{y;Q}^{-1} \\ & & & -\delta_{y;Q+1}^{-1} \end{pmatrix}.$$

Note that $\hat{\mathrm{Y}} \in \mathbb{R}^{Q\times(Q+1)}$ and $\mathrm{Y} \in \mathbb{R}^{(Q+1)\times Q}$.

System Formulation 2

We also introduce the diagonal medium matrices

$$\begin{aligned} M_\sigma &= \mathrm{diag}(\sigma(y_1), \sigma(y_2), ..., \sigma(y_Q)), \\ M_\varepsilon &= \mathrm{diag}(\varepsilon(y_1), \varepsilon(y_2), ..., \varepsilon(y_Q)), \\ M_\mu &= \mathrm{diag}(\mu(\hat{y}_1), \mu(\hat{y}_2), ..., \mu(\hat{y}_{Q+1})). \end{aligned}$$

System Formulation 2

We also introduce the diagonal medium matrices

$$\begin{aligned} \mathsf{M}_\sigma &= \mathrm{diag}(\sigma(y_1), \sigma(y_2), ..., \sigma(y_Q)), \\ \mathsf{M}_\varepsilon &= \mathrm{diag}(\varepsilon(y_1), \varepsilon(y_2), ..., \varepsilon(y_Q)), \\ \mathsf{M}_\mu &= \mathrm{diag}(\mu(\hat{y}_1), \mu(\hat{y}_2), ..., \mu(\hat{y}_{Q+1})). \end{aligned}$$

Leading to the discrete system

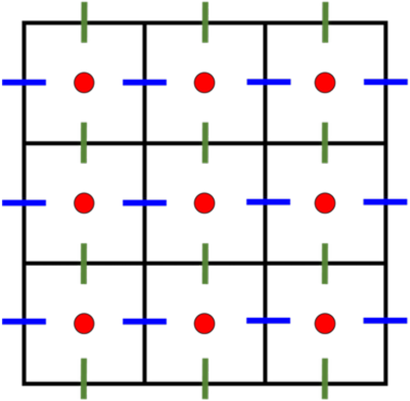
$$\left[\begin{pmatrix} 0 & \hat{\mathsf{Y}} \\ \mathsf{Y} & 0 \end{pmatrix} + \begin{pmatrix} \mathsf{M}_\sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \mathsf{M}_\varepsilon & 0 \\ 0 & \mathsf{M}_\mu \end{pmatrix}, \partial_t \right] \begin{pmatrix} \mathsf{e}_z \\ \mathsf{h}_x \end{pmatrix} = - \begin{pmatrix} \mathsf{j}_z^{\mathrm{ext}} \\ 0 \end{pmatrix}$$

or more succinctly

$$[\mathsf{D} + \mathsf{S} + \mathsf{M}\partial_t]\, \mathsf{f}(t) = -\mathsf{q}(t).$$

What happens in higher dimensions?

- Every field component $(\mathcal{H}_x, \mathcal{H}_y, \mathcal{E}_z)$ gets defined on ints own grid
- $\partial_x \mathcal{H}_y$ and $\partial_y \mathcal{H}_x$ are evaluated on the primary grid of $\mathcal{E}_z$
- We sort these 2D fields into a single vector



We still obtain a form similar to

$$[\mathsf{D} + \mathsf{S} + \mathsf{M}\partial_t]\, \mathsf{f}(t) = -\mathsf{q}(t).$$

Properties and Symmetry summary

To summarize we find

$$\mathsf{D}^T\mathsf{W} = -\mathsf{W}\mathsf{D},$$

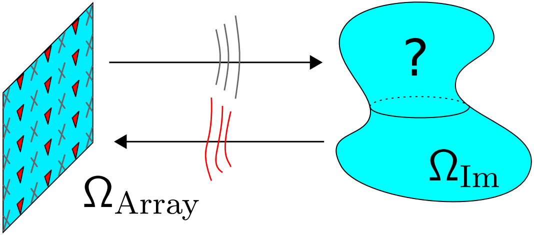
aswell with $\delta^- = \mathrm{diag}(\mathsf{I}, -\mathsf{I})$

$$\mathsf{D}^T\mathsf{W} = \delta^-\mathsf{W}\mathsf{D}.$$

This symmetry is useful to preserve in reduced order modeling.

Can we image based on finite-differences?

Assume a active array imaging configuration



Can we reconstruct the impedance $z(\mathbf{x}) = \sqrt{\frac{\mu}{\varepsilon}}$ in 1D from boundary measurements $u(0,s)$ in the Laplace (s) domain?

$$
\begin{aligned}
v_x(x,s) + s\frac{1}{z(x)}u(x,s) &= 0 \\
u_x(x,s) + sz(x)v(x,s) &= 0 \\
v(0,s) = -1 \quad u(L,s) &= 0,
\end{aligned}
\tag{3}
$$

Can we image based on finite-differences?

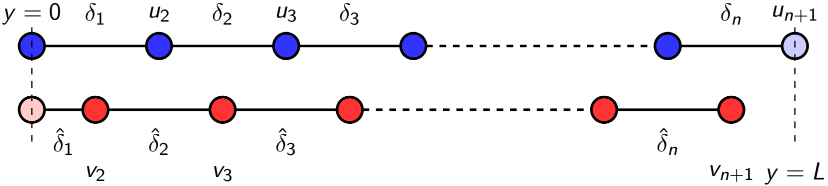
- We want to reconstruct $z(x)$ from measuring $u(x)$
- Define the data as [2m-1]/[2m] rational function

$$\Lambda(s) = \frac{u(0,s)}{v(0,s)} = -u(0,s) = \sum_{j=1}^{m} \frac{r_j}{s+\zeta_j} + \frac{\bar{r}_j}{s+\bar{\zeta}_j} \qquad (4)$$

- m residues r_j and poles ζ_j

Can we image based on finite-differences?

Can we link this rational function to a FD discretization?



Discretizing on this grid leads to

$$\frac{v_{j+1}-v_j}{\hat{\delta}_j} + s\frac{1}{z_j^{\mathrm{FD}}}u_j = 0$$

$$\frac{u_j-u_{j-1}}{\delta_{j-1}} + s\hat{z}_j^{\mathrm{FD}}v_j = 0$$

$$v_1(s) = -1 \quad u_{n+1}(s) = 0.$$

with $z_j = z(y_j)$, $\hat{z}_j = z(\hat{y}_j)$, $u_j = u(y_j)$ and $v_j = v(\hat{y}_j)$

Compare spectral and FD discretization

We have two representations for the same data. The FD data after sorting the field as $[u_1, v_2, u_2, v_3, \dots]$ reads

$$\Lambda^{\mathrm{FD}}(i\omega) = -e_1^T \left(\begin{bmatrix} 0 & \frac{1}{\hat\delta_1} & 0 & \dots & 0 \\ \frac{1}{\delta_1} & 0 & -\frac{1}{\delta_1} & \dots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & -\frac{1}{\hat\delta_n} & 0 & \frac{1}{\hat\delta_n} \\ 0 & \dots & 0 & \frac{1}{\delta_n} & 0 \end{bmatrix} + s \begin{bmatrix} \frac{1}{z_1} & & & & \\ & \hat z_1 & & & \\ & & \ddots & & \\ & & & \frac{1}{z_n} & \\ & & & & -\hat z_{n+1} \end{bmatrix} \right)^{-1} e_1,$$

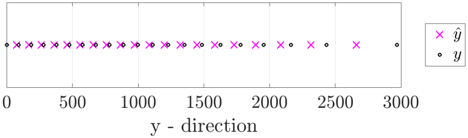
and the data can be written as

$$\Lambda^{\mathrm{FD}}(i\omega) = e_1^T \left(\begin{bmatrix} \zeta_1 & & & & \\ & \overline{\zeta}_1 & & & \\ & & \ddots & & \\ & & & \zeta_m & \\ & & & & \overline{\zeta}_m \end{bmatrix} + s \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} y_1 \\ \overline{y}_1 \\ \vdots \\ y_1 \\ \overline{y}_1 \end{bmatrix},$$

If we take $m = n$ we can find a linear algebraic transform directly from $\Lambda(s)$ to $\Lambda(s)^{\mathrm{FD}}$ and link the poles ζ_j and residues r_j to z_j and $\hat z_j$.

Compare spectral and FD discretization

The grid steps δ_j and $\hat{\delta}_j$ are known and medium independent.



The impedance can be found from the data (here $m = 20$)